Fractals and Fourier anaysis

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Application of fractals

We use fractals to describe objects or phenomena in which some sort of scale invariance exists.

Fractals appear physics, astronomy, biology, chemistry, market fluctuation analysis, and so on.

At the conference

Practical Applications of Fractals
17 - 19 November 2004
Miramare, Trieste, Italy the following main applications were discussed:

Fractals in industry and man-made fractals:

- Fractal antennae,
- Fractal sound barriers,
- Use of fractal polymeric surfaces,
- Fractal reactor design,
- Fractal studies of heterogeneous catalysis,
- Petroleum research.

Natural fractal objects:

- Fractal bronchial trees in mammals,
- Growth of fractal trees in nature,
- Optimal fractal distribution,
- Absolute limitations of tree distributive structures,
- River Networks,
- Fractals and allometry (relative growth of a part in relation to an entire organism or to a standard; also: the measure and study of such growth).

Applications of fractal concepts to the study of complex systems:

- Image analysis and compression
- Multifractal signal analysis
- Scaling topology of the internet and the www
- Fractal aviation communication network

Figure: Waclaw Sierpiński

- Born in Warsaw 1882.
- Ph.D. in 1908 at Univ. of Krakkow (Poland).
- 1919-1969 worked at the Univ of Warsaw, died: 1969
- Very important results in: set theory, real analysis and topology.

How long is the coast of Britain?

Figure: Britain coastline. 200km: 2400km, 50 km:3400km

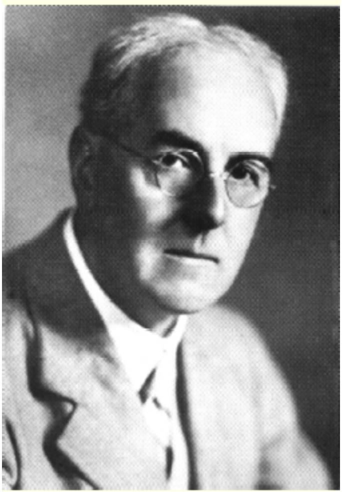


Figure: Lewis R. Richardson 1881-1953

Richardson conjectured: The measured length $L(G)$ of a geographic boarder is

$$L(G) \approx M \cdot G^{1-D},$$

M is a constant and D is the dimension . Namely:

$$L(G) = N(G) \cdot G$$

$$\frac{\log N(G)}{\log G^{-1}} \approx D \Longrightarrow L(G) \approx G^{1-D}.$$

Britain: $D = 1.25$, Germany: $D = 1.14$, South Africa $D = 1.02$.

Beniot Mandelbrot

Figure: The father of fractal geometry

- In Ecole Polytechnique, student of Julia, Lévy.
- Later post. doc. working with J. Neumann at Princeton.
- Worked for IBM for 35 years. Then moved to Yale. Books:
- Fractals: Form, Chance and Dimension 1975.
- The Fractal Geometry of Nature, 1982.

Middle-α Cantor set

Fix an $\alpha \in (0,1)$. We remove the (open) middle-α portion from the interval $[0,1]$. Repeat the same procedure for these smaller intervals ad infinitum to get the middle-α Cantor set. More precisely, let

$$\mathcal{S}_\alpha := \left\{ S_1(x) = \frac{1-\alpha}{2} \cdot x,\ S_2(x) = \frac{1-\alpha}{2} \cdot x + \frac{1+\alpha}{2} \right\}.$$

Then the middle-α Cantor set Λ_α is defined by

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,\dots,i_n)\in\{1,2\}^{\mathbb{N}}} S_{i_1\dots i_n}([0,1]). \tag{1}$$

The middle-1/2 Cantor set

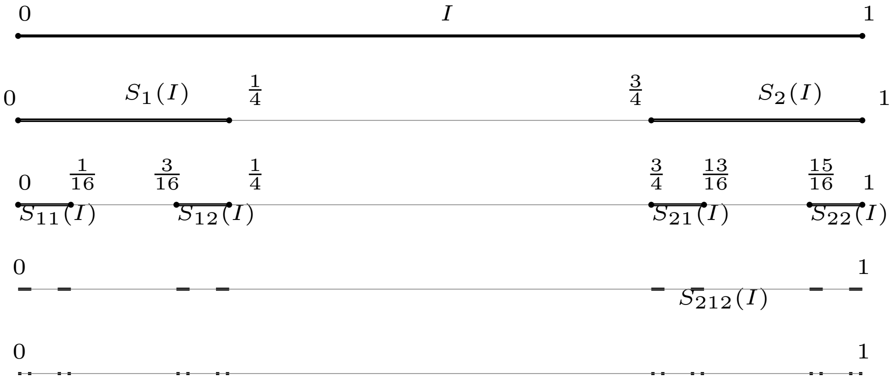


Figure: $S_{i_1 \dots i_n}(x) := S_{i_1} \circ \cdots \circ S_{i_n}(x) = S_{i_1}\left(S_{i_2}\left(\cdots\left(S_{i_n}(x)\right)\cdots\right)\right)$, where $i_k \in \{1, 2\}$.

Sierpiński Gasket

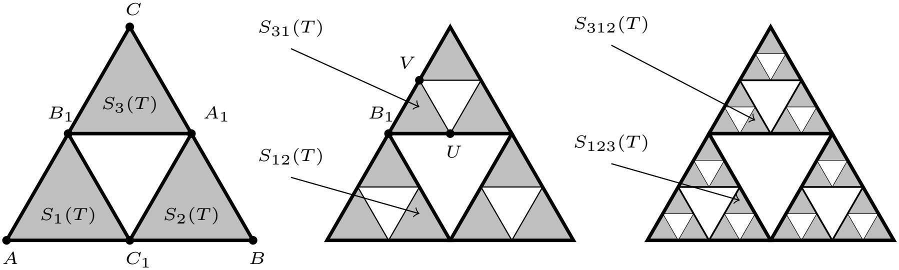


Figure: $S_{312}(x) := S_3 \circ S_1 \circ S_2(x) = S_3(S_1(S_2(x)))$

S_i are translations of the appropriate homothety-transformatons of the form:

$S_i(x) = \frac{1}{2}x + t_i$.

IFS

Let A_i be $d \times d$ non-singular matrices with $\|A_i\| < 1$ and $t_i \in \mathbb{R}^d$ for $i = 1, \dots, m$. Let

$$\mathcal{F} := \{f_i\}_{i=1}^{m} = \{A_i \cdot x + t_i\}_{i=1}^{m}, \tag{2}$$

where we always assume that

$$\|A_i\| < 1.$$

We study the attractor Λ of the IFS $\mathcal{F}$.

The attractor Λ (definition I)

Let $B = B(0, r)$ be any closed ball centered at the origin with radius r such that

$$r > \max_{1 \le i \le m} \frac{\|t_i\|}{1 - \max_{1 \le i \le m} \|A_i\|}$$

then

$$\forall i = 1, \dots, m : \qquad f_i(B) \subset B. \tag{3}$$

Thus

$$\begin{aligned} \bigcup_{i_1 \dots i_{n+1}} f_{i_1 \dots i_{n+1}}(B) &= \bigcup_{i_1 \dots i_n} f_{i_1 \dots i_n}\left(\bigcup_{i_{n+1}=1}^{m} f_{i_{n+1}}(B)\right) \\ &\subset \bigcup_{i_1 \dots i_n} f_{i_1 \dots i_n}(B) \end{aligned} \tag{4}$$

The attractor Λ (definition II)

So we can define the non-empty compact set

$$\Lambda := \bigcap_{n=1}^{\infty} \bigcup_{i_1 \dots i_n} f_{i_1 \dots i_n}(B). \tag{5}$$

The definition is independent of B. Then Λ is the only non-empty compact set satisfying

$$\Lambda = \bigcup_{i=1}^{m} f_i(\Lambda). \tag{6}$$

Coding the points of Λ

To code the elements of Λ we use the symbolic space

$$\Sigma := \{1, \dots, m\}^{\mathbb{N}} .$$

To code the elements of Λ with the infinite sequences from Σ we choose a sufficiently big closed ball B centered at the origin. We have seen that $f_i(B) \subset B$ for all $i = 1, \dots, m$. This follows that for all infinite sequence $\mathbf{i} := (i_1, i_2, \dots) \in \Sigma$ the sequence of sets

$$\{f_{i_1 \dots i_n}(B)\}_{n=1}^{\infty}$$

converge to a single point as $n \to \infty$. We call this point $\Pi(\mathbf{i})$.

Coding of the points of Λ (cont.)

For an $\mathbf{i} = (i_1, i_2, \dots) \in \Sigma$ we have

(7)
$$\begin{array}{ccc} \mathbf{i} & \xrightarrow{\ \sigma\ } & \sigma\mathbf{i} \\ \Pi \big\downarrow & & \big\downarrow \Pi \\ \Pi(\mathbf{i}) & \xleftarrow[f_{i_1}]{} & \Pi(\sigma\mathbf{i}) \end{array}$$

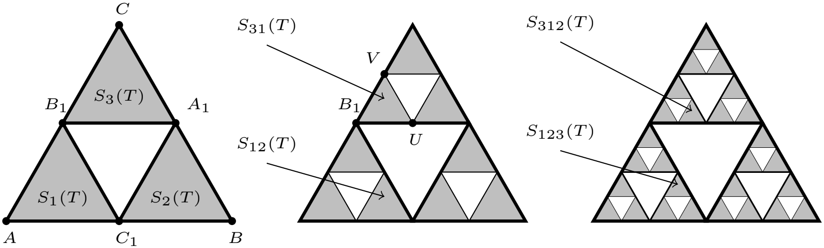


Figure: T is the "big" equilateral triangle. The first three approximations of the Sierpiński triangle.

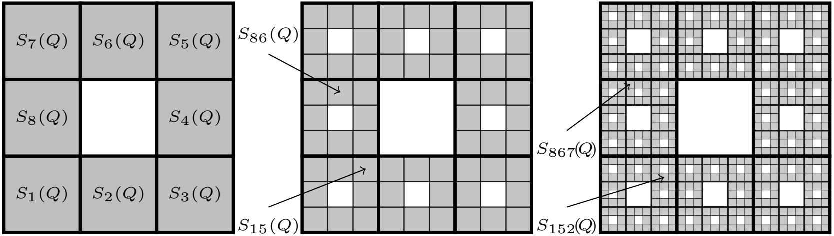


Figure: Q is the "big" square. The first three approximations of the Sierpiński carpet.

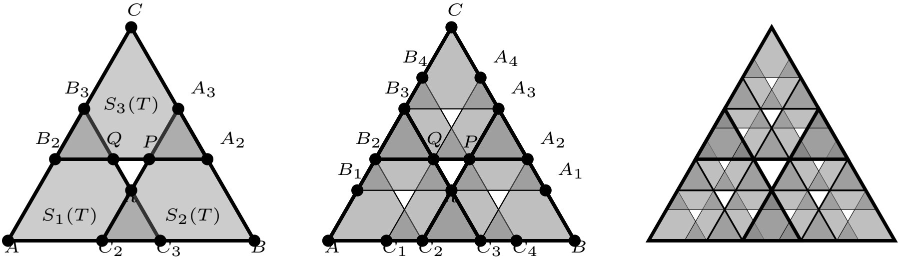


Figure: The third approximation of the golden gasket

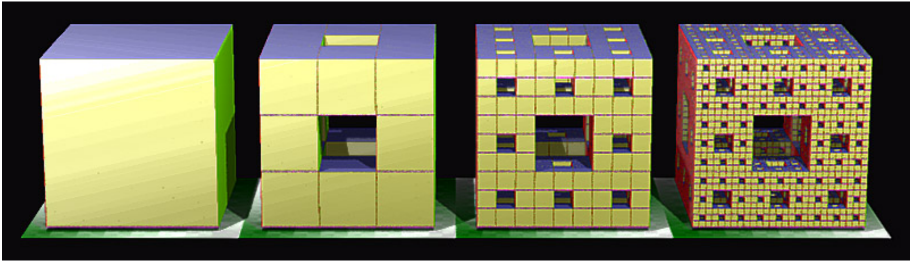


Figure: Menger Sponge (from Wikipedia)

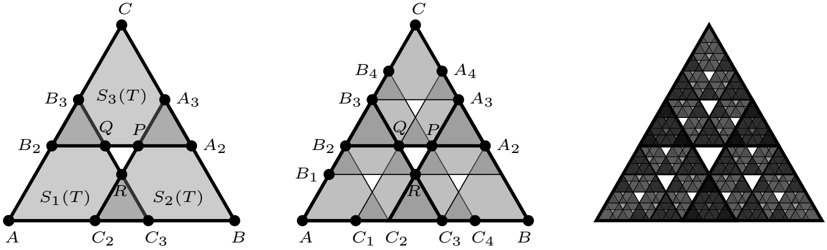


Figure: The first, the second and the fourth approximations of the golden gasket.

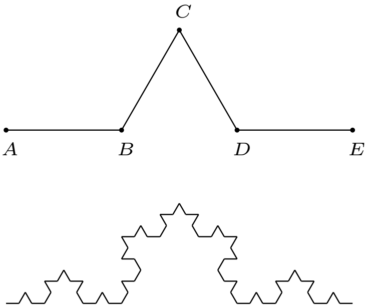


Figure: The first four approximations of the von Koch curve.

Heighway Dragon I

Click here to see a vidio on youtube how the Heighway dragon fractal builds up.

Heighway Dragon II

$$S_1(\mathbf{x}) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \mathbf{x}$$

$$S_2(\mathbf{x}) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \cdot \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\mathcal{S}_H := \{S_1(\mathbf{x}), S_2(\mathbf{x})\}. \tag{8}$$

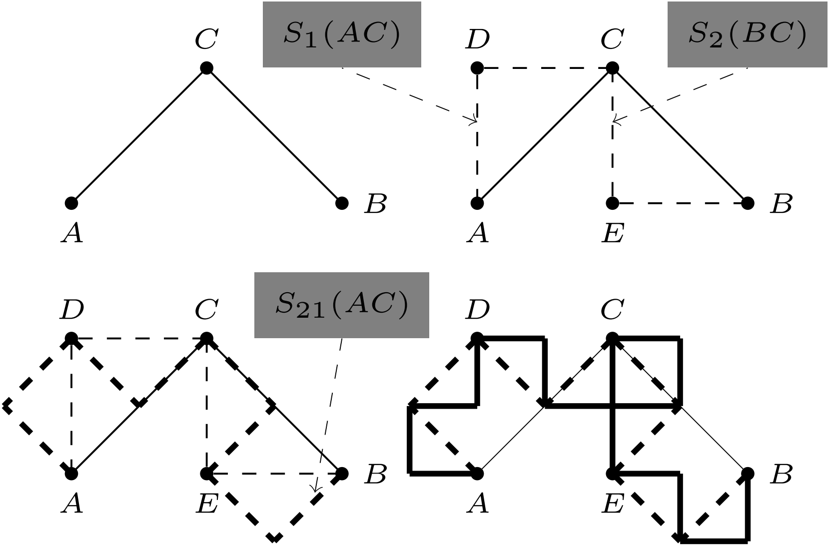


Figure: The first four approximations of the Heighway dragon.

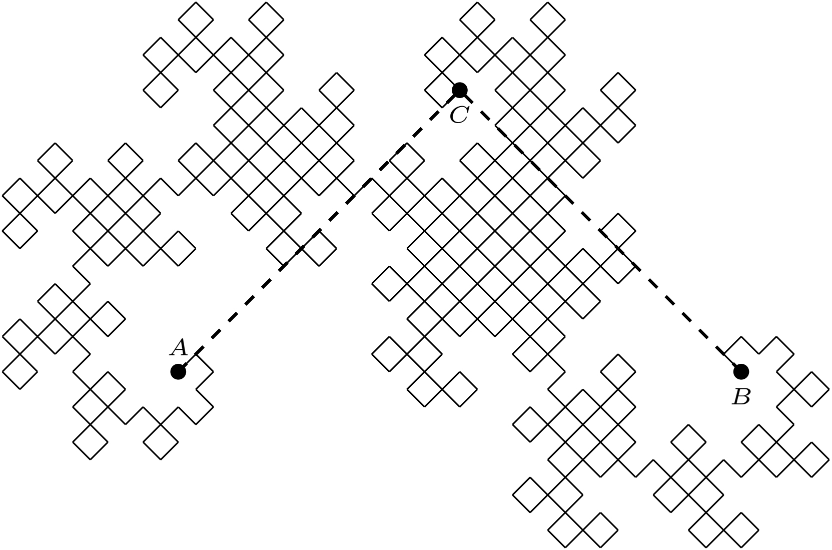


Figure: The 9-th approximation of the Heighway dragon.

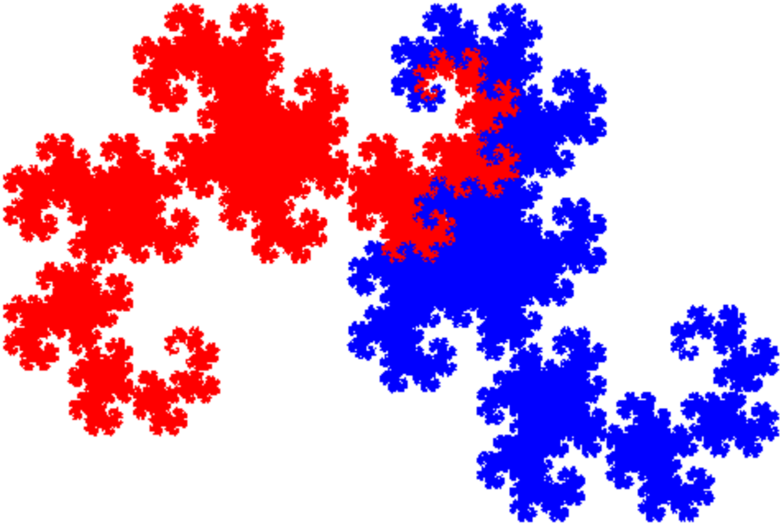


Figure: Heighway Dragon. The Figure is from the Internet.

Heighway Dragon VI

Let P_n be the broken line that we obtain after n steps. Then $\{P_n\}_{n=1}^{\infty}$ is a Cauchy sequence of compact sets in the Hausdorff metric (defined later). It converges to a set Λ (the attractor) which is called Heighway dragon .

- The interior of Λ is non-empty
- The plane can be tiled with congruent copies of Λ.
- The Hausdorff dimension (to be defined later) of the boundary is $2\log\lambda/\log 2 = 1.5236270862\ldots$, where λ is the largest real zero of $\lambda^3 - \lambda^2 - 2$.

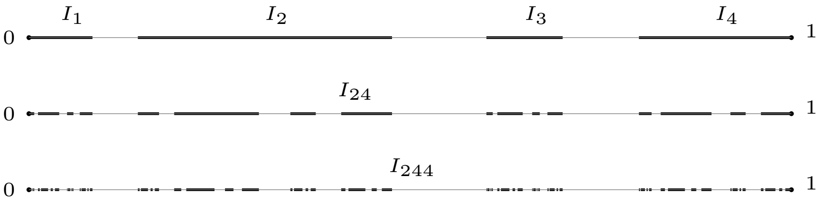


Figure: The first three approximations of a non-homogeneous self-similar IFS.

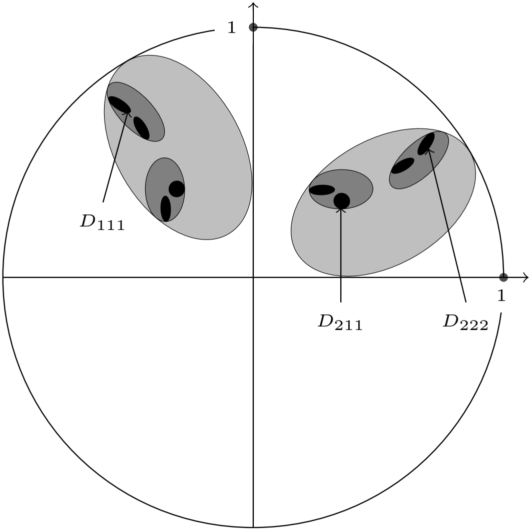


Figure: The level 1, 2 and level 3 cylinders for Example **??**.

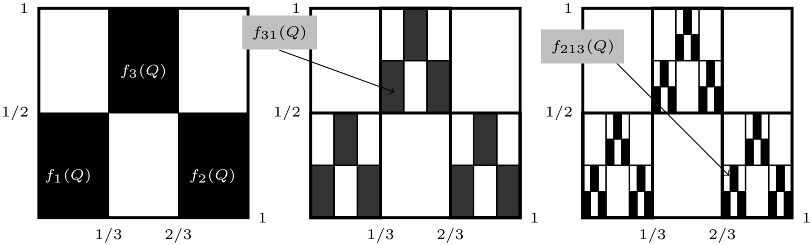


Figure: The Hironaka curve.

$$A_1 := \begin{bmatrix} 0.3464101616 & -0.1250000000 \\ 0.2 & 0.2165063510 \end{bmatrix},$$

$$A_2 := \begin{bmatrix} 0.2 & 0.2165063510 \\ -0.3464101616 & 0.1250000000 \end{bmatrix}$$

$$t_1 := [0.5196152, 0.3], t_2 := [-0.4688749, 0.5721152]$$

Let $f_1(x) := A_1 x + t_1$ and $f_2(x) := A_2 x + t_2$ and

$$D_{i_1 \dots i_n} := f_{i_1} \circ \cdots \circ f_{i_n}(D),$$

where D is the unit disk.

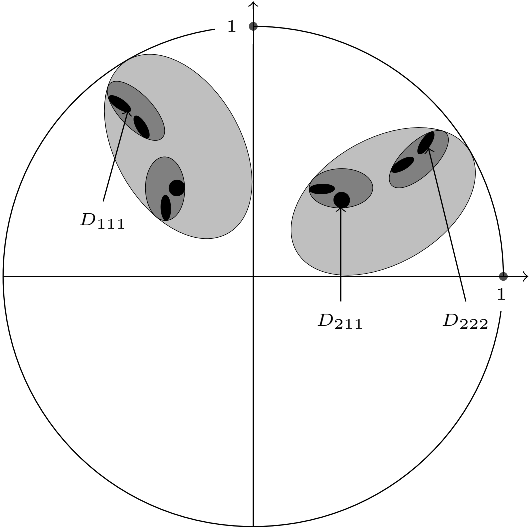


Figure: The third approximation of the attractor of the self affine IFS.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ .16 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} .2 & -.26 \\ .22 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} .23 \\ 1.6 \end{pmatrix};$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -.15 & .28 \\ .24 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} .26 \\ .44 \end{pmatrix}; \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} .75 & -.04 \\ .85 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -.04 \\ 1.6 \end{pmatrix}.$$

Figure: The Barnsly's fern

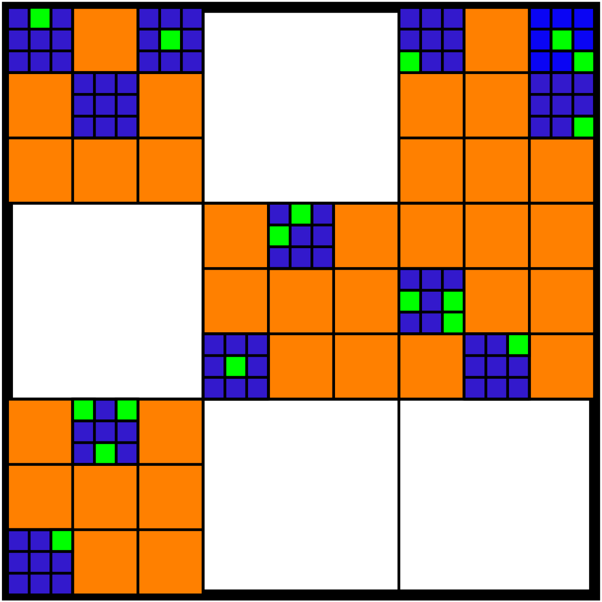


Figure: Fractal percolation

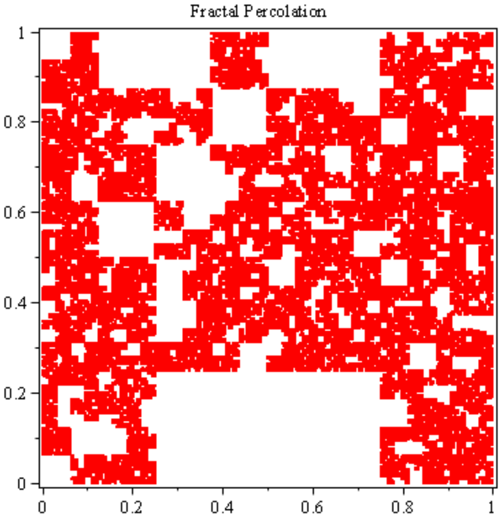


Figure: Mandelbrot Percolation.

Figure: Brownian motion

Separation conditions

Strong separation Property

$$f_i(\Lambda) \cap f_j(\Lambda) = \emptyset \text{ for all } i \neq j \tag{9}$$

Open Set Condition (OSC)

There exists a non-empty open set V such that

1. $f_i(V) \subset V$ holds for all $i = 1, \ldots, m$
2. $f_i(V) \cap f_j(V) = \emptyset$ for all $i \neq j$.

M. Keane's " $\{0, 1, 3\}$ " problem:

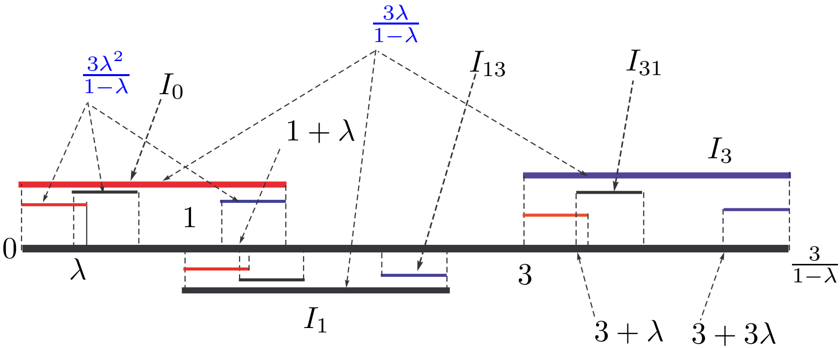
For every $\lambda \in (\frac{1}{4}, \frac{2}{5})$ consider the following self-similar set:

$$\Lambda_\lambda := \left\{ \sum_{i=0}^{\infty} a_i \lambda^i : a_i \in \{0, 1, 3\} \right\}.$$

Then Λ_λ is the attractor of the one-parameter (λ) family IFS:

$$\left\{ S_i^\lambda(x) := \lambda \cdot x + i \right\}_{i=0,1,3}$$

$\{0,1,3\}$ problem II.



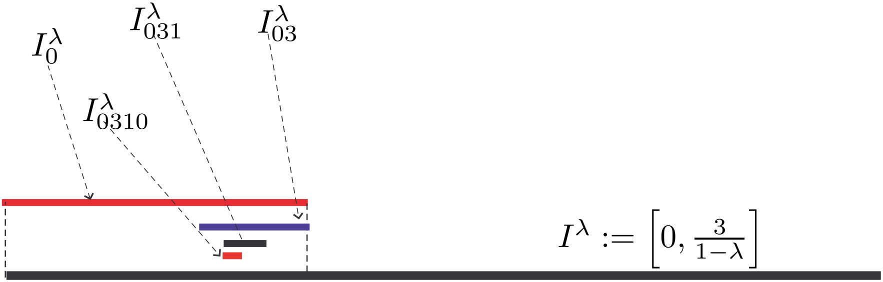
$$\Sigma := \{0,1,3\}^{\mathbb{N}}, \quad \Pi_\lambda : \Sigma \to \Lambda_\lambda,$$
$$\mathbf{i} = (i_0, i_1, i_2, \dots) \in \Sigma :$$
$$\Pi_\lambda(\mathbf{i}) := i_0 + i_1 \cdot \lambda + i_2\lambda^2 + i_3 \cdot \lambda^3 + \cdots$$

Π_λ is the natural projection which is , NOT $1-1$

Let $k \in \mathbb{N}$ and $\mathbf{i} = (i_0, i_1, \dots) \in \underbrace{\{0,1,3\}^{\mathbb{N}}}_{\Sigma}$.

$I^{\lambda}_{i_0,\dots,i_k} := S^{\lambda}_{i_0} \circ \cdots \circ S^{\lambda}_{i_k}(I^{\lambda})$ and $\Pi_{\lambda}(\mathbf{i}) := \bigcap\limits_{k=1}^{\infty} I^{\lambda}_{i_0,\dots,i_k}$.

Example: $\Pi_{\lambda}(0,3,1,0,\dots)$



Box dimension

Let $E \subset \mathbb{R}^d$, $E \neq \emptyset$, bounded. $N_\delta(E)$ be the smallest number of sets of diameter δ which can cover E. Then the lower and upper box dimensions of E:

$$\underline{\dim}_{\mathrm{B}}(E) := \liminf_{r \to 0} \frac{\log N_\delta(E)}{-\log \delta}, \tag{10}$$

$$\overline{\dim}_{\mathrm{B}}(E) := \limsup_{r \to 0} \frac{\log N_\delta(E)}{-\log \delta}. \tag{11}$$

If the limit exists then we call it the box dimension of E.

Equivalent definitions I.

The definition of the box dimension does not change if we define $N_\delta(E)$ in any of the following ways:

1. the smallest number of closed balls of radius δ that cover E,
2. the smallest number of cubes of side δ that cover E,
3. the number of δ-mesh cubes that intersect E
4. the smallest number of sets of diameter at most δ that cover E,
5. the largest number of disjoint balls of radius δ with centers in E.

Equivalent definitions II.

$$(12) \qquad \underline{\dim}_{\mathrm{B}}(E) := d - \limsup_{\delta \to 0} \frac{\log \mathrm{vol}^d([E]_\delta)}{-\log \delta},$$

$$(13) \qquad \overline{\dim}_{\mathrm{B}}(E) := d - \liminf_{\delta \to 0} \frac{\log \mathrm{vol}^d([E]_\delta)}{-\log \delta},$$

where $[E]_\delta$ is the δ parallel body of E.

Hausdorff measure on $\mathbb{R}^d$

Let $\Lambda \subset \mathbb{R}^d$ and let $t \geq 0$. We define

(14)

$$\mathcal{H}^t(\Lambda) = \lim_{\delta \to 0} \left\{ \underbrace{\inf \left\{ \sum_{i=1}^{\infty} |A_i|^t : \Lambda \subset \bigcup_{i=1}^{\infty} A_i ; |A_i| < \delta \right\}}_{\mathcal{H}^t_\delta(\Lambda)} \right\}$$

Then $\mathcal{H}^t$ is a metric outer measure. The t-dimensional Hausdorff measure is the restriction of $\mathcal{H}^t$ to the σ-field of $\mathcal{H}^t$-measurable sets which include the Borel sets.

Hausdorff dimension I.

Let $\Lambda \subset \mathbb{R}^d$ and $0 \leq \alpha < \beta$. Then

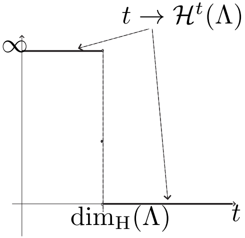
$$\mathcal{H}_\delta^\beta(\Lambda) \leq \delta^{\beta-\alpha}\mathcal{H}_\delta^\alpha(\Lambda).$$

Using that $\mathcal{H}^t(\Lambda) = \lim_{\delta \to 0} \mathcal{H}_\delta^t(\Lambda)$

$$\mathcal{H}^\alpha(\Lambda) < \infty \Rightarrow \mathcal{H}^\beta(\Lambda) = 0 \text{ for all } \alpha < \beta.$$

$$0 < \mathcal{H}^\beta(\Lambda) \Rightarrow \mathcal{H}^\alpha(\Lambda) = \infty \text{ for all } \alpha < \beta.$$

Hausdorff dimension II.



The Hausdorff dimension of Λ

$$\begin{aligned}\dim_{\mathrm{H}}(\Lambda) &= \inf\left\{t : \mathcal{H}^t(\Lambda) = 0\right\} \\ &= \sup\left\{t : \mathcal{H}^t(\Lambda) = \infty\right\}.\end{aligned}$$

Elementary properties of Hausdorff dimension I

- EP0 Every countable set has zero Hausdorff dimension.
- EP1 For every $F \subset \mathbb{R}^d$ we have $\dim_{\mathrm{H}}(F) \leq d$.
- EP2 If $\mathcal{L}^d(E) > 0$ then $\dim_{\mathrm{H}}(E) = d$.
- EP3 For any $k < d$ a k-dimensional smooth surface in $\mathbb{R}^d$ has Hausdorff dimension k.

Elementary properties of Hausdorff dimension II

- EP4 For a Lipschitz map $f : \mathbb{R}^d \to \mathbb{R}^d$ and a Borel set $E \subset \mathbb{R}^d$ we have $\dim_{\mathrm{H}}(f(E)) \leq \dim_{\mathrm{H}}(E)$.
- EP5 Let E be a Borel set and let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a bi-Lipschitz map (that is f is invertible and both f and its inverse are Lipschitz maps.) Then $\dim_{\mathrm{H}}(E) = \dim_{\mathrm{H}}(f(E))$.
- EP6 Let $\{E_i\}_{i=1}^{\infty}$ be a sequence of Borel sets in $\mathbb{R}^d$. Then $\dim_{\mathrm{H}}(\cup_{i=1}^{\infty} E_i) = \sup_i \dim_{\mathrm{H}}(E_i)$.

Measure

Definition 5.1

Given a set X and a σ-field $\mathcal{F}$ of subsets of X. μ is a measure on $\mathcal{F}$ if μ is a function $\mu : \mathcal{F} \to [0, \infty]$ such that

- $\mu(\emptyset) = 0$
- $\mu(\cup_{i=1}^{\infty} A_i) = \sum\limits_{i=1}^{\infty} \mu(A_i)$, for every countable disjoint $\{A_i\}_{i=1}^{\infty}$, $A_i \subset X$.

An outer measure ν on X is defined on all subsets of X takes values from $[0, \infty]$ such that

- $\nu(\emptyset) = 0$,
- $\nu(A) \leq \nu(B)$ if $A \subset B$,
- $\nu(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \nu(E_i)$ for all sequence of sets $\{E_i\}_{i=1}^{\infty}$.

A set E is measurable with respect to the outer measure ν if every set is dissected properly. That is for every $A \subset X$ we have

$$\nu(A) = \nu(A \cap E) + \nu(A \setminus E).$$

Measurable sets

Let $\mathcal{M}$ be the collection of all measurable set for an outer measure ν. Then $\mathcal{M}$ is a σ-field and the restriction of ν to $\mathcal{M}$ is a measure.
Further, assume that (X, d) is a metric space. We say that the outer measure ν is a metric outer measure if $\nu(A \cup B) = \nu(A) + \nu(B)$ holds for all $A, B \subset X$ with $d(A, B) := \inf \{d(a, b) : a \in A, b \in B\} > 0$.
In this case the restriction of ν to the σ-field of the measurable sets $\mathcal{M}$ is a Borel measure.

Support of a measure

Definition 5.2

Let μ be a measure on a separable metric space X. The support of μ, $\mathrm{spt}(\mu)$ is the smallest closed set F such that $\mu(X \setminus F) = 0$. In other words:

$$\mathrm{spt}(\mu) = X \setminus \{x : \exists r > 0,\ \mu(B(x,r)) = 0\}.$$

$\mathcal{M}(A)$, mass distribution

Definition 5.3

- Mass distribution : a Borel measure μ on $\mathbb{R}^d$ of compact support with $0 < \mu(\mathbb{R}^d) < \infty$.
- Let $A \subset \mathbb{R}^d$ for a $d \geq 1$. We write $\mathcal{M}(A)$ for the collection of Borel measures μ
  - whose support $\mathrm{spt}(\mu) \subset A$ and
  - $\mathrm{spt}(\mu)$ is compact and
  - $0 < \mu(A) < \infty$.

Definitions

Let (X, d) be a separable metric space and let μ be a measure on X.

1. μ is locally finite if $\forall x \in X, \ \exists r > 0$, such that $\mu(B(x, r)) > 0$.
2. μ is a Borel measure if all Borel sets are μ measurable. (The family of Borel sets in X is the smallest σ-algebra containing all open sets.)
3. The measure μ is Borel regular if
   (a) Borel measure and
   (b) $\forall A \subset X, \ \exists A \subset B \subset X$ Borel set s.t. $\mu(A) = \mu(B)$.

Radon measure definition

μ is a Radon measure if

(a) Borel measure,

(b) $\forall K \subset X$ compact: $\mu(K) < \infty$,

(c) $\forall V \subset X$ open:
$\mu(V) = \sup \{\mu(K) : K \subset V \text{ is compact }\}$

(d) $\forall A \subset X$:
$\mu(A) = \inf \{\mu(V) : A \subset \text{ and } V \text{ is open }\}$.

Theorem 5.4

A measure μ on $\mathbb{R}^d$ is a Radon measure if and only if it is locally finite and Borel regular

Proof: See Mattila's book [2, p. 11-12].

Radon measure examples

1. The Lebesgue measure $\mathcal{L}\mathrm{eb}_d$ on $\mathbb{R}^d$ is a Radon measure.
2. The Dirac measure $\delta_a(A) := 1$ if $a \in A$ and $\delta_a(A) = 0$ if $a \notin A$ is a Radon measure.
3. For every $s \geq 0$ the Hausdorff measure is a Borel regular measure but it need not be locally finite. So, in general the Hausdorff measure is not a Radon measure. However, for an $A \subset \mathbb{R}^d$, $\mathcal{H}^s(A) < \infty$ the restriction $\mathcal{H}^s|_A$ is a Radon measure. (See Mattila's book: [2, p. 57].)

Push forward measure

Let X, Y be separable metric spaces and $f : X \to Y$ continuous and let μ be Radon measure on X. Then the push forward measure of μ defined by

$$f_*\mu(A) := \mu(f^{-1}A)$$

is also a Radon measure and the support of $f_*\mu(A)$ is

$$\mathrm{spt} f_*\mu = f(\mathrm{spt}\mu).$$

Change of variable formulae

Theorem 5.5

Let X, Y be separable metric spaces and let $f : X \to Y$ be a Borel map and μ is a Borel measure on X. Further let $g : Y \to \mathbb{R}$ be a non-negative Borel function. Then

$$\int_Y g(y) d(f_* \mu)(y) = \int_X (g \circ f)(x) d\mu(x).$$

Mass Distribution Principle

Lemma 5.6 (Mass Distribution Principle)

If $A \subset X$ supports a mass distribution μ such that for a constant C and for every Borel set D we have

$$\mu(D) \leq \mathrm{const} \cdot |D|^t$$

Then $\dim_{\mathrm{H}}(A) \geq t$.

Proof For all $\{A_j\}_{j=1}^{\infty}$

$$A \subset \bigcup_{j=1}^{\infty} A_j \Rightarrow \sum_j |A_j|^t \geq C^{-1} \sum_j \mu(A_j) \geq \frac{\mu(A)}{C}.$$

Frostman's Energy method

Let μ be a mass distribution on $\mathbb{R}^d$. The t-energy of μ is defined by

$$I_t(\mu) := \iint |x-y|^{-t} d\mu(x) d\mu(y).$$

Lemma 5.7 (Frostman (1935))

For a Borel set $\Lambda \subset \mathbb{R}^d$ and for a mass distribution μ supported by Λ we have

$$I_t(\mu) < \infty \Longrightarrow \dim_{\mathrm{H}}(\Lambda) \geq t.$$

In this case $\mathcal{H}^t(\Lambda) = \infty$.

Proof of Frostman Lemma I

This proof if due to Y. Peres. Let

$$\Phi_t(\mu, x) := \int \frac{d\mu(y)}{|x-y|^t}.$$

Then $I_t(\mu) = \int \Phi_t(\mu, x) d\mu(x)$. Let

$$\Lambda_M := \{x \in \Lambda : \Phi_t(\mu, x) \leq M\}.$$

Since $\int \Phi_t(\mu, x) d\mu(x) = I_t(\mu) < \infty$ we have M such that $\mu(\Lambda_M) > 0$. Fix such an M.

Proof of Frostman Lemma II

Let

$$\nu := \mu|_{\Lambda_M}$$

Then ν is a mass distribution supported by Λ. (That is ν satisfies one of the assumptions of the Mass Distribution Principle above.) Now we show that for every bounded set D:

$$(15) \qquad \nu(D) < \mathrm{const} \cdot |D|^t.$$

If $D \cap \Lambda_M = \emptyset$ then (15) holds obviously. From now we assume that D is a bounded set such that $D \cup \Lambda_m \neq \emptyset$.

Proof of Frostman Lemma III

Pick an arbitrary $x \in D \cap \Lambda_M$. We define

$$m := \max\left\{k \in \mathbb{Z} : B(x, 2^{-k}) \supset D\right\}.$$

Then

$$|D| \geq 2^{-(m+1)} \text{ and } |D| < 2 \cdot 2^{-m}. \tag{16}$$

Proof of Frostman Lemma IV

Observe that from the right hand side of (16): $y \in D$ we have $|x-y|^{-t} \geq |D|^{-t} \geq 2^{-t} \cdot 2^{mt}$. So,

$$M \geq \int \frac{d\nu(y)}{|x-y|^t} \geq \int_D \frac{d\nu(y)}{|x-y|^t} \geq \nu(D) \cdot 2^{-t} \cdot 2^{m \cdot t}.$$

Using this and the left hand side of (16) we obtain

$$\nu(D) \leq M \cdot 2^t \cdot 2^t \cdot 2^{-(m+1)t} \leq M \cdot 2^{2t} \cdot |D|^t.$$

So, the mass distribution ν satisfies the assumptions of the Mass Distribution Principle which completes the proof of the Lemma.

Example: Bold Play from Edgar's book I

The following is literary copied from Eddgar's book [1]: This is an exercise which goes back to Cesaro 1906 and this is about a gambling system known as "bold play" . The gambler wants to increase his holdings to a certain amount by repeatedly playing a game at even money, but under unfavorable odds. He attempts to do this by always placing the maximum sensible bet. The probability of eventual success is a function $Q(x)$ of the fraction x of the goal that the gambler currently holds. Let p be the probability of winning on any given play; we are told the odds are unfavorable, that is, $0 < p < 1/2$. To analyze the function Q, consider two cases.

Example: Bold Play from Edgar's book II

If $x \geq 1/2$ then the bet to be placed should be the fraction $1-x$ of the goal; if he wins he has reached the goal, and if he loses, he continues with stakes reduced to the fraction $x-(1-x)=2x-1$ of the goal. Thus

$$Q(x) = p + (1-p)Q(2x-1), \quad \text{if } x \geq 1/2$$

On the other hand, if $x < 1/2$, then the bet to be placed should be the fractio x of the goal; if he wins, he increases his stake to fraction $2x$ of the goal an continuesi if he loses, he is broke and that is that. Thus

$$Q(x) = pQ(2x), \quad \text{if } x < 1/2.$$

Example: Bold Play from Edgar's book II

So what we know about the unknown function $Q : [0,1] \to [0,1]$ are as follwos:

$$
\begin{aligned}
(17)\; Q(x) &= pQ(2x), \qquad \text{if } x < 1/2,\\
(18)\; Q(x) &= p + (1-p)Q(2x-1), \qquad \text{if } x \geq 1/2.
\end{aligned}
$$

Let $F_1, F_2 : [0,1]^2 \to [0,1]^2$,

$$(19) \qquad F_1(x,y) := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & p \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix},$$

$$(20) \qquad F_2(x,y) := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1-p \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ p \end{pmatrix}$$

Example: Bold Play from Edgar's book III

By definition,

$$\mathrm{Graph}(Q) = \left\{(x, Q(x)) \in [0,1]^2 : x \in [0,1]\right\}.$$

Using (17) and (18), one can easily check that

$$\mathrm{Graph}(Q) = F_1(\mathrm{Graph}(Q)) \cup F_2(\mathrm{Graph}(Q)). \tag{21}$$

That is $\mathrm{Graph}(Q)$ is the attractor of the IFS $\{F_1, F_2\}$.

Example: Bold Play from Edgar's book IV

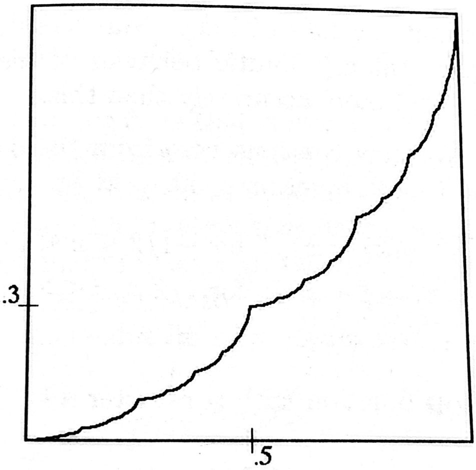


Figure: $Q(x)$ when $p = 0.3$. The picture is copied (in a terrible way) from Edger's book.

Example: Bold Play from Edgar's book V

The function $Q(x)$ is

1. Self affine function graph,
2. continuous,
3. strictly increasing
4. $Q'(x) = 0$ for Lebesgue a.e. $x \in (0, 1)$.

Theorem 5.8

If a function is strictly increasing then the Hausdorff dimension of its graph is equal to 1.

Equivalent definitions of the box-dimension

For an arbitrary $F \subset \mathbb{R}^d$ we say that $\{A_i\}_{i=1}^{\infty}$ is a cover of F if $F \subset \bigcup\limits_{i=1}^{\infty} A_i$. We denote the family of all covers of a set $F \subset \mathbb{R}^d$ by $\mathscr{C}(F)$. Moreover, we write

$$\mathscr{C}_U(F) := \left\{ \{A_i\}_{i=1}^{\infty} \in \mathscr{C},\ |A_i| = |A_j|\ \forall i,j \right\}.$$

Then

$$\dim_{\mathrm{H}} F = \inf \left\{ s \geq 0 \colon \forall \varepsilon > 0, \exists \{A_i\} \in \mathscr{C}(F), \sum_{i=1}^{\infty} |A_i|^s \leq \varepsilon \right\}.$$

$$\underline{\dim}_{\mathrm{B}} F = \inf \left\{ s \geq 0 \colon \forall \varepsilon > 0, \exists \{A_i\} \in \mathscr{C}_U(F), \sum_{i=1}^{\infty} |A_i|^s \leq \varepsilon \right\}$$

Equivalent definitions of the Hausdorff dimension

Let $F \subset \mathbb{R}^d$ be a Borel set. Then we get the Hausdorf dimensin of F by any of the following expressions

(22) $\dim_{\mathrm{H}} F =$

$$\sup\left\{s \geq 0 : \exists \mu \in \mathcal{M}(F),\ \mu(B(x,r)) \leq r^s, \forall x \in \mathbb{R}^d, r > 0\right\},$$

(23) $$\dim_{\mathrm{H}} F = \sup\left\{s \geq 0 : \exists \mu \in \mathcal{M}(F),\ I_s(\mu) < \infty\right\},$$

(24)

$$\dim_{\mathrm{H}} F = \sup\left\{s \geq 0 : \exists \mu \in \mathcal{M}(F),\ \int |x|^{s-d} |\widehat{\mu}(x)|^2 < \infty\right\}.$$

Hausdorff dimension of a measure

Let $\mu \in \mathcal{M}(A)$.

Definition 5.9

$\dim_{\mathrm{H}}(\mu) := \inf\left\{\dim_{\mathrm{H}}(A) : \mu(\mathbb{R}^d \setminus A) = 0\right\}.$

Lemma 5.10

$$\dim_{\mathrm{H}}(\mu) = \mathrm{ess\ sup}_x \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

Roughly speaking, $\dim_{\mathrm{H}}(\mu) = \delta$ if for a μ-typical x we have

$$\mu(B(x,r)) \approx r^\delta$$

for small $r > 0$.

Lower Hausdorff dimension of a measure

Let $\mu \in \mathcal{M}(A)$.

Definition 5.11

$\underline{\dim}_{\mathrm{H}}(\mu) := \inf \left\{ \dim_{\mathrm{H}}(A) : \mu(A) > 0 \right\}.$

Lemma 5.12

$$\underline{\dim}_{\mathrm{H}}(\mu) = \mathrm{ess\ inf}_x \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

Self-similar sets with OSC

Assume that $\mathcal{F} := \{f_i\}_{i=1}^{m}$ is a self-similar IFS on $\mathbb{R}^d$. The similarity dimension $s = s(\mathcal{F})$ is defined as the only positive solution of the equation

$$r_1^s + \cdots + r_m^s = 1, \tag{25}$$

where r_i is the similarity ratio for f_i.

Hutchinson Theorem

Hutchinson (1981)

Theorem 6.1

Given a self similar IFS $\mathcal{F}$ which satisfies the OSC. Let $s = s(\mathcal{F})$ be the similarity dimension. Then

$$0 < \mathcal{H}^s(\Lambda) < \infty. \tag{26}$$

Further,

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} = s.$$

Hausdorff measure for self-similar attractors

We cannot easily estimate the appropriate dimensional Hausdorff measure of a self similar-set in the plane or higher dimension. If Λ is the Sierpinski triangle then the we know that $s = \dim_{\mathrm{H}} \Lambda = \frac{\log 3}{\log 2}$. The best estimate for s-dimensional Hausdorff measure:

$$0.77 \leq \mathcal{H}^s(\Lambda) \leq 0.81$$

The upper bound is old (proved in 1999) but the lower bound is new. It was given by Peter Móra.

Lemma 6.2

For every $E \subset \mathbb{R}^d$ we have

$$\dim_{\mathrm{H}}(E) \leq \underline{\dim}_{\mathrm{B}}(E) \leq \overline{\dim}_{\mathrm{B}}(E).$$

Theorem 6.3 (Moran, Hutchinson)

Assume that the self-similar IFS $\mathcal{S} = \{S_1, \dots, S_m\}$ *acts on* $\mathbb{R}^d$ *and satisfies the* *OSC*. *The similarity ratio of* S_i *is* $0 < r_i < 1$, $i = 1, \dots, m$. *Let* s *be the similarity dimension, that is,* $r_1^s + \cdots + r_m^s = 1$. *Then for the attractor* Λ *of the IFS* $\mathcal{S}$ *we have*

$$0 < \mathcal{H}^s(\Lambda) < \infty. \tag{27}$$

Moreover,

$$\dim_{\mathrm{H}}(\Lambda) = \dim_{\mathrm{B}}(\Lambda) = s, \tag{28}$$

in particular the box dimension exists.

Infinite Bernoulli convolution I

For a $\lambda \in (0,1)$ we define the random variable

$$Y_\lambda := \sum_{n=0}^{\infty} \pm\lambda^n.$$

ν_λ be the distribution of Y_λ. On the other hand ν_λ is the self similar measure of the IFS. That is for $\lambda \in (0,1)$, $x \in [0, 1/(1-\lambda)]$

$$S_1^\lambda(x) := \lambda x + 1, \qquad S_{-1}^\lambda(x) := \lambda x - 1,$$

with weights $1/2, 1/2$
$(\nu_\lambda(A) = \frac{1}{2}\nu_\lambda((S_1^\lambda)^{-1}(A)) + \frac{1}{2}\nu_\lambda((S_{-1}^\lambda)^{-1}(A)))$.

Infinite Bernoulli convolution II

$$\nu_\lambda = (\Pi_\lambda)_*(\{1/2, 1/2\}^{\mathbb{N}}),$$

$$\Pi_\lambda(i_0, i_1, i_2, \dots) = i_0 + i_1\lambda + i_2\lambda^2 + \cdots$$

Let $I_\lambda := \left[-\frac{1}{1-\lambda}, \frac{1}{1-\lambda}\right]$. Yet again we write

$$I^\lambda_{i_0\dots i_k} := S_{i_0\dots i_k}(I^\lambda).$$

Then

$$\Pi_\lambda(i_0, i_1, \dots) = \bigcap_{k=0}^{\infty} I^\lambda_{i_0\dots i_k}.$$

Infinite Bernoulli convolution III

Cylinders for $\lambda \in (0.5, 1)$

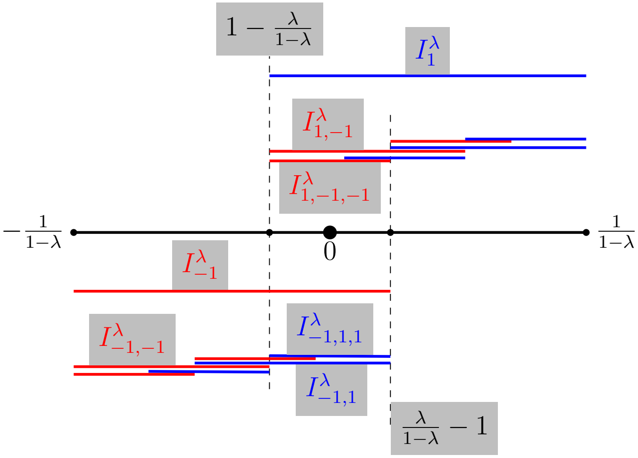


Figure: $\lambda = \frac{\sqrt{5}-1}{2}$ is the golden mean

Infinite Bernoulli convolution IV

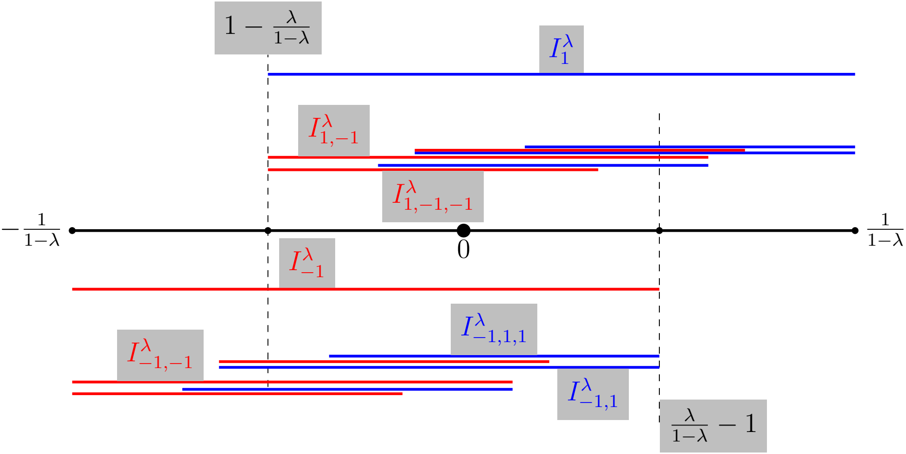


Figure: General $\lambda \in (0.5, 1)$

Law of pure type

Theorem 7.1 (Jensen, Wintner 1935)

Either $\nu_\lambda \ll \mathcal{L}\mathrm{eb}$ *or* $\nu_\lambda \perp \mathcal{L}\mathrm{eb}$

It was proved by Parry and York that for every λ we have

$$\text{Either } \nu_\lambda \sim \mathcal{L}\mathrm{eb} \text{ or } \nu_\lambda \perp \mathcal{L}\mathrm{eb}. \tag{29}$$

Solomyak's Theorem (1995)

After 60 years after that in 1930's P. Erdős started to investigate the infinite Bernoulli convolutions Boris Solomyak made the following major achievement:

Theorem 7.2 (Solomyak (1995))

1. *$\nu_\lambda \ll \mathcal{L}eb$ with a density in $L^2(\mathbb{R})$ for a.e. $\lambda \in (1/2, 1)$.*
2. *$\nu_\lambda \ll \mathcal{L}eb$ with a density in $\mathcal{C}(\mathbb{R})$ for a.e. $\lambda \in (2^{-1/2}, 1)$.*

$$\widehat{\nu}_\lambda(x) := \int_{\mathbb{R}} e^{itx} d\nu_\lambda(t) = \prod_{n=0}^{\infty} \cos(\lambda^n x).$$

Hence

$$\widehat{\nu}_{\sqrt{\lambda}}(x) = \widehat{\nu}_\lambda(x) \cdot \widehat{\nu}_\lambda(\sqrt{\lambda} \cdot x) \tag{30}$$

From Plancherel Theorem:

$$\text{if } \nu_\lambda \ll \mathcal{L}\mathrm{eb} \text{ with } L^2 \text{ density } \Longrightarrow \widehat{\nu}_\lambda \in L^2.$$

On the other hand by (30)

$$\widehat{\nu}_\lambda \in L^2 \Longrightarrow \widehat{\nu}_{\sqrt{\lambda}} \in L^1 \Longrightarrow \nu_{\sqrt{\lambda}} \text{ has continuous density.}$$

That is, if $\nu_\lambda \ll \mathcal{L}eb$ with a density in $L^2(\mathbb{R})$ for a.e. $\lambda \in \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ then

(a) $\nu_\lambda \ll \mathcal{L}eb$ with a density in $L^2(\mathbb{R})$ for a.e. $\lambda \in (1/2, 1)$. Moreover,

(b) $\nu_\lambda \ll \mathcal{L}eb$ with a density in $\mathcal{C}(\mathbb{R})$ for a.e. $\lambda \in (2^{-1/2}, 1)$.

Erdős Results form the 1930's

Theorem 7.3 (Pál Erdős 1940)

There exists a $t < 1$ (rather close to 1) such that for a.e. $\lambda \in (t, 1)$ we have $\nu_\lambda \ll \mathcal{L}eb$. More precisely,

$$\exists a_k \uparrow 1 \text{ s.t. } \quad \frac{d\nu_\lambda}{dx} \in \mathcal{C}^k(\mathbb{R}) \text{ for } \lambda \in (a_k, 1).$$

Problem 7.4 (Erdős)

Is it true that $\nu_\lambda \ll \mathcal{L}eb$ *holds for a.e.* $\lambda \in (1/2, 1)$*?*

The only known counter examples are the reciprocals of the so-called PV number or Pisot or Pisot Vayangard numbers (they are the same but nobody can pronounce Vayangard properly so people avoid using his name).

Solomyak's Theorem (1995)

After 60 years after that in 1930's P. Erdős started to investigate the infinite Bernoulli convolutions Boris Solomyak made the following major achievement:

Theorem 7.5 (Solomyak (1995))

1. $\nu_\lambda \ll \mathcal{L}eb$ *with a density in* $L^2(\mathbb{R})$ *for a.e.* $\lambda \in (1/2, 1)$.
2. $\nu_\lambda \ll \mathcal{L}eb$ *with a density in* $\mathcal{C}(\mathbb{R})$ *for a.e.* $\lambda \in (2^{-1/2}, 1)$.

Methods of Proving Solomyak's theorem

The proof uses the so-called transversality condition plus
There are two approaches:

- either Derivative of measures method or
- Fourier analysis.

First we study the drivaive of measures method.

Derivative of a measure I

Definition 7.6

Let μ, η be Radon measures on $\mathbb{R}^d$. We define the upper and lower derivatives of μ with respect to η:

$$\overline{D}(\mu, \eta, x) := \overline{\lim}_{r \to 0} \frac{\mu(B(x,r))}{\eta(B(x,r))}.$$

If the limit exists then we write $D(\mu, \eta, x)$ for this common value and we call it the derivative of the measure μ with respect to the measure η.

Derivative of a measure II

Theorem 7.7

Let μ, η be Radon measures on $\mathbb{R}^d$.

(i) The derivative $D(\mu, \eta, x)$ exists and is finite for η almost all $x \in \mathbb{R}^d$. [2, Theorem 2.12]

(ii) For all Borel sets $B \subset \mathbb{R}^d$ we have

$$\int_B D(\mu, \eta, x) d\eta(x) \leq \mu(B) \tag{31}$$

with equality if $\mu \ll \eta$. [2, Theorem 2.12]

Derivative of a measure III

Theorem 7.7 (Cont.)

(iii) $\mu \ll \eta$ if and only if $\underline{D}(\mu, \eta, x) < \infty$ for μ almost all $x \in \mathbb{R}^d$. [2, Theorem 2.12]

(iv) If $\mu \ll \eta$ then

$$\int D(\mu, \eta, x)^2 d\eta(x) = \int D(\mu, \eta, x) d\mu(x).$$

This is [2, Exercise 6 on p. 43]

Derivative of a measure IV

Theorem 7.7 (Cont.)

(v) Assume that $\mu \ll \eta$. Then $\underline{D}(\mu, \eta, x)$ is a version of the Radon-Nikodym derivative $\frac{d\mu(x)}{d\eta(x)}$. So, we know that $\int\limits_{\mathbb{R}} \underline{D}(\mu, \eta, x) d\eta(x) < \infty$. Further, by (iv) above, we have:

(32)

$$\int\limits_{\mathbb{R}} \underline{D}(\mu, \eta, x) d\mu(x) < \infty \Longrightarrow \frac{d\mu(x)}{d\eta(x)} \in L^2(\eta).$$

This argument appears in [4, p.233].

[1] . G.A. Edgar Integral Probability and Fractal measures

[2] P. Mattila Geometry of sets and measure in Euclidean spaces. Cambridge, 1995.

[3] P. Mattila Fourier analysis and Hausdorff dimension Cambridge, 2015.

[4] Y. Peres and B. Solomyak Absolute continuity of Bernoulli convolutions, a simple proof. *Mathathematics Research Letters* vol. 3, 231-239, (1996).