Markov Processes and Martingales

Károly Simon

Department of Stochastics
Institute of Mathematics
Technical University of Budapest
www.math.bme.hu/~simonk

A file

Martingales, the definition

Definition 1.1 (Filtered space)

Here we follow the Williams' book. [21] A filtered space is $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{\mathcal{F}_n\}_{n=0}^{\infty}$ is a filtration. This means:

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \cdots \subset \mathcal{F}$$

is an increasing sequence of sub σ-algebras of $\mathcal{F}$. Put

(1) ?a1? $$\mathcal{F}_\infty := \sigma\left(\bigcup_n \mathcal{F}_n\right) \subset \mathcal{F}.$$

The reason that we use filtration so often is

Martingales, the definition (cont.)

Theorem 1.2

?⟨a2⟩? *Given the r.v. $X_1, \dots, X_n$ and Y on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define $\mathcal{F} := \sigma(X_1, \dots, X_n)$. Then*

(2) $$Y \in \mathcal{F} \iff \exists g : \mathbb{R}^n \to \mathbb{R}, \text{ Borel s.t. } Y(\omega) = g(X_1(\omega), \dots, X_n(\omega)).$$

This means that if $X_1, \dots, X_n$ are outcomes of an experiment then the value of Y is predictable based on we know the values of $X_1, \dots, X_n$ iff $Y \in \mathcal{F}$, where $Y \in \mathcal{F}$ means that Y is $\mathcal{F}$-measurable.

Martingales, the definition (cont.)

When we say simply "process" in this talk, we mean "Discrete time stochastic process".

Definition 1.3 (Adapted process)

We say that the process $M = \{M_n\}_{n=0}^{\infty}$ is adapted to the filtration $\{\mathcal{F}_n\}$ if $M_n \in \mathcal{F}_n$.

Martingales, the definition (cont.)

Definition 1.4

⟨a61⟩? Let $M = \{M_n\}_{n=0}^{\infty}$ be an adaptive process to the filtration $\{\mathcal{F}_n\}$. We say that X is a martingale if

(i) $\mathbb{E}[|M_n|] < \infty$, $\forall n$

(ii) $\mathbb{E}[M_n|\mathcal{F}_n] = M_{n-1}$ a.s. for $n \geq 1$

X is supermartingale if we substitute (ii) with

$$\mathbb{E}[M_n|\mathcal{F}_{n-1}] \leq M_{n-1} \text{ a.s. } n \geq 1.$$

Finally, M is a submartingale if we substitute (ii) with

$$\mathbb{E}[M_n|\mathcal{F}_{n-1}] \geq M_{n-1} \text{ a.s. } n \geq 1.$$

Functions of MC

Remark 2.1

⟨a10⟩ Given a Markov chain $X = (X_n)$ with transition probability matrix $\mathbf{P} = (p(x,y))_{x,y}$. We are also give a function $f : S \times \mathbb{N} \to \mathbb{R}$ satisfying

(5) a8 $$f(x,n) = \sum_{y \in S} p(x,y) f(y, n+1).$$

Then $M_n = f(X_n)$ is a martingale w.r.t. X. (We verified this in the Stochastic Processes course. See [4, Theorem 5.5].)

Functions of MC (cont.)

Given a Markov chain $X = (X_n)$ with transition probability matrix $\mathbf{P} = (p(x,y))_{x,y}$.

Functions of MC (cont.)

Definition 2.2 (P-harmonic functions)

⟨a12⟩? For an $f : S \to \mathbb{R}$:

(6) ?a6? $$Pf(x) := \sum_{y \in S} p(x,y) f(y).$$

We say that such an f is harmonic if

(i) $\sum_{y \in S} p(x,y)|f(y)| < \infty$, $\forall x \in S$ and

(ii) $\forall x \in S, \quad h(x) = Ph(x)$

if we replace (ii) with $\forall x, \ f(x) \le Pf(x)$ then f is subharmonic.

Functions of MC (cont.)

f is called superharmonic if $-f$ is subharmonic. It follows from Remark 2.1 that

Theorem 2.3

Let $X = (X_n)$ be a Markov chain with transition probability matrix $\mathbf{P} = (p(x,y))_{x,y}$ and let h be a $\mathbf{P}$-harmonic function. Then $h(X_n)$ is a Martingale w.r.t. X.

Functions of MC (cont.)

Example 2.4

?⟨a9⟩? Let $X_1, X_2, \dots$ be iid with

$$\mathbb{P}(X_i = 1) = p \text{ and } \mathbb{P}(X_i = -1) = 1 - p,$$

$p \in (0,1)$, $p \neq 0.5$. Let $S_n := X_1 + \cdots + X_n$. Then

(7) ?a11? $$M_n := \left(\frac{1-p}{p}\right)^{S_n}$$

is a martingale. Namely, $h(x) = ((1-p)/p)^x$ is harmonic.

Functions of MC (cont.)

Example 2.5 (Simple Symmetric Random Walk)

Let $Y_1, Y_2, \dots$ be iid with

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2,$$

We write $S_n := S_0 + Y_1 + \cdots + Y_n$. Then $M_n := S_n^2 - n$ is a martingale. Namely, $f(x,n) = x^2 - n$ satisfies (5).

Theorem 2.6

⟨a14⟩? *Let h be a subharmonic function for the Markov chain $X = (X_n)$. Then $M_k := h(X_k)$ is a submartingale.*

Polya's Urn,

One can find a nice account with more details at `http://www.math.uah.edu/stat/urn/Polya.html` or click here

Given an urn with initially contains: $r > 0$ red and $g > 0$ green balls.

(a) draw a ball from the urn randomly,

(b) observe its color,

(c) return the ball to the urn along with c new balls of the same color.

- If $c = 0$ this is sampling with replacement.
- If $c = -1$ sampling without replacement.

Polya's Urn, (cont.)

From now we assume that $c \geq 1$. After the n-th draw and replacement step is completed:

- the number of green balls in the urn is: G_n.
- the number of red balls in the urn is: R_n.
- the fraction of green balls in the urn is X_n.
- Let $Y_n = 1$ if the n-th ball drawn is green. Otherwise $Y_n := 0$.
- Let $\mathcal{F}_n$ be the filtration generated by $Y = (Y_n)$.

Polya's Urn, (cont.)

Claim 1

X_n is a martingale w.r.t. $\mathcal{F}_n$.

Proof Assume that

$$R_n = i \text{ and } G_n = j$$

Then

$$\mathbb{P}\left(X_{n+1} = \frac{j+c}{i+j+c}\right) = \frac{j}{i+j},$$

and

$$\mathbb{P}\left(X_{n+1} = \frac{j}{i+j+c}\right) = \frac{i}{i+j}.$$

Polya's Urn, (cont.)

Hence

$$(8)\quad \mathbb{E}\left[X_{n+1}|\mathcal{F}_n\right] = \frac{j+c}{i+j+c}\cdot\frac{j}{i+j} + \frac{j}{i+j+c}\cdot\frac{i}{i+j} = \frac{j}{i+j} = X_n.$$

□

Corollary 3.1

There exists an X_∞ s.t. $X_n \to X_\infty$ a.s..

This is immediate from Theorem 1.10.
In order to find the distribution of $X\infty$ observe that:

Polya's Urn, (cont.)

- The probability $p_{n,m}$ of getting green on the first m steps and getting red in the next $n-m$ steps is the same as the probability of drawing altogether m green and $n-m$ red balls in any particular redescribed order.
-

$$p_{n,m} = \prod_{k=0}^{m-1} \frac{g+kc}{g+r+kc} \cdot \prod_{\ell=0}^{n-m-1} \frac{r+\ell c}{g+r+(m+\ell)c}$$

Polya's Urn, (cont.)

If $c=g=r=1$ then

$$\mathbb{P}(G_n = 2m+1) = \binom{n}{m}\frac{m!(n-m)!}{(n+1)!} = \frac{1}{n+1}.$$

That is X_∞ is uniform on $(0,1)$: In the general case X_∞ has density

$$\frac{\Gamma((g+r)/c)}{\Gamma(g/c)\Gamma(r/c)} x^{(g/c)-1}(1-x)^{(r/x)-1}.$$

That is the distribution of X_∞ is $\mathrm{Beta}\left(\frac{g}{c},\frac{r}{c}\right)$

Games

Imagine that somebody plays games at times $k = 1, 2, \dots$. Let $X_k - X_{k-1}$ be the net winnings per unit stake in game n.
In the martingale case

$$\mathbb{E}\left[X_n - X_{n-1}|\mathcal{F}_{n-1}\right] = 0, \quad \text{the game is fair.}$$

In the supermartingale case

$$\mathbb{E}\left[X_n - X_{n-1}|\mathcal{F}_{n-1}\right] \le 0, \quad \text{the game is unfavorable.}$$

Games (cont.)

Definition 4.1

⟨a19⟩ Given a process $C = (C_n)$. We say that:

(i) C is previsible or predictable if

$$\forall n \ge 1, \quad C_n \in \mathcal{F}_{n-1}.$$

(ii) C is bounded if $\exists K$ such that $\forall n, \forall \omega,\ |C_n(\omega)| < K$.

(iii) C has bounded increments if $\exists K$ s.t. $\forall n \ge 1, \forall \omega \in \Omega,\ |C_{n+1}(\omega) - C_n(\omega)| < K$

Games (cont.)

C_n is the player's stake at time n which is decided based upon the history of the game up to time $n-1$. The winning on game n is $C_n(X_n - X_{n-1})$. The total winning after n game is

$$(9)\ \boxed{\text{a18}}\ Y_n := \sum_{1\le k\le n} C_k(X_k - X_{k-1}) =: (C \bullet X)_n.$$

By definition:

$$(C \bullet X)_0 = 0.$$

Clearly,

$$Y_n - Y_{n-1} = C_n(X_n - X_{n-1}).$$

Games (cont.)

We say that

$C \bullet X$ is the martingale transform of X by C.

Games (cont.)

Theorem 4.2 (You cannot beat the system)

⟨a20⟩ *Given $C = (C_n)_{n=1}^{\infty}$ satisfying:*

(a) $C_n \geq 0$ *for all n (otherwise the player would be the Casino),*

(b) *C is previsible (that is $C_n \in \mathcal{F}_{n-1}$),*

(c) *C is bounded.*

Then $C \bullet X$ is a supermartingale (martingale) if X is a supermartingale (martingale) respectively.

Games (cont.)

Proof.

$$(10) \quad \mathbb{E}\left[Y_n - Y_{n-1} | \mathcal{F}_{n-1}\right] = C_n \mathbb{E}\left[X_n - X_{n-1} | \mathcal{F}_{n-1}\right] < 0.$$

□

Theorem 4.3

⟨a21⟩ *Assume that C is a bounded and previsible process and X is a martingale then $C \bullet X$ is a martingale which is null at 0.*

Games (cont.)

Theorem 4.4

In the previous two theorems the boundedness can be replaced by $C_n \in L^2$, $\forall n$ if $X_n \in L^2$, $\forall n$.

The proofs of the one but last theorem is obvious. The proof of the last theorem immediately follows from (f) on slide 133 of file "Some basic facts from probability theory".

Stopping Times, definitions

Definition 5.1

A map $T : \Omega \to \{0, 1, \dots, \infty\}$ is called stopping time if

$$(11) \quad ?a16? \{T \leq n\} = \{\omega : T(\omega) \leq n\} \in \mathcal{F}_n, \quad n \leq \infty.$$

equivalent definition:

$$(12) \quad ?a17? \{T = n\} = \{\omega : T(\omega) = n\} \in \mathcal{F}_n, \quad n \leq \infty.$$

We say that the stopping time T is bounded if $\exists K$ s.t. $T(\omega) < K$ holds for all $\omega \in \Omega$.

Stopping Times, definitions (cont.)

E.g. T is the time when we stop plying the game. We can decided at time n if we stop at that moment based on the history up to time n.

Stopping Times, definitions (cont.)

Example 5.2

Given a process (X_n) which is adapted to the filtration $\{\mathcal{F}_n\}$, further given a Borel set B. Let

$$T := \inf\left\{n \geq 0 : X_n \in B\right\}.$$

By convention: $\inf \emptyset := \infty$. Then

$$\{T \leq n\} = \bigcup_{k \leq n} \{T = k\} \in \mathcal{F}_n.$$

Stopping Times, definitions (cont.)

Lemma 5.3

⟨a37⟩ *Assume that T is a stopping time w.r.t. the filtration $\{\mathcal{F}_n\}$. Let*

$$C_n^T := \mathbb{1}_{n \le T}.$$

Then C_n^T is previsible. That is

$$C_n^T \in \mathcal{F}_{n-1}. \quad (13) \text{ ?a40?}$$

Proof.

$\{C_n^T = 0\} = \{T \le n-1\} \in \mathcal{F}_{n-1}$. □

Stopped martingales

⟨a60⟩? Let T be a stopping time for an $\{\mathcal{F}_n\}$ filtration. For a process $X = (X_n)$ we write X^T for the process stopped at T:

$$X_n^T(\omega) := X_{T(\omega)\wedge n}(\omega),$$

where $a \wedge b := \min\{a, b\}$.

Assume that Kázmér always bets 1\$ and stops playing at time T. Then Kazmér's stake process is:

$$C_n^{(T)} = \mathbb{1}_{n \le T} \quad (14) \text{ a23}$$

Stopped martingales (cont.)

In Lemma 5.3 we proved that $C^{(T)}$ is previsible (the notion "previsible" was defined on slide # 23).
By (9), Kázmér's winning's process:

$$(C^{(T)} \bullet X)_n = X_{T\wedge n} - X_0.$$

That is

$$C^{(T)} \bullet X = X^T - X_0.$$

So, by Theorems 4.2 and 4.3 we obtain

Stopped martingales (cont.)

Theorem 6.1

⟨a22⟩ *Let T be a stopping time*

(i)

$$X \text{ supermartingale} \Longrightarrow X^T \text{ supermartingale}.$$

So, in this case $\forall n$, $\mathbb{E}[X_{T\wedge n}] \le \mathbb{E}[X_0]$

(ii)

$$X \text{ martingale} \Longrightarrow X^T \text{ martingale}.$$

So, in this case $\forall n$, $\mathbb{E}[X_{T\wedge n}] = \mathbb{E}[X_0]$

Stopped martingales (cont.)

Proof

We define $C_n^{(T)}$ as in (14). Clearly, $C^{(T)} \ge 0$ and bounded. As we saw in Lemma 5.3, $C^{(T)}$ is previsible. So, we can apply Theorem 4.2 for

$$(15) \quad (C \bullet X)_n = \sum_{K=1}^{n} C_k \cdot (X_k - X_{k-1})$$

$$= \begin{Bmatrix} X_n - X_0, & \text{on } \{T \ge n\}; \\ \sum_{k=1}^{T}(X_k - X_{k-1}) = X_T - X_0, & \text{on } \{T < n\}. \end{Bmatrix}$$

$$= X_{T\wedge n} - X_0.$$

Stopped martingales (cont.)

That is by Theorem 4.2 we get that $X_{T\wedge n} - X_0$ is a supermartingale (martingale) if (X_n) is a supermartingale (martingale) respectively. Which yields the assertion of the theorem. ■

Remark 6.2

⟨a26⟩? It can happen for a martingale X that

$$\mathbb{E}[X_n] \ne \mathbb{E}[X_0]. \quad (16) \text{ a32}$$

The most popular counter example uses the Simple Symmetric Random Walk (SSRW). First we recall its definition and a few of its most important properties.

Stopped martingales (cont.)

Example 6.3 (Simple Symmetric Random Walk (SSRW))

⟨a27⟩? The Simple Symmetric Random Walk (SSRW) on $\mathbb{Z}$ is $S = (S_n)_{n=0}^{\infty}$, where

$$S_n = X_0 + X_1 + \cdots + X_n, \quad (17) \text{ ?a33?}$$

where $X_0 = 0$ and $X_1, X_2, \dots$ are iid with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$.

We have seen that

Stopped martingales (cont.)

Lemma 6.4 (SSRW)

⟨a34⟩? *The Simple Symmetric Random Walk on $\mathbb{Z}$ is*

(i) *Null recurrent,*

(ii) *martingale.*

The second part follows from Example 1.7. We proved that SSRW is null recurrent in the course Stochastic processes. To give an example where (16) happens:

Stopped martingales (cont.)

Example 6.5

⟨a35⟩? $S = (S_n)$ be the SSRW and let $T := \inf\{n : S_n = 1\}$. Then by Theorem 6.1, $\mathbb{E}[X_{T\wedge n}] = \mathbb{E}[X_0]$. However,

$$\mathbb{E}[X_T] = 1 \neq 0 = X_0 = \mathbb{E}[X_0].$$

Question 1

Let X be a martingale and let T be a stopping time. Under which conditions can we say that

$$(18)\ ?a25? \qquad \mathbb{E}[X_T] = \mathbb{E}[X_0]?$$

Stopped martingales (cont.)

Theorem 6.6 (Doob's Optional Stopping Theorem)

⟨a28⟩ *Let X be a supermartingale and T be a stopping time. If any of the following conditions holds*

(i) *T is bounded.*

(ii) *X is bounded and $T < \infty$ a.s..*

(iii) *$\mathbb{E}[T] < \infty$ and X has bounded increments.*

then

(a) *$X_T \in L^1$ and $\mathbb{E}(X_T) \leq \mathbb{E}[X_0]$.*

(b) *If X is a martingale then $\mathbb{E}(X_T) = \mathbb{E}[X_0]$.*

Stopped martingales (cont.)

Proof.

By Thm: 6.1 $\forall n$, $X_{T\wedge n} \in L^1$ and $\mathbb{E}[X_{T\wedge n} - X_0] \leq 0$.
If (i) holds then $\exists N$ s.t. $T \leq N$. Then for $n = N$, we have $X_{T\wedge n} = X_T$. Hence (a) follows.
If (ii) holds then $\lim\limits_{n\to\infty} X_{T\wedge n} = X_T$. So, by Dominated Convergence Theorem: $\lim\limits_{n\to\infty} \mathbb{E}[X_{n\wedge T}] = \mathbb{E}[X_T]$. On the other hand, by Theorem 6.1, $\mathbb{E}[X_{T\wedge n}] \leq \mathbb{E}[X_0]$.
If (iii) holds The answer comes from Dom. Conv. Thm. $|X_{T\wedge n} - X_0| = \left|\sum\limits_{k=1}^{T\wedge n}(X_k - X_{k-1})\right| \leq KT < \infty$. If X is a martingale, apply everything above also for $-X$. □

Stopped martingales (cont.)

Corollary 6.7

⟨a41⟩? *Assume that*

(a) *$M = (M_n)$ is a martingale.*

(b) *$\exists K_1$ s.t. $\forall n$, $|M_n - M_{n-1}| < K_1$,*

(c) *$C = \{C_n\}$ is a previsible process with $|C_n(\omega)| < K_2$, $\forall \omega, \forall n$.*

(d) *T is a stopping time with $\mathbb{E}[T] < \infty$.*

Then

$$(19)\ ?a42? \qquad \mathbb{E}[(C \bullet M)_T] = 0.$$

Stopped martingales (cont.)

Proof.

Put together Theorem 4.3 and Theorem 6.6. □

A corollary of the Optional Stopping Theorem is:

Theorem 6.8

⟨a43⟩? *Assume that*

(i) *$M = (M_n)$ is a non-negative supermartingale,*

(ii) *T is a stopping time s.t. $T < \infty$ a.s..*

Then $\mathbb{E}[X_T] \leq \mathbb{E}[X_0]$.

Stopped martingales (cont.)

Proof.

We know that $\lim\limits_{n\to\infty} X_{T\wedge n} = X_T$ a.s. and $X_{T\wedge n} \geq 0$. So we can apply Fatou Lemma :

$$\liminf_{n\to\infty} \mathbb{E}[X_{T\wedge n}] \geq \mathbb{E}[X_T].$$

On the other hand, by Theorem 6.1 the left hand side is smaller than or equal to $\mathbb{E}[X_0]$. □

Awaiting for the (almost) inevitable

In order to apply the previous theorems we need a machinery to check if $\mathbb{E}[T < \infty]$ a.s. holds.

Theorem 6.9

⟨a44⟩ *Assume that $\exists N \in \mathbb{N}, \varepsilon > 0$ s.t. $\forall n \in \mathbb{N}$,*

$$(20)\ \boxed{a45} \qquad \mathbb{P}(T \leq n + N | \mathcal{F}_n) > \varepsilon, \quad a.s.$$

then

$$\mathbb{E}[T] < \infty.$$

Awaiting for the (almost) inevitable (cont.)

Proof.
We apply (20) for $n = (k-1)N$. Then the assertion follows by mathematical induction from Homework 11. □

ABRACADABRA

The following exercise is named as "Tricky exercise" in Williams' book [21, p.45].

Problem 6.10 (Monkey at the typewriter)

⟨a46⟩? *Assume that a monkey types on a typewriter. He types only capital letters and he chooses equally likely any of the 26 letters of the English alphabet independently of everything. What is the expected number of letters he needs to type until the word "ABRACADABRA" appears in his typing for the first time?*

The same problem formulated in a more formal way:

ABRACADABRA (cont.)

Problem 6.11 (Monkey at the typewriter)

⟨a48⟩? *Let $X_1, X_2, \dots$ be iid r.v. taking values from the set* $\text{Alphabet} := \{A, B, \dots, Z\}$ *of cardinality 26. We assume that the distribution of X_k is uniform. Let T be*

$$T := \min \{n + 10 : (X_n, X_{n+1}, \dots, X_{n+10}) = (A, B, R, A, C, A, D, A, B, R, A)\} \tag{21}$$

Find $\mathbb{E}[T] = ?$

We associate a players in a Casino to the monkey:

ABRACADABRA (cont.)

Example 6.12 (Players associated to the monkey)

⟨a47⟩? Imagine that for every $\ell = 1, 2, \dots,$ on the ℓ-th day a new gambler arrives in a Casino. He bets:
$1\$$ on the event: "$X_\ell = A$".
If he loses he leaves. If he wins he receives $26\$$. Then he bets his
$26\$$ on the event: "$X_{\ell+1} = B$".
If he loses he leaves. If he wins then he receives $26^2\$$ and then he bets all of his
$26^2\$$ on the event: "$\ell + 2$-th letter will be R".
and so on until he loses or gets ABRACADABRA.

[1] Balázs Márton, Tóth Bálint
Valószínûségszámítás 1. jegyzet matematikusoknak és fizikusoknak
Bálázs Márton Honlapja, 2012. Az internettes változatért kattintson ide.

[2] P. Billingsley
Convergence of probability measures
Wiley, 1968

[3] B. Driver
Analysis tools with examples
Lecturenotes, 2012. Click here.

[4] R. Durrett
Essentials of Stochastic Processes, Second edition
Springer, 2012. Click here

[5] R. Durrett
Probability: Theory with examples, 4th edition
Cambridge University Press, 2010.

[6] R. Durrett
Probability: Theory and Examples
Click here

[7] D.H. Fremlin
Measure Theory Volume I
Click here

[8] D.H. Fremlin
Measure Theory Volume II
Click here

[9] O. van Gaans
Probability measures on metric spaces
Click here

[10] I.I. Gihman, A.V. Szkorohod
Bevezetés a sztochasztikus folyamatok elméletébe
Mûszaki Könyvkiadó1975, Budapest, 1985

[11] S. Karlin, H.M. Taylor
A first course in stochastic processes
Academic Press, New York, 1975

[12] S. Karlin, H.M. Taylor
Sztochasztikus Folyamatok
Gondolat, Budapest, 1985

[13] S. Karlin, H.M. Taylor
A second course in stochastic processes
, Academic Press, 1981

[14] G. Lawler
Intoduction to Stochastic Processes
Chapmann & Hall 1995.

[15] D.A. Levin, Y. Peres,E.L. Wilmer
Markov chains and mixing times
American Mathematical Society, 2009.

[16] Major Péter
Folytonos idejû Markov láncok http://www.renyi.hu/~major/debrecen/debrecen2008a/markov3.html

[17] P. Mattila Geometry of sets and measure in Euclidean spaces. Cambridge, 1995.

[18] Rényi Alfréd
Valószínûségszámítás, (negyedik kiadás)
Tankönyvkiadó Budapest, 1981.

[19] S. Ross
A First Course in Probability, 6th ed.
Prentice Hall, 2002

[20] Tóth Bálint *Sztochasztikus folyamatok jegyzet*
Tóth Bálint Jegyzetért kattintson ide

[21] D. Williams
Probability with Martingales
Cambridge 2005