A CHARACTERISATION OF THE SOURCE IN A SEMIMODULAR LATTICE

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ABSTRACT. The semimodular lattices are are very special join-homomorphic images of finite distributive lattices, these are the so called cover-preserving join-congruences. A cover-preserving join-congruence Θ of a distributive lattice L is determined by a subset S of L, which is the *source* of Θ. We characterise these subsets.

1. PRELIMINERIES

This paper is a continuation of [2] and [3].

There is a trivial "representation theorem for finite lattices: each of them is a join-homomorphic image of a finite distributive lattice D. This follows from the fact that the finite free join semilattices with zero are the finite Boolean lattices. The semimodular lattices are are very special join-homomorphic images of finite distributive lattices.

A join-homomorphism $\varphi : L \to K$ is said to be *cover-preserving* iff it preserves the relation $\preceq$. Similarly, a join-congruence Φ of L is called cover-preserving if the natural join-homomorphism $L \to L/\Phi$, $x \mapsto [x]\Phi$ is cover-preserving.

As usual, $\mathbf{J}(L)$ stands for the poset of all nonzero join-irreducible elements of L. The *width* $\mathrm{w}(P)$ of a (finite) poset P is defined to be $\max\{n$: P has an n-element antichain$\}$. We define the dimension of a finite semimodular lattice: $\mathbf{dim}(L) = \mathrm{w}(\mathbf{J}(L))$.

Definition 1. *A* grid *of a semimodular lattice L is $G = C_1 \times C_2 \times ... \times C_n$, where the C_i-s are subchains of L, $\mathbf{J}(L) \subseteq C_1 \cup C_2 \cup ... \cup C_n$, $n = \mathbf{dim}(L)$.*

The grid is not uniquely determined, we fix one of these. We choose maximal chains, these have the same length, m. $\mathcal{C}_n$ denotes the chain $0 < 1 < .... < n-1$ of natural numbers. Then we can take as grid $G = {\mathcal{C}_m}^n$. The direct product $G = C_1 \times C_2 \times C_3$, where C_1, C_2 and C_3 are chains can be considered as a 3D *hypermatrix* (this is a square cuboid), in the $n = 2$ case a matrix, this has a row and two columns. A sublattice $\{a_1 \wedge a_2, a_1, a_2, a_1 \vee a_2\}$ of a lattice is called a *covering square* if $a_1 \wedge a_2 \prec a_i \prec a_1 \vee a_2$ for $i = 1, 2$.

In [1] we proved the following lemma and theorem:

Lemma 1. *Let Φ be a join-congruence of a finite semimodular lattice M. Then Φ is cover-preserving if and only if for any covering square $S = \{a \wedge b, a, b, a \vee b\}$ if $a \wedge b \not\equiv a$ (Φ) and $a \wedge b \not\equiv b$ (Φ) then $a \equiv a \vee b$ (Φ) implies $b \equiv a \vee b$ (Φ).*

Date: January 24, 2016.

2000 *Mathematics Subject Classification.* Primary: 06C10, Secondary: 06B15.

Key words and phrases. Semimodular lattice, cover-preserving.

Theorem 1. *Each finite semimodular lattice L is a cover-preserving join-homomorphic image of the direct product of $n = \mathrm{w}(\mathbf{J}(L))$ maximal (0, 1)-subchains of L, $\mathbf{J}(L) \subseteq C_1 \cup C_2 \cup .... \cup C_n$.*

2. The source

We define the source in paragraph 2.2.

2.1. **The source element.** To describe the cover-preserving join- congruences of a distributive lattice G we need the notion of source elements of G. Czédli and E. T. Schmidt [3]. Let Θ be a cover-preserving join-congruence of G.

Definition 2. *An element $s \in G$ is called a* source element *of Θ if there is a $t, t \prec s$ such that $s \equiv t$ (Θ) and for every prime quotient u/v if $s/t \searrow u/v, s \neq u$ imply $u \not\equiv v$ (Θ). The set $\mathcal{S}_\Theta$ of all source elements of Θ is the* source *of Θ.*

The source elements are top element of the cells.

Lemma 2. *Let x be an arbitrary lower cover of a source element s of Θ. Then $x \equiv s$ (Θ). If $s/x \searrow v/z$, $s \neq v$, then $v \not\equiv z$ (Θ).*

The following results are proved in [3]. The source $\mathcal{S}$ satisfies an independence property:

Definition 3. *Two elements s_1 and s_2 of a distributive lattice are s-*independent *if $x \prec s_1, y \prec s_2$ there is no $v \prec s_2$ such that $s_1/x \searrow s_2/v$ and there is no $u \prec s_1$ such that $s_2/y \searrow s_1/u$. A subset S is s-independent iff every pair $\{s_1, s_2\}$ is s-independent.*

Lemma 3. *Every row/column contains at most one source element.*

Proof. This is trivial by the definition of the source element. □

The semimodular lattice L is determined by (G, Θ) or $(G, \mathcal{S})$, where $\mathcal{S}$ is an s-independent subset and therefore we write:

$$L = \mathcal{L}(G, \mathcal{S}).$$

Determined means, if $L \ncong L'$ then $\mathcal{S} \ncong \mathcal{S}'$ (order isomorphic subsets of G).

The meet of two cover-preserving join-congruence is in generally not cover-preserving.

Take $\mathcal{S}$ a subset of G. and the set of all lower covers of $s \in \mathcal{S}$, ${s'}_i \prec s$ $(i \in \{1, 2, 3\})$. Then we have the following set of primintervals of G:

$$P = \{[{s'}_i, s]|, s \in \mathcal{S}\}.$$

Let $\Theta_\mathcal{S}$ be the join congruence generated by this set of primintervals, i.e. for a priminterval $[a, b]$, $a \equiv b$ (Θ_S) if and only if there is a $s \in \mathcal{S}$ priminterval $[{s'}_i, s]$ such that $[a, b]$ is upper perspective to a $[{s'}_i, s]$.

Let Θ be a cover-preserving join-congruence of an n-dimensional grid G and let $\mathcal{S}$ be the source of Θ. Then $\Theta = \Theta_\mathcal{S}$ (if $\mathcal{S}$ is an s-independent set then Θ_S is generally not a cover-preserving join-congruence). Θ_s denotes the cover-preserving join-congruence determined by s, see in Figure 9. The source of Θ_s is $\{s\}$.

It is easy to prove that in the 2D case every s-independent subset $\mathcal{S}$ determinate a cover-preserving join-congruence:

Lemma 4. *Let G be a 2-dimensional grid, i.e. the direct product of two chains. Let $\mathcal{S}$ be an s-independent subset of G. Then there exists a cover-preserving join-congruences Θ of G with the source $\mathcal{S}$.*

2.2. **Two Takes All: the TTA-property.** If we have source element in two directions then we have in all directions.

In the n-dimensional case, $n > 2$ the source satisfies the following additionally property:

The TTA–property: Let G be a 3D grid and $(x, y, z) \in G$. If the intervals $[(x, y, 0), (x, y, z)]$ and $[(x, 0, z), (x, y, z)]$ contains a source elements then there is a source element in the interval $[(0, y, z), (x, y, z)]$.

In the special case, if $(x, y, z) = (1, 1, 1)$, then if $(x_1, 1, 1), x_1 < 1$ and $(1, x_2, 1), x_2) < 1$ are source elements, then $(1, 1, x_3), x_i < 3$ is a source element for some $x_3) < 1$.

The TTA property is illustrated in Figure. We generaize Lemma 3.

From every grid point starts half lines parallel to the axes.

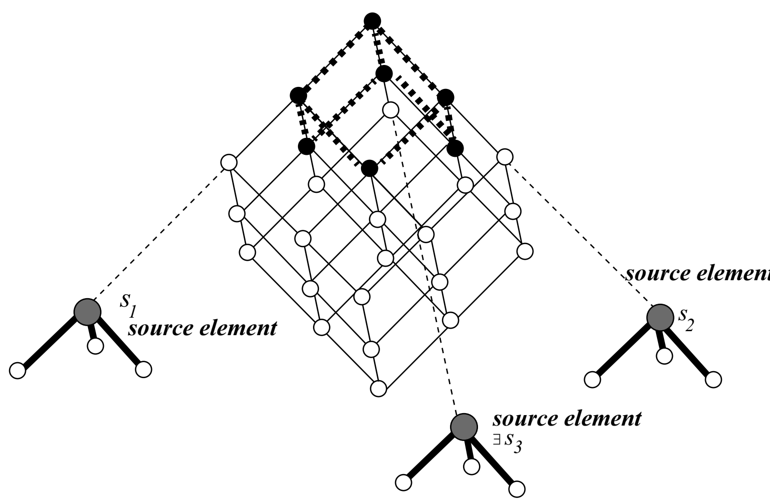

FIGURE 1. The TTA property: $s_1, s_2 \in S$ then there exists a $s_3 \in S$

Lemma 5. *Let S be a source of 3D semimodular lattice L, then S satisfies the TTA property.*

Proof. $G = (\mathcal{C}_2)^3$. Assume that $(1, 1, 0)$ and $(1, 0, 1))$ are source elements, then $(1, 0, 0) \equiv (1, 1, 0)(\Theta)$, $(0, 1, 0) \equiv (1, 1, 0)(\Theta)$,$(0, 1, 0) \equiv (0, 1, 1)(\Theta)$ and $(0, 0, 1) \equiv (0, 1, 1)(\Theta)$. These imply $(1, 1, 1) \equiv (1, 1, 0)(\Theta)$, $(1, 1, 1) \equiv (0, 1, 1)(\Theta)$, $(1, 1, 1) \equiv (1, 0, 1)(\Theta)$, $(0, 1, 0) \equiv (1, 1, 0)(\Theta)$, By the transitivity we obtain $(0, 0, 1) \equiv (1, 0, 1)(\Theta)$ and $(0, 1, 0) \equiv (1, 0, 1)(\Theta)$, i.e. $(0, 1, 1)$ is a source element.

For $G = (\mathcal{C}_n)^3$ the counting is similar but the notation is more complicated.

□

Theorem 2. *Let $\mathcal{S}$ be a s-independent subset of a 3D grid G, which satisfies the TTA property. Then there exists a cover-preserving join-congruence Θ such that the source of Θ is $\mathcal{S}$.*

Proof. We take a s-independent subset $\mathcal{S}$ of G, which satisfies the TTA property and define a join-congruence $\Theta_{\mathcal{S}}$ of G. In Figure we see the possible congruence-classes on the eight-element boolean lattice.

Define for $a \prec b$, $a, b \in G$, $a \equiv b \ (\Theta)$ if and only if for a pair s, t, $s \in \mathcal{S}$, $t \prec s$ $a \vee s = b$ and $a \wedge s = t$. We prove that $\Theta_{\mathcal{S}}$ is cover-preserving and its source is $\Theta_{\mathcal{S}}$. Let $a \wedge b, a, b, a \vee b$ a covering square of G, $a \equiv a \vee b \ (\Theta)$, $b \not\equiv a \vee b \ (\Theta)$, $a \wedge b \not\equiv a$ (Θ), $a \wedge b \not\equiv b \ (\Theta)$. By the definition of $\Theta_{\mathcal{S}}$ there is a a pair s, t, $s \in \mathcal{S}$, $t \prec s$ such that $a \vee s = a \vee b$ and $a \wedge s = t$. Then $s \not\leq b$, otherwise $b \equiv s \vee a \ (\Theta)$, this woud imply $b \equiv a \vee b \ (\Theta)$, contradiction. This proves $s \vee b = a \vee b$. Take $s \wedge b$ and $s \ a \wedge b$. The elements $s \wedge a \wedge b, t, s \wedge b, s$ is a covring square. By Lemma 2 $s \wedge b \equiv s \ (\Theta)$ and therefore $b \equiv a \vee b \ (\Theta)$

□

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