A SHORT TOUR OF CONGRUENCE LATTICES

From the point of view my collaboration with George

Budapest, June 2006
Part I

THE CONGRUENCE LATTICES OF UNIVERSAL ALGEBRAS
Part I. The congruence lattices of universal algebras

- Once upon a time…..
Part I. The congruence lattices of universal algebras

in 1948 Garett Birkhoff and Orrin Frink proved that the congruence lattice of an algebra is an algebraic lattice.

Is the converse true?
My collaboration with George began …

(the two rascals)
Part I. The congruence lattices of universal algebras

in 1952.

In 1963 we solved the characterizations problem of congruence lattices of universal algebras:
Part I. The congruence lattices of universal algebras

- **Theorem 1** (G. Grätzer, E. T. Schmidt, 1963). *Let \( L \) be an algebraic lattice. Then there exists an algebra \( A \) whose congruence lattice is isomorphic to \( L \).*

It is perhaps one of the most famous open problem of universal algebra whether every finite lattice is isomorphic to the congruence lattice of a finite algebra. Pálfy-Pudlák proved:

- **Theorem 2** (P. P. Pálfy, P. Pudlák, 1980). *The following two statements are equivalent: (1) every finite lattice isomorphic to an interval in a subgroup lattice of a finite group; (2) every finite lattice isomorphic to the congruence lattice of some finite algebra.*

It is unknown whether these equivalent statements are actually true.
Part I. The congruence lattices of universal algebras

- **Theorem 3** (W. A. Lampe, 1972). *Given algebraic lattices \( L_c \) and \( L_a \) with more than one element each and a group \( G \), there exist an algebra \( A \) whose congruence lattice is isomorphic to \( L_c \) whose subalgebra lattice is isomorphic to \( L_a \) and whose automorphism group is isomorphic to \( G \).*

  Lampe proved in 1977 that in Theorem 1 no upper bound can be placed on the number of operations. A stronger statement is:

- **Theorem 4** (R. Freese, W. A. Lampe, W. Taylor, 1980). *For every similarity type \( \tau \) there is there exists a modular algebraic lattice \( L \) such that \( L \) is not isomorphic to the congruence lattice of any algebra of type \( \tau \).*
Part I. The congruence lattices of universal algebras

And here is a nice new result:

- **Theorem 5** (Růzička, Tůma, Wehrung, 2005). *There is a distributive algebraic lattice that is not isomorphic to the congruence lattice of any algebra with permutable congruences.*
Lattice theory conference

In 1963 George went to America
Part II

COMPLETE CONGRUENCES OF COMPLETE LATTICES
Part II. Complete congruences of complete lattices

For complete lattices we have complete congruences, and the complete lattice of complete congruences. These lattices were characterized by G. Grätzer, 1991:

- **Theorem 6.** Every complete lattice $K$ can be represented as the lattice of complete congruence relations of a complete lattice $L$.

The next step:

- **Theorem 7** (Grätzer, Freese, Schmidt, 1994). Every complete lattice $K$ can be represented as the lattice of complete congruence relations of a complete modular lattice $L$. 
Part II. Complete congruences of complete lattices

In a series of papers, much sharper results have been obtained, culminating in:

- **Theorem 8** (Grätzer, Schmidt, 1995). *Every complete lattice \( L \) can be represented as the lattice of complete congruence relations of a complete distributive lattice \( D \).*
Part II. Complete congruences of complete lattices

A complete lattice is *complete-simple* if it has only the two trivial complete congruences. The crucial step in the proof of Theorem 8 is the following:

- *There exists a complete-simple distributive lattice S with more than two elements.*

The smallest such lattice is of size $c$, where $c$ denotes the power of the continuum.
Part III

CONGRUENCE LATTICES OF LATTICES
Part III. Congruence lattices of lattices

Let $L$ be a *distributive algebraic lattice*. We want to represent $L$ as the congruence lattice of a special algebra with finite many operations.

It is an long standing open problem:

Is every distributive algebraic lattice isomorphic to the congruence lattice of some algebra $A$ *with finite many operations*?

The most important case is if $A$ is a lattice. This problem was solved last year by F. Wehrung. Here we present the short history of this problem.
Part III. Congruence lattices of lattices

- By a result of Funayama and Nakayama, $\text{Con } L$ is a distributive lattice. We formulate the question:

  is every distributive algebraic lattice isomorphic to the congruence lattice of a suitable lattice?

This was one of the most famous open questions of the lattice theory for more than fifty years.

- It is more convenient to consider $\text{Comp } L$, the distributive semilattice of compact congruences of the lattice $L$. The original question can be rephrased: is every distributive semilattice $S$ isomorphic to the semilattice of all compact congruences of a lattice $L$? In this case we say $S$ is representable.
Part III. Congruence lattices of lattices

Each one of the following conditions implies that $S$ is representable:

- $S$ is a lattice (E. T. Schmidt, 1968; and P. Pudlak, 1985),

- $S$ is locally countable (that is for every $s$ in $S$, $(s]$ is countable, A. P. Huhn, 1983, H. Dobbertin),

- $|S| \leq \aleph_1$ (A. P. Huhn, 1984).

András Huhn (1947-1985)
Part III. Congruence lattices of lattices

It was hoped for a long time that the two approaches (Schmidt resp. Pudlak) solving the case for a lattice $S$ can be used to answer the general question.

*M. Ploscica, J. Tuma, F. Wehrung* (1999) proved that neither method can answer the general question even the lattices of size $\mathcal{K}_2$
Part III. Congruence lattices of lattices

After sixty years the problem is solved!

- **Theorem 9** (F. Wehrung, 2005). There exists a distributive algebraic lattice which is not isomorphic to the congruence lattice of any lattice. This lattice has \( \kappa_{\omega+1} \) compact elements.

Pavel Růžička, 2006:

- There is a distributive join-semilattice \( S \) of size \( \kappa_2 \) which is not isomorphic to the semilattice of compact congruences of any lattice.
Part IV

CONGRUENCE LATTICES OF FINITE LATTICES
Part IV. Congruence lattices of finite lattices

Dilworth theorem: every finite distributive lattice \( D \) is isomorphic to the congruence lattice of a finite lattice.

R. P. Dilworth (1914 -1993)

We want:

Every finite distributive lattice \( D \) can be represented as the congruence lattice of a “nice” finite lattice.
Part IV. Congruence lattices of finite lattices

We have the following two types of theorems:

- The straight representation theorems
- The congruence-preserving extension results

Let $K$ be a finite lattice. A finite lattice $L$ is a *congruence-preserving extensions* of $K$, if $L$ is an extension and every congruence $\Theta$ of $K$ has exactly one extension $\Phi$ to $L$ – that is $\Phi|_K = \Theta$.

Of course, the congruence lattice of $K$ is isomorphic to the congruence lattice of $L$. See the next figure:
Part IV. Congruence lattices of finite lattices

Congruence-preserving extension:
Part IV. Congruence lattices of finite lattices

For these theorems we introduced three constructions:

- **Chopped lattice** (introduced by Grätzer, Lakser, and generalized for the infinite case in Grätzer, Schmidt, 1995),
- **Cubic extension** (Grätzer, Schmidt, 1995)
- **The $M_3[D]$ construction**

I present here the $M_3[D]$ construction:
Part IV. Congruence lattices of finite lattices

The $M_3[D]$ construction is the key to congruence-preserving extensions.

Let $D$ be a bounded distributive lattice and let $M_3$ be the five element, nondistributive, modular lattice. Let $M_3[D]$ denote the subposet of $D^3$ consisting all $(x,y,z)$ satisfying

$$x \cap y = y \cap z = z \cap x,$$

we call such a triple \textit{balanced} (Schmidt, 1962).

Grätzer and Wehrung (1999) generalized this construction for arbitrary lattice $D$, this is the \textit{Boolean triple construction}, which is a special case of the lattice tensor product.
Part IV. Congruence lattices of finite lattices

$D$ is an ideal and $M_3[D]$ is a congruence-preserving extension of $D$
Part IV. Congruence lattices of finite lattices

In this example $D$ is the three element chain:
Part IV. Congruence lattices of finite lattices

In the last twelve years we prove theorems of the following type:

- **Theorem 10** (G. Grätzer, E. T. Schmidt). *Every finite distributive lattice $D$ can be represented as the congruence lattice of a finite “nice” lattice $L$.*

- **Theorem 11** (G. Grätzer, E. T. Schmidt). *Every finite lattice $K$ has a congruence-preserving embedding into a finite “nice” lattice $L$.***
Part IV. Congruence lattices of finite lattices

“Nice” is one of the following properties:

- sectionally complemented,
- semimodular,
- regular,
- uniform,
- isoform,
- small size.
Part IV. Congruence lattices of finite lattices

Let $L$ be a lattice. We call a congruence relation $\Theta$ of $L$ isoform, if any two congruence classes of $\Theta$ are isomorphic (as lattices). Let us call the lattice isoform, if all congruences of $L$ are isoform.

- **Theorem 12** (G. Grätzer, E. T. Schmidt, 2002). *Every finite distributive lattice $D$ can be represented as the congruence lattice of a finite, isoform lattice $L$.*

- **Theorem 13** (G. Grätzer, R. W. Quackenbush and E. T. Schmidt, 2004). *Every finite lattice $K$ has a congruence-preserving extension to a finite isoform lattice $L$.***
Part IV. Congruence lattices of finite lattices

There are some other “nice” properties:

The lattice $L$ constructed by R. P. Dilworth to represent $D$ is very large, it has $O(2^{2n})$ elements. We can find small $L$.

**Theorem 14** (G. Grätzer, H. Lakser and E. T. Schmidt 1996). Let $D$ be a finite distributive lattice with $n$ join-irreducible elements. Then there exists a planar lattice $L$ of $O(n^2)$ elements with $\text{Con } L \approx D$. 
The independence theorem:

**Theorem 15** (V. A. Baranskii, A. Urquhart, 1979). *Let $D$ be a finite distributive lattice with more than one element, and let $G$ be a finite group. Then there exists a finite lattice $L$ such that the congruence lattice of $L$ is isomorphic to $D$ and the automorphism group of $L$ is isomorphic to $G$.***

This is a representation theorem. There is also a congruence-preserving extension variant for this result.
Part IV. Congruence lattices of finite lattices

Theorem 16 (G. Grätzer, E. T. Schmidt, 1995). Let $K$ be a finite lattice with more than one element and let $G$ be a finite group. Then $K$ has a congruence-preserving extension $L$ whose automorphism group is isomorphic to $G$. 
Let $L$ a lattice. We call a congruence relation $\Theta$ *regular*, if any congruence class of $\Theta$ determines the congruences. Let us call the lattice $L$ *regular*, if all congruences of $L$ are regular. Sectionally complemented lattices are regular, so we already have a representation theorem for finite lattices (Theorem 11), but the following theorem holds for arbitrary lattices:

**Theorem 17** (Grätzer, Schmidt, 2001). *Every lattice has a congruence-preserving embedding into a regular lattice.*
Part IV. Congruence lattices of finite lattices

Let \( L \) be a lattice and let \( K \) be a sublattice of \( L \). Then the extension map \( \text{ext}: \text{Con } K \) into \( \text{Con } L \) is a join-homomorphism.

**Theorem 18** (Huhn, 1974). Let \( D \) and \( E \) be finite distributive lattices, and let \( \varphi \) be a \( \{0\} \)-separating join-homomorphism of \( D \) into \( E \). Then there are finite lattices \( L \) and a sublattice \( K \) of \( L \) and isomorphisms \( \alpha: D \rightarrow \text{Con } K \), and \( \beta: E \rightarrow \text{Con } L \) satisfying \( \varphi \beta = \alpha(\text{ext id}_K) \), where \( \text{id}_K \) is embedding of \( K \) into \( L \); that is the diagram.

\[
\begin{array}{ccc}
D & \longrightarrow & E \\
\downarrow & & \downarrow \\
\text{Con } K & \rightarrow & \text{Con } L \\
\end{array}
\]

is commutative.
G. Grätzer and H. Lakser, E. T. Schmidt (1996) proved a much stronger version:

**Theorem 19.** Let $K$ be a finite lattice, let $E$ be finite distributive lattices, and let $\psi : \text{Con } K \to E$ be a $\{0\}$-separating join-homomorphism. Then there is a finite lattice $L$, an embedding $\text{id}_K : K \to L$ and an isomorphism $\beta : E \to \text{Con } L$ with $\text{ext } \text{id}_K = \psi \beta$ that is such that the diagram

\[
\begin{array}{ccc}
\text{Con } K & \longrightarrow & E \\
\downarrow & & \downarrow \\
\text{Con } K & \longleftarrow & \text{Con } L
\end{array}
\]

is commutative.
Part IV. Congruence lattices of finite lattices

The congruence lattice of a finite modular lattice is a Boolean lattice. But we have:

- **Theorem 20** (Schmidt, 1984). *Every finite distributive lattice is isomorphic to the congruence lattice of a complemented modular lattice.*
Finally…

and here are the old-timers.
Thank you all for coming.

Thank the organizers László Márki and Péter Pál Pálfy