

# A NEW LOOK AT THE SEMIMODULAR LATTICES A GEOMETRIC APPROACH

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ABSTRACT. In this paper we consider the finite semimodular lattices as geometric shapes. In 2009 I published a paper with *Gábor Czédli*, [10], we proved that every semimodular lattice  $L$  can be obtained from a direct power of a chain  $G = C^n$  – geometrically a cube – on an easy way:  $L$  is the cover-preserving join-homomorphism of  $G$ .

We introduce the concept of rectangular semimodular lattices and prove that the building stones of finite semimodular lattices are special rectangulars, the **patch lattices (or pigeonholes)**. The building tool is a special S-verklebte sum (introduced by *Christian Herrmann* [21]) this is the **patchwork**. We prove a structure theorem for finite semimodular lattices: *every finite semimodular lattice is the patchwork of patch lattices*. The 2-dimensional case was proved in [5],

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*Date:* February 20, 2016.

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## Part 1. Geometric shapes

### 1. INTRODUCTION

We have the following structure theorem: *every semimodular lattice is the patchwork of patch lattices*. In the class of distributive lattices a good example is the Rubik's cube (this gave the idea to define the patchwork construction), where 27 small "unit" cubes are glued together by faces (we obtain every finite distributive lattices on this way from "unit" cubes). Our goal is to extend this construction to all semimodular lattice.

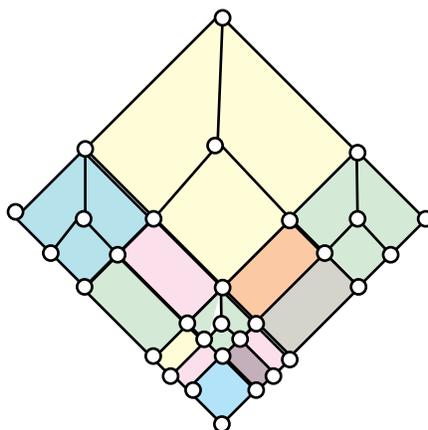


FIGURE 1. A patchwork in the two-dimensional case

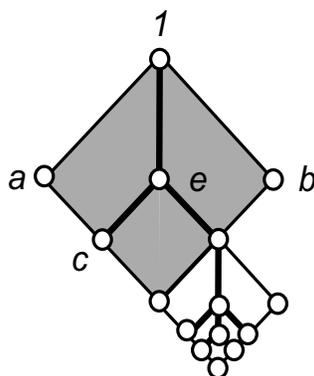


FIGURE 2. Patchwork of two patch lattices

### 2. PRELIMINARIES

**2.1. Distributive lattices.** We consider the diagrams of semimodular lattices as geometric shapes, the eight element boolean lattice is a cube. First, the simplest case is the class of planar distributive lattices. Every planar distributive lattice  $D$  is the cover-preserving sublattice of a direct product of two finite chains, which is

geometrically a rectangle. On Figure 1 you can see a typical example ( $D$  is the shaded part), this lattice contains three maximal rectangles  $A = [c, 1]$ ,  $B = [a, d]$  and  $C = [0, b]$  which are glued together. The intersection (which is a rectangle)  $A \cap B$  is a dual ideal of  $A$  and an ideal of  $B$ . *Every planar distributive lattice  $D$  can be covered by rectangles, that is, in other words it is a "glued system" of rectangles.* This gluing is a special case of the S-verklebte sum, introduced by Christian Herrmann [21] for modular lattices. We can do this in various ways such as with largest rectangular components (on Figure 3 there are three components). If we cut up the lattice defined on Figure 3 into smaller pieces, we may assume that the intersection of two "small" rectangles is either empty or is a part of a side of the rectangular components. The smallest rectangles are obviously the unit squares. This means a planar distributive lattice  $D$  looks like the mosaic pavement in the kitchen, the only one difference is that on some places we may have "degenerate" unit squares (blocks) which are unit sections or one element (see Figure 2). On Figure 1. 17 blocks  $M_1, \dots, M_{17}$ . This covering has the following special property: if  $M_i \cap M_j \neq \emptyset, i \neq j$ , then the union,  $M_i \cup M_j$  is the Hall-Dilworth gluing via an edge or corner, i.e. the "dimension" of  $M_i \cap M_j$  is smaller than the "dimension" of  $M_i$  and  $M_j$ . In  $G = 3 \times 3$  on Figure 4 we have four blocks  $M_1, \dots, M_4$ . If we take the cover-preserving join-homomorphic image,  $S_7$ , then we have again four mosaics  $M_1, \dots, M_4$ , but  $M_4$  is in this case a degenerate mosaic, the edge  $c, 1$ . A *1-narrow*s of  $L$  is an element  $a \in L - \{0, 1\}$  such that  $L = \downarrow a \cup \uparrow a$ . A *2-narrow*s of  $L$  is a priminterval  $[u, v]$  such that  $u$  is meet-irreducible and  $v$  is join-irreducible. If the planar distributive lattice  $D$  has no  $n$ -narrow,  $n = 1, 2$  we say that  $D$  is *narrow free*.

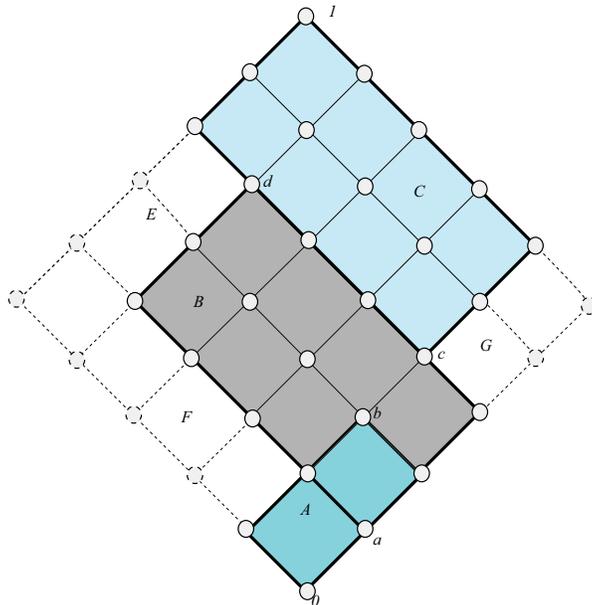


FIGURE 3. A "typical" planar distributive lattice  $D$  (the shaded part)

Every finite distributive lattice  $L$  (not only the planar) can be considered as a shape which we obtain from cover-preserving cubes (bricks) applying gluing by

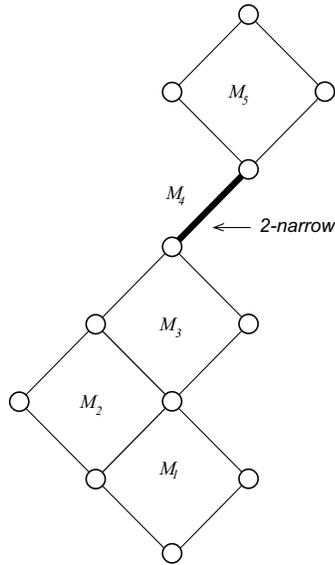


FIGURE 4. A planar distributive lattice with a narrow

faces. We will work with such coverings by semimodular lattices and we will call them *patchwork*. The patchwork was introduced for planar semimodular lattices in [10].

We will define special semimodular lattices the *patch lattices* (or *pigeonholes*. In the class of finite distributive lattices these are the Boolean lattices. Using these terminologies we can say:

*Every finite distributive lattice is the patchwork of patch lattices.*

Our goal to extend these trivial statement for all semimodular lattices. As a non-distributive example let us take the lattice (patchwork [10]) on Figure 1.

Let us take again Figure3. If we start with  $D$  (the shaded part) then we glue to this further rectangulars to  $E$ ,  $F$  and  $G$  to get a "narrowest big" rectangle. This is a rectangle hull of  $D$ . We extend these trivial statement for all semimodular lattices.

Let  $L$  and  $K$  be finite lattices. A join-homomorphism  $\varphi : L \rightarrow K$  is said to be *cover-preserving* iff it preserves the relation  $\preceq$ . Similarly, a join-congruence  $\Phi$  of  $L$  is called cover-preserving if the natural join-homomorphism  $L \rightarrow L/\Phi$ ,  $x \mapsto [x]\Phi$  is cover-preserving.  $J(L)$  denotes the order of all nonzero join-irreducible elements of  $L$  and  $J_0(L)$  is  $J(L) \cup \{0\}$ .

The concept of the *dimension* of a semimodular lattice is a sensitive step in this paper. There are different possibilities. The *width*  $w(P)$  of a (finite) order  $P$  is defined to be  $\max\{n : P \text{ has an } n\text{-element antichain}\}$ . If  $L$  is a lattice the number  $k = w(J(L))$  is called the *J-width* of  $L$ . This will be denoted by  $\mathbf{dim}(L)$ .

In virtue of Dilworth [12],  $P$  is the union of  $k$  appropriate chains. This concept is a kind of geometric *dimension*. The Kurosh-Ore dimension  $\mathbf{Dim}(L)$  of  $L$  is the minimal number of join-irreducibles needed to span the unit element of  $L$ . The dimension of a finite semimodular lattice  $L$  is the greatest natural number  $n$  such that there is an interval  $I$  with the Kurosh-Ore dimension  $n = \mathbf{Dim}(I)$ .

It was proved by G. Czédli and E. T. Schmidt [4], (for planar semimodular lattices see G. Grätzer, E. Knapp [15], [16],[17],[18]):

**Theorem 1.** *Each finite semimodular lattice  $L$  is a cover-preserving join-homomorphic image of a distributive lattice  $G$  which is the direct product of  $\mathbf{dim}(L)$  finite chains.*

**Definition 1.**  *$G$  is called a grid of the lattice.*

**Remark.** The theorem was proved originally (and independently) in the first edition of M. Stern, [28] (Th6.3.14, p. 240).

$G$  can be interpreted in two different ways:

- (1)  $G$  is a **grid** (i.e. a coordinate system),
- (2)  $G$  is a geometric **rectangle** (other names: a  $n$ -dimensional cube, a shape or cuboid).

By Theorem 1 we can get every semimodular lattice  $L$  from a rectangle, using as tool a cover-preserving join homomorphism. We "carve" the semimodular lattice  $L$  from  $G$ .

In [4] we proved an other similar theorem:

**Theorem 2.** *Every finite semimodular lattice  $L$  is a cover-preserving join-homomorphic image of the unique distributive lattice  $D$  determined by  $J(D) \cong J(L)$ . Moreover, the restriction of an appropriate cover-preserving join-homomorphism from  $D$  onto  $L$  is a  $J(D) \rightarrow J(L)$  order isomorphism.*

From the proof of Theorem 2 we can see that  $F = H(J(L))$ , which is the distributive lattice determined by the order  $P = J(L)$  and the poset  $H(P)$  denotes the lattice of all downsets of the order  $P$ .

A subset  $\{a, b, a \wedge b, a \vee b\}$  of a semimodular lattice is called a *covering square* if  $a \wedge b \prec a$  and  $a \wedge b \prec b$ . The semimodularity implies  $a \prec a \vee b$  and  $b \prec a \vee b$ .

If we have a grid  $G = C_k^n$  ( $C_k = \{1, 2, \dots, k\}$  is the chain of natural numbers). then we will denote its elements as vectors  $g = (x_1, x_2, \dots, x_n)$ .

We need the following lemma [4], which characterize the cover-preserving join-homomorphisms:

**Lemma 1. (Covering square lemma.)** *Let  $\Phi$  be a join-congruence of a finite semimodular lattice  $M$ . Then  $\Phi$  is cover-preserving if and only if for any covering square  $S = \{a \wedge b, a, b, a \vee b\}$  if  $a \wedge b \not\equiv a$  ( $\Phi$ ) and  $a \wedge b \not\equiv b$  ( $\Phi$ ) then*

$$a \equiv a \vee b \text{ } (\Phi) \text{ implies } b \equiv a \vee b \text{ } (\Phi).$$

This lemma states: a join-congruence of a finite semimodular lattice  $M$  is cover-preserving if and only if its restriction to any covering square is a cover-preserving join-congruence.

$l(L)$  denotes the length of  $L$ . Let  $a/b$  and  $c/d$  prime quotients of a lattice  $L$ . If  $b \vee c = a$ ,  $b \wedge c = d$  we say that  $a/b$  is perspective up to  $c/d$  and we write  $a/b \searrow c/d$ .

**2.2. The breakdown procedure.** There is an other way to represent a finite distributive lattice  $L$ , this is the *breakdown procedure*: we cut out (carve) from the "big" cuboid  $E = \{0, c_5, e_3, 1\}$  (we call *envelop* distributive lattice is the cover-preserving sublattice of a rectangle (cuboid)  $R$  such that  $Jw(R) = Jw(L)$ ). In the planar case, see Figure 3, we cut out  $S_1$  and  $S_2$ . It is easy to define the *corner* and

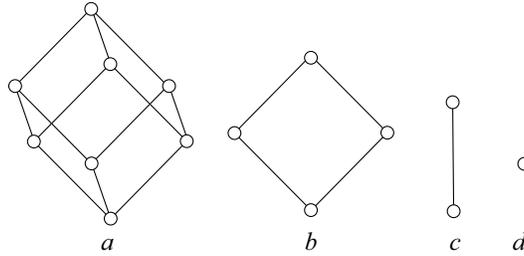


FIGURE 5. The building stones of distributive lattices in the 3-dimensional case)

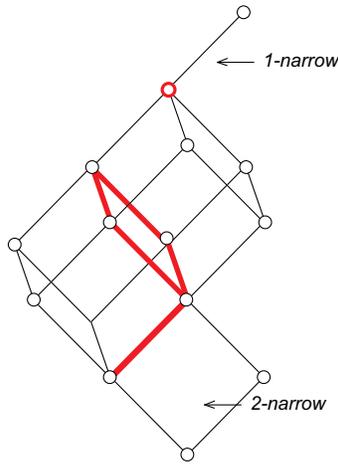


FIGURE 6. A distributive lattice  $D$ , as patchwork,  $\mathbf{dim}(D) = 3$

other geometric concepts for arbitrary finite distributive lattice which was defined originally in planar case in [17].

Our goal is to give "similar" representations for all semimodular lattices. We ask, which properties of special gluings are inherited for semimodular lattices? First, in section 3 we define distributive lattices which are associated to a semimodular lattice.

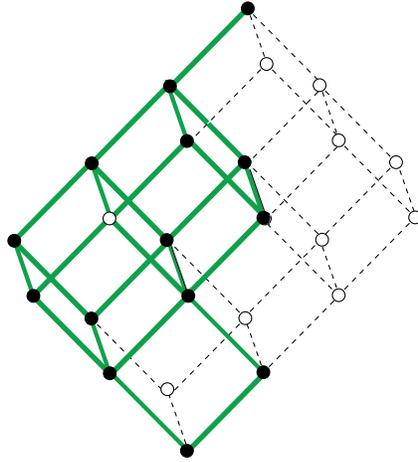
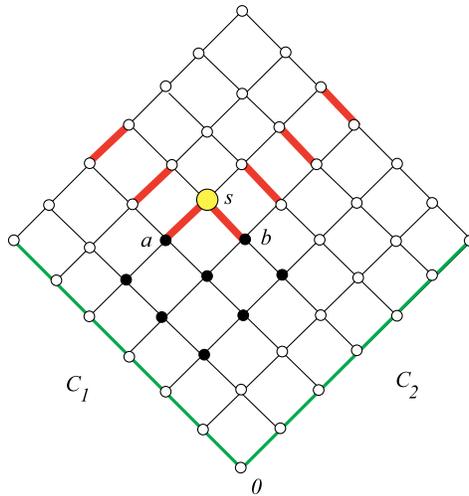
**2.3. Minimal cover-preserving join-congruences.** Let  $s, t$  be elements of a semimodular lattice  $L$ . Then  $\text{con}^{\vee\text{cp}}(s, t)$  denotes the smallest cover-preserving join-congruence  $\alpha$  of  $L$  where  $s \equiv t (\alpha)$ . Let  $t \prec s$  be a covering pair. It is clear that for a covering pair  $d \prec c$ ,  $c \equiv d (\alpha)$  if and only if there is a lower cover  $u$  of  $s$  such that  $c/d$  is perspective up to  $s/u$ . That means  $\alpha$  is determined by the element  $s$  and we write:

$$\text{con}^{\vee\text{cp}}(s) := \text{con}^{\vee\text{cp}}(s, t)$$

On Figure 8 you can see the planar case. Obviously,

$$\text{con}^{\vee\text{cp}}(s) = \bigvee_{t_i \prec s} (\text{con}^{\vee}(s, t_i)).$$

where  $\text{con}^{\vee}(s, t)$  denotes the smallest join-congruence  $\alpha$  of  $L$  where  $s \equiv t (\alpha)$ .

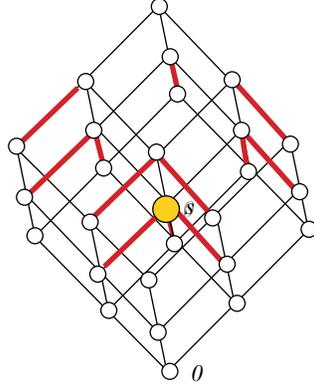
FIGURE 7.  $L$  in a rectangular box (envelop)  $E$ FIGURE 8.  $\text{con}^{\text{vcp}}(s)$  in a direct product of two chains

If we have three chains the  $\text{con}^{\text{vcp}}(s)$  is given on Figure 9.

### 3. DISTRIBUTIVE LATTICES ASSOCIATED TO A SEMIMODULAR LATTICE

Let  $S$  be a semimodular lattice. By Theorem 1 and Theorem 2 there are different distributive lattices such that  $S$  is the cover-preserving join-homomorphic image of these distributive lattices. How can we get these lattices and what is the connection between these lattices?

**3.1. The grids and the frame.** Let  $S$  be a finite semimodular lattice and let  $C_1, \dots, C_n$  be maximal chains of  $S$  such that  $n = \mathbf{dim}(S)$  and  $C_1 \cup C_2 \cup \dots \cup C_n \supseteq J(S)$ , see Definition 1.

FIGURE 9.  $\text{con}^{\vee\text{cp}}(s)$  in a direct product of three chains

The Jordan-Hölder theorem implies that the grid is determined up to isomorphism. Theorem 1 asserts that  $S$  is a cover-preserving join-homomorphic image of  $G$ .

Let  $D_1, \dots, D_n$  be pairwise disjoint subchains of the order  $J_0(S)$  such that  $n = w(J(S))$  and  $D_1 \cup D_2 \cup \dots \cup D_n = J_0(S)$ . We may assume that  $D_i \subset C_i$ , i.e. the  $D_i$  is the restriction of  $C_i$  to  $J_0(S)$ .

**Definition 2.** The direct product  $G_\Gamma = D_1 \times \dots \times D_n$  is called a lower grid of  $S$ .

$G_\Gamma$  is a sublattice of  $G$ . Let us note that the lower grid is not determined uniquely. Obviously, geometrically the grid is a cuboid (rectangular). More results on grids see in G. Czédli, [3]. If  $G$  is fixed then  $G_\Gamma$  is determined.

It is clear that  $l(G) = n \cdot l(S)$  and  $l(G_\Gamma) = |J(S)|$ . The last equality means that every finite distributive lattice  $D$  is the cover-preserving sublattice of the direct product of  $\dim(D)$  chains, this is the lower grid. The lower grid of  $M_3$  is the  $2^3$  boolean lattice and the grid is isomorphic to  $3^3$ .

**Definition 3.** The frame of the semimodular lattice  $S$  is  $H(J(S))$ .

It is clear that  $J(H(J(S)))$  is order isomorphic to  $J(S)$ . The frame is a cover-preserving sublattice of the lower grid,  $\text{Frame}(S) \subseteq G_\Gamma \subseteq G$ . It is easy to see that  $w(\text{Frame}(S)) = w(S)$ .

**3.2. The skeleton.** The skeleton was introduced by G. Grätzer and R. W. Quackenbush [20] for planar modular lattices (they called them frame). Let  $S$  be a planar modular lattice with a planar diagram  $P$ , and let  $M = [b, t]$  be an interval isomorphic to  $M_n$  with exterior atoms  $\{a_1, a_2\}$  and interior atoms  $\{a_2, \dots, a_{n-1}\}$  between  $a_1$  and  $a_2$  in the planar diagram  $P$ . These are doubly irreducible elements. We say that  $a \in S$  is an internal element (with respect to  $P$ ) if  $a$  is an external element. Let  $\text{Skeleton}(S)$  be the sublattice of  $S$  consisting of all external elements of  $S$ , which is unique determined up to isomorphism.

We extend this concept for semimodular lattices.

**Definition 4.** A maximal cover-preserving complemented sublattice  $C$  in a distributive lattice  $D$  is called a cell. If  $C \cong 2^k$  we say that  $C$  is a  $k$ -cell.

**Definition 5.** The skeleton,  $\text{Skeleton}(S)$  of a finite semimodular lattice  $S$  is a maximal cover-preserving distributive sublattice  $D$  of  $S$  such that every element of  $S$  is in the interval generated by a cell of  $D$ .

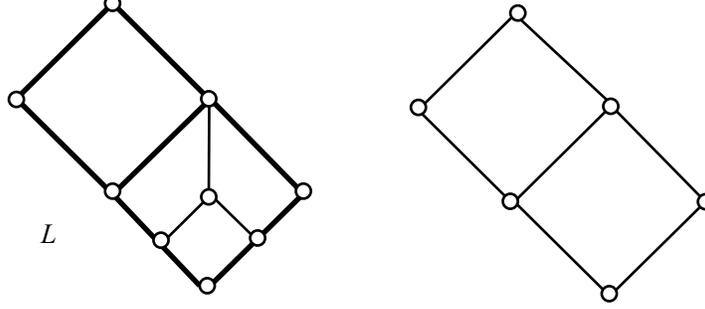


FIGURE 10. The skeleton of a semimodular lattice  $S$

**3.3. Mappings associated to grids and frames.**  $\text{con}^\vee(y, x)$  denotes the principal join-congruence, i.e. the smallest join-congruence – under which  $y \equiv x$ .

**Definition 6.** A join-congruence  $\Phi$  of a distributive lattice  $D$  is called distributive join-congruence if  $\Phi = \bigvee \text{con}^\vee(p_i, x_i)$ , where  $p_i \in D$  are join-irreducible elements.

This notion was introduced in [23].

Let  $S$  be a semimodular lattice, then we have the distributive lattices  $G, G_\perp$ , and the  $\text{Frame}(S) = \text{H}(\text{J}(S))$ . There are several mappings between these lattices:

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & S \\ \downarrow \alpha & & \uparrow \beta \\ G_\perp & \xrightarrow{\psi} & \text{Frame}(S) \end{array}$$

(I.)  $\varphi$  from  $G$  onto  $S$ . Theorem 1 states that there is cover-preserving join-homomorphism  $\varphi$  from  $G$  onto  $S$ :

$$\varphi : G \rightarrow S, \quad (x_1, \dots, x_k) \mapsto x_1 \vee \dots \vee x_k.$$

(II.)  $\alpha : G \rightarrow G_\perp$ . We may assume that  $D_i \subseteq C_i$  for every  $i \in \{1, \dots, n\}$ . In this case to every  $c \in C_i$  there is a largest  $c^* \in D_i$  such that  $c \geq c^*$ .

It is clear that  $\alpha$  is a lattice homomorphism of  $G$  into  $G_\perp$  (which is determined by the congruence  $(c_1, \dots, c_n) \equiv (c_1^*, \dots, c_n^*)(\Theta)$ ).

(III.)  $\psi : G_\perp \rightarrow \text{H}(\text{J}(S)) = \text{Frame}(S)$ . This is a distributive if join-congruence.

Take for simplicity the  $n = 2$  case. By the definition of  $G_\perp$  every  $p \in \text{J}(S)$  appears in one of the  $D_i$ -s. If  $p, q \in D_i$ ,  $p < q$ , then  $p < q$  in  $\text{J}(S)$  too. If  $p \in D_i$  and  $q \in D_j$ ,  $i \neq j$  but  $p < q$  in  $\text{J}(S)$  then take the elements  $(p, 0), (p, q)$  of  $G_\perp$  and the join-congruence induced by this pair

$$(1) \quad \text{con}^\vee((p, 0), (p, q))$$

On this way we get a join-homomorphism from  $G_l$  to  $\text{Frame}(S)$ , i.e. the frame  $\text{Frame}(S)$  is distributive join-homomorphic image of the lower grid.

(IV.)  $\beta : \text{Frame}(S) = \text{H}(\text{J}(S)) \longrightarrow S$  exists by Theorem 2,  $\beta$  is a cover-preserving join-homomorphism.

**Lemma 2.**  $\varphi = \beta\psi\alpha$ .

*Proof.* Easy computation. □

### 3.4. Horizontal and vertical edges of an order. 15.050

Let us consider  $P = \text{J}(D)$ , where  $D$  is the distributive lattice defined on Figure 1. In the diagram (graph) of  $P$  (see Figure 10) we have two type of edges:  $\overline{x, y}$ ,  $x \prec y$ ,  $x, y \in D_i$  for some  $i$ , these are the *vertical edges* (these are in the two chains) and all others the *horizontal edges*. We omit (cut out) the horizontal edges of  $\text{J}(D)$  we get the diagram of  $\text{J}(G_l)$ , i.e. we have two maximal chains  $0 < c_1 < c_2 < c_3 < c_4 < c_5$  and  $0 < e_1 < e_2 < e_3$  and the relations,  $e_2 > c_3, c_4 > e_1$  which are given by two horizontal edges,  $\overline{e_2, c_3}$  and  $\overline{c_4, e_1}$  (red lines).

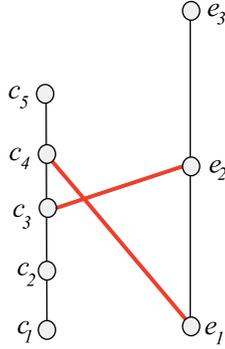


FIGURE 11.  $\text{J}(D)$

On  $G_l$  take the corresponding join congruence

$$(2) \quad \Theta = \text{con}^\vee((c_4, 0), (c_4, e_1)) \vee \text{con}^\vee((0, e_2), (c_2, e_2))$$

Then  $G_l/\Theta = P$ . The natural join-homomorphism  $G_l \rightarrow G_l/\Phi$ , preserves boundary chains, this observation makes possible to introduce such notions in an easy way in the  $\text{J-width} > 2$  cases. To the horizontal edge  $\overline{c_4, e_1}$  we associate the cuboid  $S_1$  and similarly,  $\overline{e_2, c_3}$  represents  $S_2$ . We have a bijection between the horizontal edges and "corner" cuboids of  $G_l$ .

We are going to the  $w(P) = 3$  case and take the following poset  $P$  see on Figure 11:

We can visualize this as follows, see Figure 12. From the "big" cuboid we cut out two "small" cuboid-s, which are determined by the horizontal edges.

It is easy to prove that every finite distributive lattice  $D$  of  $\text{J-width} = n$  is

- (1) a cover-preserving sublattice of a rectangular lattice  $R$  of  $\mathbf{dim}(D) = n$  (envelop),
- (2) the S-glued sum of rectangular components of  $\text{J-width} k \leq n$ .

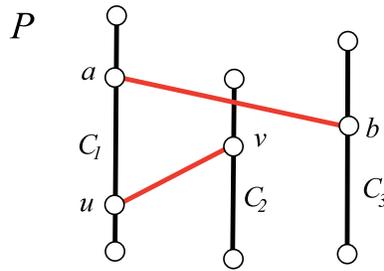


FIGURE 12. Two horizontal edges

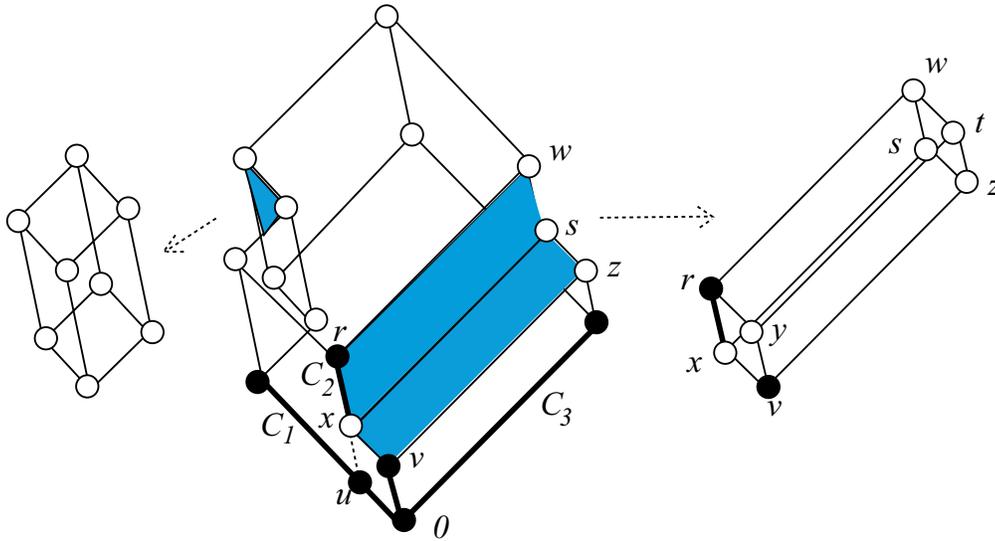


FIGURE 13. Removing corners from a cuboid

It is easy to define the volume and surface area of an arbitrary finite distributive lattice.

Is something like true for all semimodular lattices?

**3.5. Rectangular lattices.** Rectangular lattices were introduced by G. Grätzer and E. Knapp [15] for planar semimodular lattices: a *left corner* (resp. *right corner*) of a planar lattice  $K$  in a double-irreducible element in  $K - \{0, 1\}$  on the left (resp., right) boundary of  $K$ . A *rectangular* lattice  $L$  is a planar semimodular lattice which has exactly one left corner,  $u_l$  and exactly one right corner,  $u_r$  and they are complementary – that is,  $u_l \vee u_r = 1$  and  $u_l \wedge u_r = 0$ . The direct product of two chains is rectangular.

We introduce this notion for arbitrary finite semimodular lattice. Let  $X, Y$  be posets. The *cardinal sum*  $X + Y$  of  $X$  and  $Y$  is the set of all elements in  $X$  and  $Y$  considered as disjoint. The relation  $\leq$  keeps its meaning in  $X$  and in  $Y$ , while neither  $x \geq y$  nor  $x \leq y$  for all  $x \in X, y \in Y$ .

**Definition 7.** A rectangular lattice  $L$  is a finite semimodular lattice in which  $J(L)$  is the cardinal sum of chains.

Geometric lattices are rectangular. In [7] we introduced the *almost geometric lattices* these are lattices in which  $J(L)$  is the cardinal sum of at most two element chains. In the class of finite distributive lattices the rectangular lattices are the direct products of chains. The lattices  $M_3[C_n]$  are modular rectangular lattices ( $C_n$  is the chain  $\{0, 1, 2, \dots, n-1\}$  of integers).

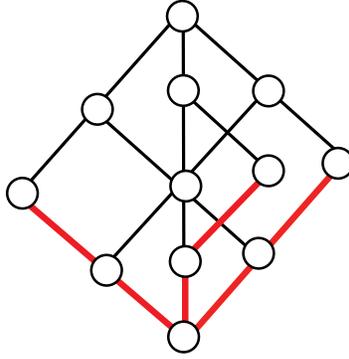


FIGURE 14. A modular non distributive rectangular lattice  $M_3[C_3]$

We prove the following:

**Lemma 3.** Let  $G$  be the direct product of chains and let  $\Theta$  be a cover-preserving join-congruence of  $G$ . Then  $G/\Theta$  is a rectangular lattice with the property  $J(G) \cong J(G/\Theta)$  if and only if every join-irreducible element of  $G$  is a one-element  $\Theta$ -class.

*Proof.* Let  $g/\Theta$  be a  $\Theta$ -class containing  $g \in G$ . Assume that  $g$  is a minimal element of these class. If  $g/\Theta$  is a join-irreducible element of  $G/\Theta$  then  $g$  is a join-irreducible element of  $G$ , i. e.  $g = (0, \dots, 0, g_i, 0, \dots, 0)$  for some  $i$ . But  $J(G) \cong J(G/\Theta)$  and therefore  $g/\Theta$  must be a one-element class. The converse is trivial,  $J(G/\Theta)$  is order isomorphic to the paset  $\{(0, \dots, 0, g_i, 0, \dots, 0)\}$ , i.e.  $G/\Theta$  is rectangular.  $\square$

The modular lattices  $\mathcal{M}_n$ ,  $n > 2$  are two dimensional lattices and  $w(J(\mathcal{M}_n)) = n$ , i.e.  $\mathbf{dim}(\mathcal{M}_n) < w(J(\mathcal{M}_n))$ .

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20 february

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## 4. THE SOURCE

First, we define the source element:

**4.1. The source element.** To describe the cover-preserving join-congruences of a distributive lattice  $G$  we need the notion of source elements of  $G$ . Czédli and E. T. Schmidt [?]. Let  $\Theta$  be a cover-preserving join-congruence of  $G$ .

**Definition 8.** *An element  $s \in G$  is called a source element of  $\Theta$  if there is a  $t, t \prec s$  such that  $s \equiv t \pmod{\Theta}$  and for every prime quotient  $u/v$  if  $s/t \searrow u/v, s \neq u$  imply  $u \not\equiv v \pmod{\Theta}$ . The set  $\mathcal{S}_\Theta$  of all source elements of  $\Theta$  is the source of  $\Theta$ .*

The source elements are top element of the cells.

**Lemma 4.** *Let  $x$  be an arbitrary lower cover of a source element  $s$  of  $\Theta$ . Then  $x \equiv s \pmod{\Theta}$ . If  $s/x \searrow v/z, s \neq v$ , then  $v \not\equiv z \pmod{\Theta}$ .*

*Proof.* Let  $s$  be a source element of  $\Theta$  then  $s \equiv t \pmod{\Theta}$  for some  $t, t \prec s$ . If  $x \prec s$  and  $x \neq t$  then  $\{x \wedge t, x, t, s\}$  form a covering square. Then  $x \not\equiv x \wedge t \pmod{\Theta}$ . This implies  $x \wedge t \not\equiv t \pmod{\Theta}$ . By Lemma 1 we have  $x \equiv s \pmod{\Theta}$ .

To prove that  $v \not\equiv z \pmod{\Theta}$ , we may assume that  $v \prec s$ . Take  $t, t \prec s$ , then we have three (pairwise different) lower covers of  $s$ , namely  $x, v, t$ . These generate an eight-element boolean lattice in which  $s \equiv t \pmod{\Theta}$ ,  $s \equiv x \pmod{\Theta}$  and  $s \equiv v \pmod{\Theta}$ . By the choice of  $t$  we know that  $v \not\equiv v \wedge t \pmod{\Theta}$ ,  $x \not\equiv x \wedge t \pmod{\Theta}$  and  $z \not\equiv x \wedge t \wedge v \pmod{\Theta}$ . It follows that  $x \not\equiv t \pmod{\Theta}$ , otherwise by the transitivity  $x \equiv v \pmod{\Theta}$ . This implies  $t \wedge x \not\equiv t \wedge x \wedge v \pmod{\Theta}$ . Take the covering square  $\{x \wedge v \wedge t, z, t \wedge x, x\}$  then by Lemma 1  $z \not\equiv x \pmod{\Theta}$ , which implies  $z \not\equiv v \pmod{\Theta}$ .  $\square$

**4.2. The independence.** The following results are proved in [?]. The source  $\mathcal{S}$  satisfies an independence property:

**Definition 9.** *Two elements  $s_1$  and  $s_2$  of a distributive lattice are  $s$ -independent if  $x \prec s_1, y \prec s_2$  there is no  $v \prec s_2$  such that  $s_1/x \searrow s_2/v$  and there is no  $u \prec s_1$  such that  $s_2/y \searrow s_1/u$ . A subset  $S$  is  $s$ -independent iff every pair  $\{s_1, s_2\}$  is  $s$ -independent.*

**Lemma 5.** *Every row/column contains at most one source element.*

*Proof.* This is trivial by the definition of the source element.  $\square$

The semimodular lattice  $L$  is determined by  $(G, \Theta)$  or  $(G, \mathcal{S})$ , where  $\mathcal{S}$  is an  $s$ -independent subset and therefore we write:

$$L = \mathcal{L}(G, \mathcal{S}).$$

Determined means, if  $L \cong L'$  then  $\mathcal{S} \cong \mathcal{S}'$  (order isomorphic subsets of  $G$ ).

The meet of two cover-preserving join-congruence is in generally not cover-preserving.

Take  $\mathcal{S}$  a subset of  $G$ . and the set of all lower covers of  $s \in \mathcal{S}$ ,  $s'_i \prec s$  ( $i \in \{1, 2, 3\}$ ). Then we have the following set of primintervals of  $G$ :

$$P = \{[s'_i, s], s \in \mathcal{S}\}.$$

Let  $\Theta_{\mathcal{S}}$  be the join congruence generated by this set of primintervals, i.e. for a priminterval  $[a, b]$ ,  $a \equiv b \pmod{\Theta_{\mathcal{S}}}$  if and only if there is a  $s \in \mathcal{S}$  priminterval  $[s'_i, s]$  such that  $[a, b]$  is upper perspective to a  $[s'_i, s]$ .

Let  $\Theta$  be a cover-preserving join-congruence of an  $n$ -dimensional grid  $G$  and let  $S$  be the source of  $\Theta$ . Then  $\Theta = \Theta_S$  (if  $S$  is an  $s$ -independent set then  $\Theta_S$  is generally not a cover-preserving join-congruence).  $\Theta_s$  denotes the cover-preserving join-congruence determined by  $s$ , see in Figure 9. The source of  $\Theta_s$  is  $\{s\}$ .

It is easy to prove that in the 2D case every  $s$ -independent subset  $S$  determinate a cover-preserving join-congruence:

**Lemma 6.** *Let  $G$  be a 2-dimensional grid, i.e. the direct product of two chains. Let  $S$  be an  $s$ -independent subset of  $G$ . Then there exists a cover-preserving join-congruence  $\Theta$  of  $G$  with the source  $S$ .*

**4.3. Two Takes All: the TTA-property.** If we have source element in two directions then we have in all directions.

In the  $n$ -dimensional case,  $n > 2$  the source satisfies the following additionally property:

**The TTA-property:** Let  $G$  be a 3D grid and  $(x, y, z) \in G$ . If the intervals  $[(x, y, 0), (x, y, z)]$  and  $[(x, 0, z), (x, y, z)]$  contains a source elements then there is a source element in the interval  $[(0, y, z), (x, y, z)]$ .

In the special case, if  $(x, y, z) = (1, 1, 1)$ , then if  $(x_1, 1, 1), x_1 < 1$  and  $(1, x_2, 1), x_2 < 1$  are source elements, then  $(1, 1, x_3), x_3 < 1$  is a source element for some  $x_3 < 1$ .

The TTA property is illustrated in Figure. We generalize Lemma 3.

From every grid point starts half lines parallel to the axes.

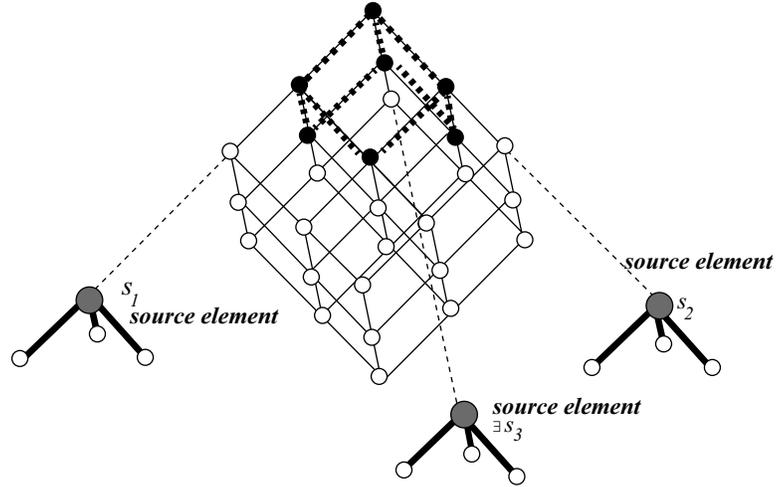


FIGURE 15. The TTA property:  $s_1, s_2 \in S$  then there exists a  $s_3 \in S$

**Lemma 7.** *Let  $S$  be a source of 3D semimodular lattice  $L$ , then  $S$  satisfies the TTA property.*

*Proof.*  $G = (\mathbb{C}_2)^3$ . Assume that  $(1, 1, 0)$  and  $(1, 0, 1)$  are source elements, then  $(1, 0, 0) \equiv (1, 1, 0)(\Theta)$ ,  $(0, 1, 0) \equiv (1, 1, 0)(\Theta)$ ,  $(0, 1, 0) \equiv (0, 1, 1)(\Theta)$  and  $(0, 0, 1) \equiv (0, 1, 1)(\Theta)$ . These imply  $(1, 1, 1) \equiv (1, 1, 0)(\Theta)$ ,  $(1, 1, 1) \equiv (0, 1, 1)(\Theta)$ ,  $(1, 1, 1) \equiv (1, 0, 1)(\Theta)$ ,  $(0, 1, 0) \equiv (1, 1, 0)(\Theta)$ . By the transitivity we obtain  $(0, 0, 1) \equiv (1, 0, 1)(\Theta)$  and  $(0, 1, 0) \equiv (1, 0, 1)(\Theta)$ , i.e.  $(0, 1, 1)$  is a source element.

For  $G = (\mathbb{C}_n)^3$  the counting is similar but the notation is more complicated. □

**Theorem 3.** *Let  $\mathcal{S}$  be a  $s$ -independent subset of a 3D grid  $G$ , which satisfies the TTA property. Then there exists a cover-preserving join-congruence  $\Theta$  such that the source of  $\Theta$  is  $\mathcal{S}$ .*

*Proof.* We take a  $s$ -independent subset  $\mathcal{S}$  of  $G$ , which satisfies the TTA property and define a join-congruence  $\Theta_{\mathcal{S}}$  of  $G$ . In Figure we see the possible congruence-classes on the eight-element boolean lattice.

Define for  $a \prec b, a, b \in G, a \equiv b (\Theta)$  if and only if for a pair  $s, t, s \in \mathcal{S}, t \prec s, a \vee s = b$  and  $a \wedge s = t$ . We prove that  $\Theta_{\mathcal{S}}$  is cover-preserving and its source is  $\Theta_{\mathcal{S}}$ . Let  $a \wedge b, a, b, a \vee b$  a covering square of  $G, a \equiv a \vee b (\Theta), b \not\equiv a \vee b (\Theta), a \wedge b \not\equiv a (\Theta), a \wedge b \not\equiv b (\Theta)$ . By the definition of  $\Theta_{\mathcal{S}}$  there is a pair  $s, t, s \in \mathcal{S}, t \prec s$  such that  $a \vee s = a \vee b$  and  $a \wedge s = t$ . Then  $s \not\leq b$ , otherwise  $b \equiv s \vee a (\Theta)$ , this would imply  $b \equiv a \vee b (\Theta)$ , contradiction. This proves  $s \vee b = a \vee b$ . Take  $s \wedge b$  and  $s \wedge a \wedge b$ . The elements  $s \wedge a \wedge b, t, s \wedge b, s$  is a covering square. By Lemma 2  $s \wedge b \equiv s (\Theta)$  and therefore  $b \equiv a \vee b (\Theta)$  □

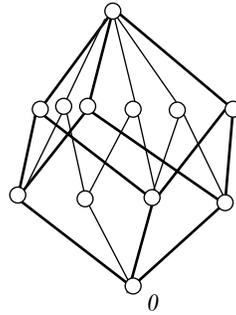


FIGURE 16. Figure 36  $\mathbb{B}_4$

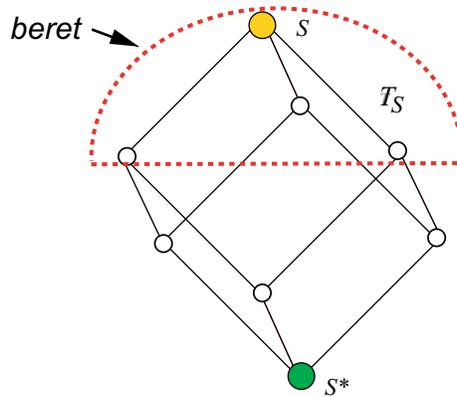


FIGURE 17. The source cell  $B_s$  with the beret  $T_s$

In the 2-dimensional case this means that  $s$  or at least the lower covers of  $s$ ,  $a$  and  $b$  is join-irreducible.

**Remark.** Let  $\varphi : G \rightarrow L$  be a cover-preserving join-homomorphism. We denote by  $\Theta$  the cover-preserving join-congruence induced by  $\varphi$ . Take the source  $S$  of  $\Theta$ . A source element  $s \in S$  is called *bastard* if  $s$  itself or at least one of its lower covers  $t$  is join-irreducible. Let  $S' \subseteq S$  the set of all non bastard source elements and  $\Theta'$  denotes the corresponding cover-preserving join-congruence. Then  $R = G/\Theta'$  is a rectangular lattice (envelop) and  $L = G/\Theta$  is a cover-preserving sublattice of  $R$ .

### 5. SOURCE LATTICES

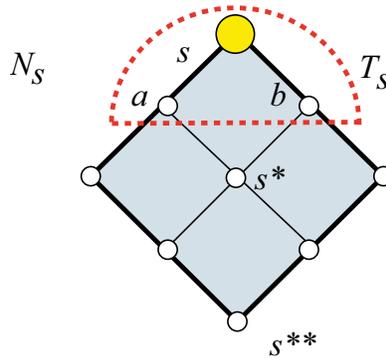


FIGURE 18. A source cell with the neighborhood,  $N_s$  and the beret  $T_s$

The join-congruence of  $N_s$ , where  $B_s$  is the only one non-trivial congruence class will be denoted by  $\Phi_s$ .

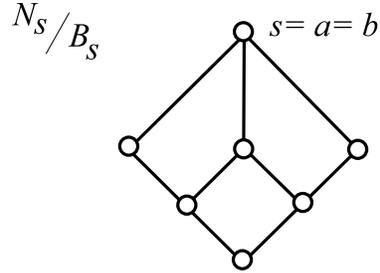


FIGURE 19. The factor of  $N_s$  in the  $Jw(D) = 2$  case,  $\mathbb{L}_2 = \mathbb{L}_{2,1}^H$

**Definition 10.** The source lattice is the factor lattice  $C_3^n / \Phi_n$ , where  $\Phi_n$  is the join-congruence with only one non-trivial congruence class containing the unit element and the dual atoms of  $C_3^n$ .

$\mathbb{L}_2 \cong S_7$  and  $\mathbb{L}_3$  is presented on Figure 11.

The skeleton of the source lattice  $\mathbb{L}_n$  is the  $2^n$  Boole lattice. We defined *Kuros-Ore dimension* of a lattice  $L$  is the minimal number of join-irreducibles needed to span the unit element of  $L$ . The Kuros-Ore dimension of  $\mathbb{L}_n$  is  $n$ .

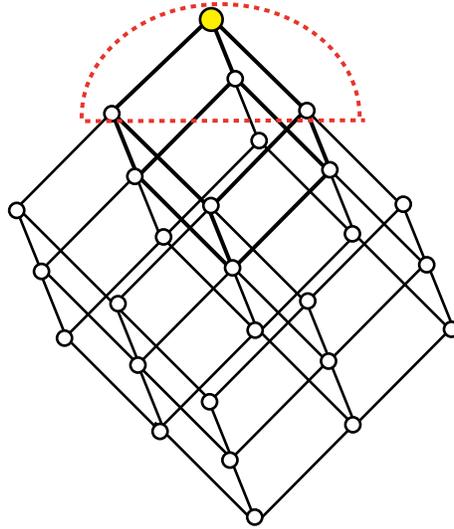
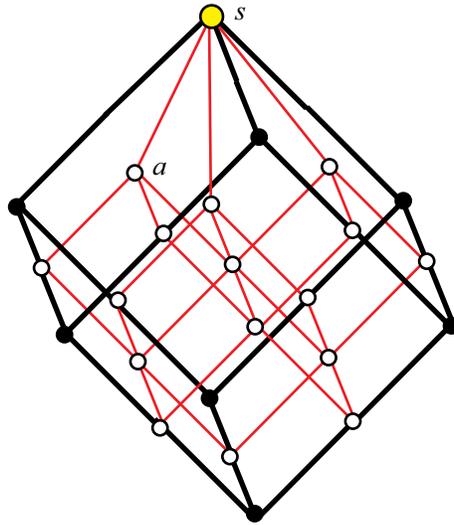


FIGURE 20. A source cell with the neighborhood and the beret

FIGURE 21. The source lattice  $\mathbb{L}_3$  and skeleton

**Lemma 8.**  $\mathbb{L}_n$  is a subdirect irreducible rectangular semimodular lattice.

*Proof.* It is an easy exercise, in the J-width = 3 case, Figure 19,  $\text{con}(s, a)$  is the smallest non trivial congruence relation.  $\square$

**5.1. The matrix of a cover-preserving join-congruence.** Let  $L$  be a semi-modular lattice. By Theorem 1. we have a grid  $G = C_k^n$  ( $C_k = \{1, 2, \dots, k\}$  is the chain of natural numbers) and a cover-preserving join-congruence  $\Theta$  of  $G$  such that  $G/\Theta \cong L$ . On Figure 19 we have a 2-dimensional case a matrix. We write into a cover-preserving square an "1" entry if at the bottom we have a source element.

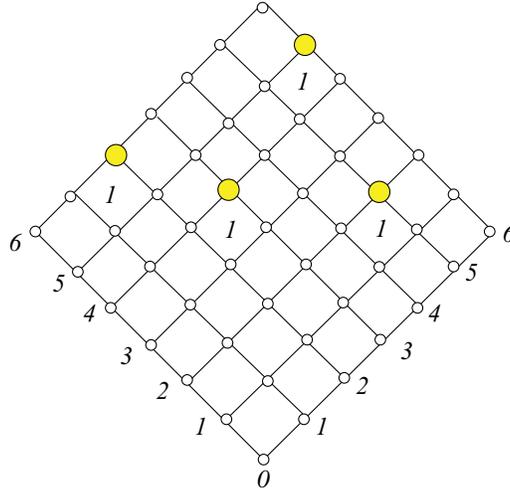


FIGURE 22. A grid and four sours elements

Otherwise we write "0". This is a  $(0,1)$ -matrix and every row/column contains at most "1" entry. Put 1 into a cell if its top element is in  $S$ , otherwise put zero. What we get is an  $n \times n$  matrix,  $\mathbf{M}_L$ , which determines  $L$  (if you like you can turn this grid with 45 degrees to see the matrix in the traditional form).

The first matrix is a  $n \times n$   $(0,1)$ -matrix, where every row/column contains at most 1 entry:

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

There is an other way to define a matrix using the coordinates of the source elements:

$$\begin{vmatrix} 6 & 5 & 4 & 2 \\ 2 & 6 & 3 & 5 \end{vmatrix}$$

For semimodular lattices  $S$  with  $\mathbf{dim}(S) \geq 3$  we use hypermatrices. The matrix of  $M_3$  is a  $(0,1)$ -matrix of type  $3 \times 3 \times 3$ :  $[a_{i,j,k}]$ ,  $a_{1,1,1} = 1$  and  $a_{i,j,k} = 0$  otherwise.

A column  $C(3)_{i,j}$  is  $\{a_{i,j,k}; k = 1, 2, \dots, n\}$  of a  $3 \times 3 \times 3$  matrix:  $[a_{i,j,k}]$  and similarly,

$$C(1)_{i,k} \text{ is } \{a_{i,j,k}; j = 1, 2, \dots, n\},$$

$$C(2)_{i,j} \text{ is } \{a_{j,k}; i = 1, 2, \dots, n\}.$$

We use  $(0,1)$ -matrices, where every column contains at most one entry 1.

See Czédli, Schmidt [8], Czédli [2], Czédli, Ozsvárt, Udvari [11].

## Part 2. Constructions

### 6. CONSTRUCTIONS OF SEMIMODULAR LATTICES

6.1. **The  $n$ -fork construction (the zipper).** This general construction was introduced in [6] for slim planar semimodular lattices (semimodular lattices of dimension 2), but it was included in some older papers, see in [14] and [24]. Let  $S$  be a 4-cell of a slim semimodular lattice  $L$ . Then  $S$  is a covering square  $\{a = b \wedge w, b, w, c = b \vee w\}$ , see on Figure 20. We change  $L$  to a new lattice  $L'$  as follows:

Step1. Firstly, we replace  $S$  by a copy of  $S_7 \cong \mathbb{L}_2$ . This way we get three new 4-cells instead of  $S$ .

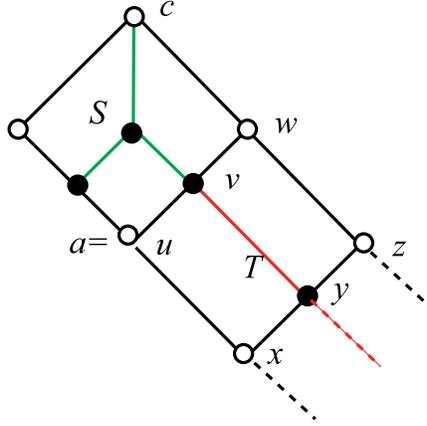


FIGURE 23. The downward-going procedure and a teeth of a fork

Step 2. Secondly, as long as there is a chain  $u \prec v \prec w$  such that  $v$  is a new element and  $T = \{x = u \wedge z, z, u, w = z \vee u\}$  is a 4-cell in the original lattice  $L$  but  $x \prec z$  at the present stage, see Figure 22, we insert a new element  $y$  such that  $x \prec y \prec z$  and  $y \prec v$ . (This way we get two 4-cells instead of  $T$ .) When this “downward-going” procedure terminates, we obtain  $L'$ . The collection of all new elements, which is a poset, will be called a *fork*. We say that  $L'$  is obtained from  $L$  by *adding a fork to  $L$  (at the 4-cell  $S$ )*, see Figure 21 for an illustration. If we add several forks to  $L$  one by one, then we simply speak of adding forks to  $L$ .

**Theorem 4.** [6] *Let  $L$  be a slim semimodular lattice consisting of at least three elements. Then  $L$  can be obtained from the direct product of two nontrivial finite chains such that*

- *first we add finitely many forks one by one,*
- *and then we remove corners, one by one, finitely many times.*

**Remark.** The second condition means we have a distributive join-congruence on the grid  $G$ .

On Figure 24. you can see a refinement of a grid  $G$ , we insert the black filled elements.

Let  $S$  be a semimodular lattice and let  $G$  be a grid of  $S$  with the corresponding cover-preserving join-congruence  $\Theta$ . If we add a fork to  $S$  then this procedure can

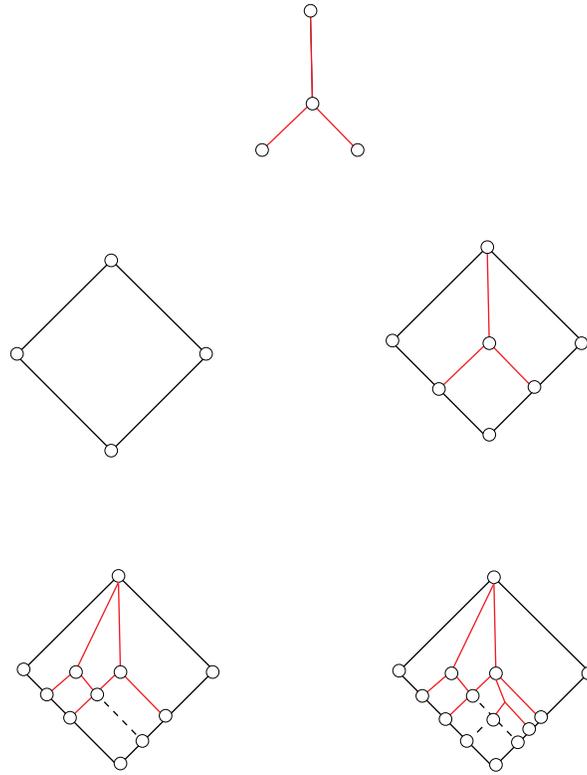


FIGURE 24. 2-fork with the skeleton

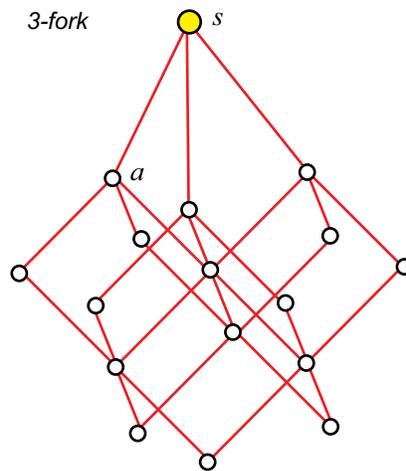


FIGURE 25. 3-fork

be obtains on the following way: we take a refinement of  $G$  and select a new source element (which is a new grid element.) See Figure 29-32.

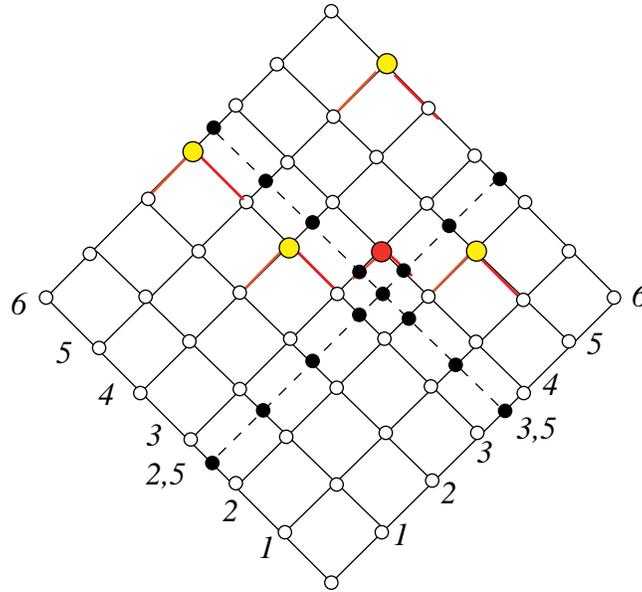


FIGURE 26. Refinement of a grid

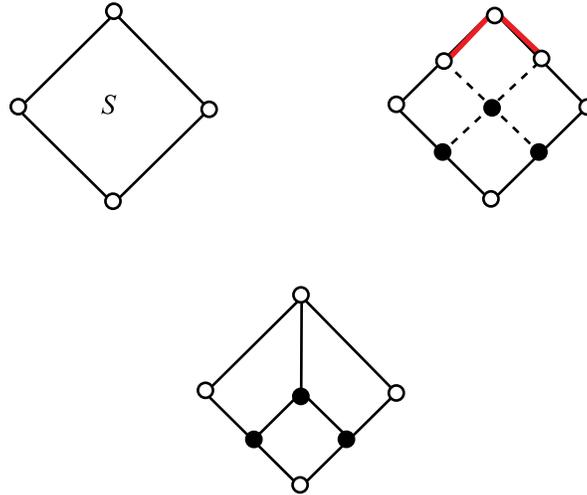


FIGURE 27. Refinement I.

It is easy to generalize this construction for the dimension  $n > 2$ , we take an  $n$ -cell which is isomorphic to  $2^n$  replace by  $\mathbb{L}_n$ , see Figure 28.

Step 1'. Firstly, we replace a covering cube  $S \cong 2^3$  by a copy of  $S_{24}$ .

Step 2'. Secondly, as long as there is an other neighbor covering cube we insert to the elements  $x, y, z, u, v$  the new elements  $x', y', z', u', v'$ . When this “downward-going” procedure terminates, we obtain  $L'$ . The collection of all new elements, which is a poset, will be called a *3-fork*. We say that  $L'$  is obtained from  $L$  by

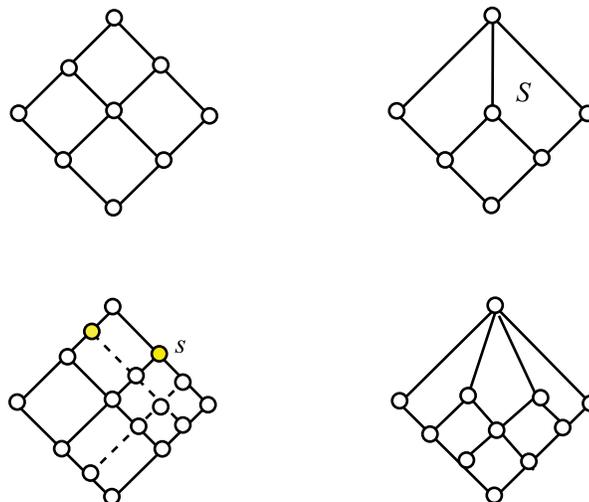
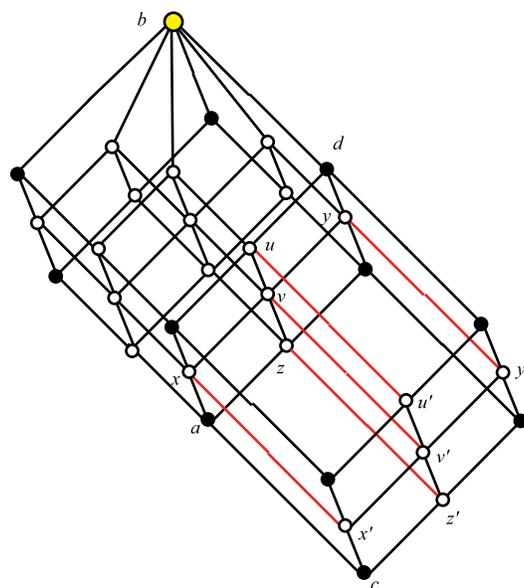


FIGURE 28. Refinement II.

FIGURE 29. The source lattice  $\mathbb{L}_3 = S_{24}$  and the “downward-going” procedure

adding a fork to  $L$ , see Figure 20 for an illustration. If we add several forks to  $L$  one by one, then we simply speak of adding forks to  $L$ .

On Figure 24. you can see a 3-fork.

6.2. **The patch lattice (pigeonhole).** To add a fork we can make the following procedure:

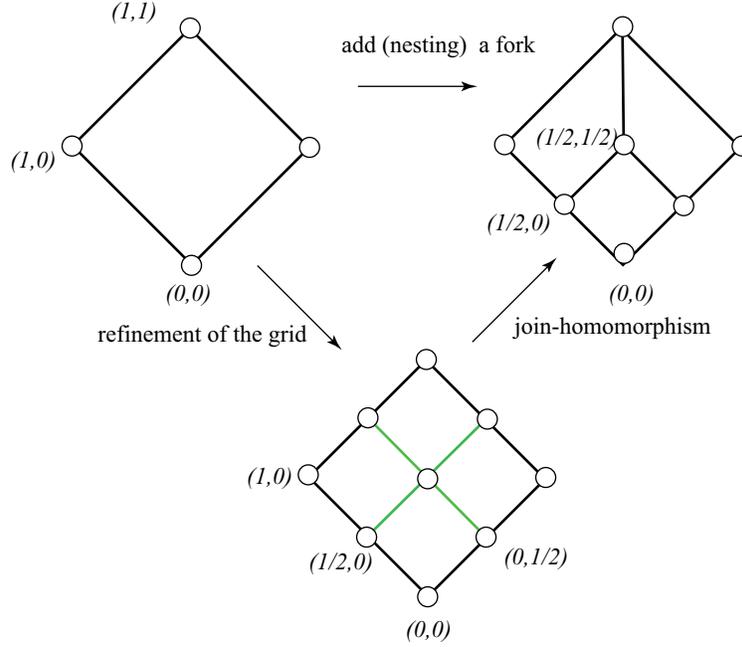


FIGURE 30. Figure 47 The iterative procedure

6.2.1. *The  $\text{Jw}(L) = 2$  case.* We define the nested slim lattices inductively, these are lattices where copies of  $S_7$ -s are nested into each other:

**Definition 11.** A slim semimodular lattice  $N$  is called a 2-nested lattice if we obtain with the following interactive procedure

- (1) The four element Boolean lattice. i.e. the 4-cell  $C_2^2$ , this is a 4-cell,
- (2) We add a fork to this 4-cell  $C_2^2$ , we obtain the source lattice  $S_7 = \mathbb{L}_2$  (on Figure 30 (III) or (V)), and apply the "downward-going" procedure if it is necessary (this two steps gives eight different 2-nested lattices),
- (3)  $S_7$  has three 4-cells. To some of these 4-cells we add further forks, i.e. we replace some covering square by a copy of  $S_7$
- (4) We continue this procedure in finite many steps (on Figure 30 (II)).

The set of all 2-nested lattices will be denoted by  $\mathfrak{N}_1$ .

**Theorem 5.** Every 2-nested lattice  $N$  satisfies the following properties:

- (1)  $N$  is rectangular,
- (2)  $N$  has a complementary pair  $a, a'$  ( $a \wedge a' = 0$  and  $a \vee a' = 1$ ) such that  $a \prec 1$  and  $a' \prec 1$ ,
- (3)  $N$  contains as ideal two chains  $C_1$  and  $C_2$ , such that  $C = C_1 \cup C_2$  is congruence-determining.

*Proof.* .

□

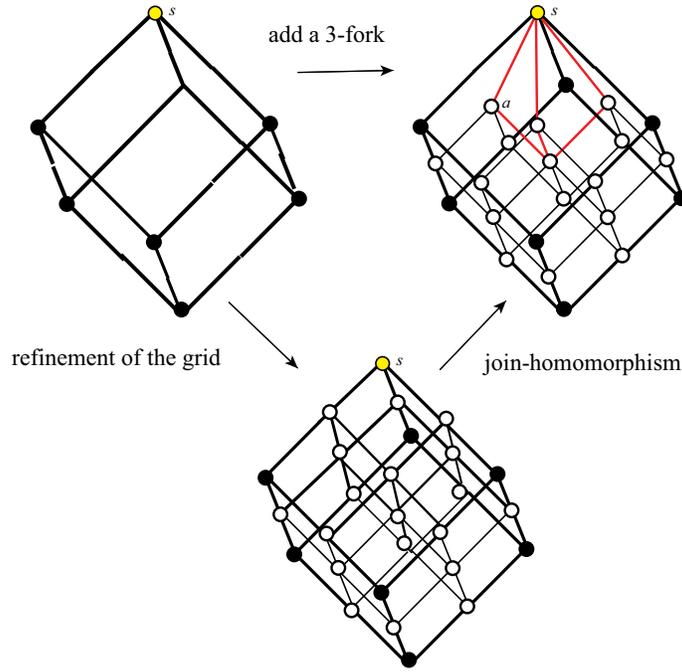


FIGURE 31. Figure 47 The iterative procedure

On the grid  $G$  there is a natural distance,  $\delta(a, b)$  of two elements  $a, b$  in a grid  $G = C^m$  is the smallest natural number  $n$  such that there is a sequence  $a = c_1, c_2, \dots, c_{n-1} = b$ ,  $c_i < c_{i+1}$  or  $c_i > c_{i+1}$  for every  $i$ . This defines an equivalence relation  $\mathcal{E}$  on  $S$ . Two source elements are of the same block iff their are "close to each other".

**Definition 12.** Two incomparable elements  $a$  and  $b$  of a distributive lattice  $D$  are horizontal-adjacent if  $\delta(a, b) = 2$ , i.e. if  $\{a \wedge b, a, b, a \vee b\}$  is a covering square. Two elements  $a$  and  $b$ ,  $b < a$  of a distributive lattice  $D$  are vertical-adjacent  $b = a^{**}$ .

**Definition 13.**  $a \equiv b$  ( $\mathcal{E}$ ) if and only if there is a sequence  $a = s_1, s_2, \dots, s_n = b$  of source elements in a grid  $G$  such that for every  $i$   $(s_i, s_{i+1})$  is either a horizontal-adjacent pair or a vertical-adjacent pair.

**Definition 14.**  $\{a, b\}$  is a remote pair if either (1)  $a$  and  $b$  are incomparable and  $\delta(a, b) > 2$  or (2)  $b < a^{**}$ .

**Definition 15.** A The element  $s$  isolated in respect of  $S$  if  $\delta(s, t) > 2$  for every  $t \in S, t \neq s$ .

**Lemma 9.**  $\mathcal{E}$  is an equivalence relation on the source  $S$ .

*Proof.* . □

Let  $A$  be a block of  $\mathcal{E}$ , this is a subset of the source  $S$ . Take the interval  $\overline{A}$  of  $G$  generated by  $A$ . We prove that the restriction of  $S$  to  $A$  defines a nested lattice  $N_A$ .

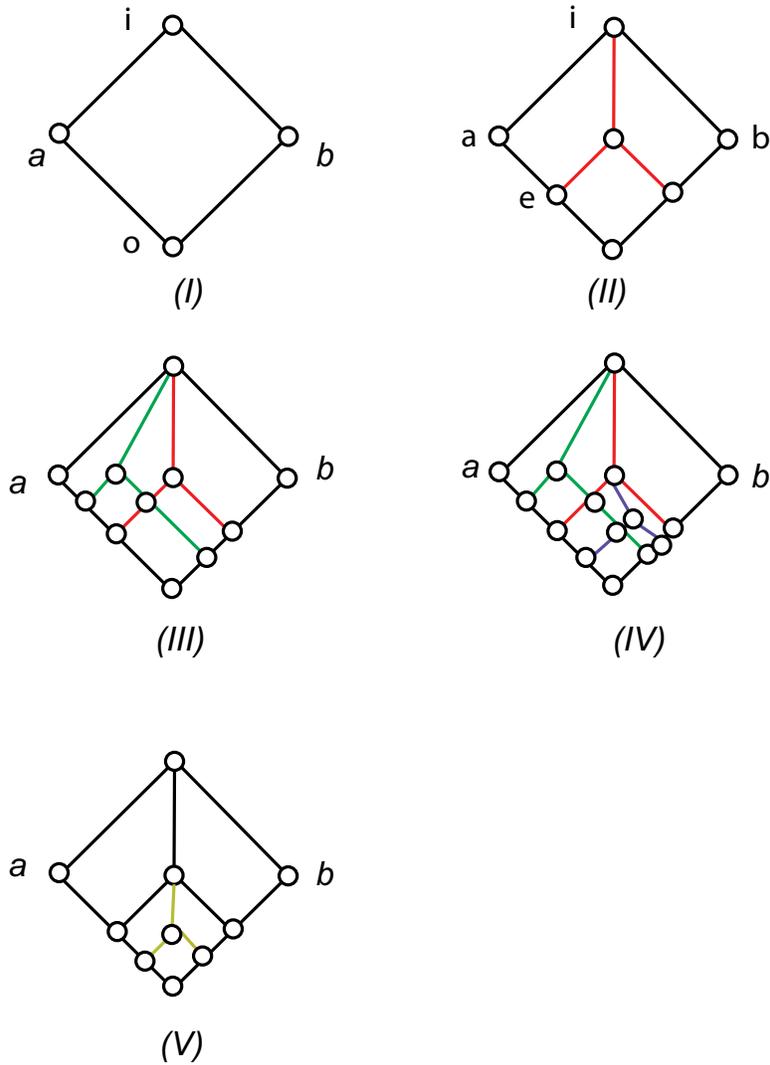


FIGURE 32. The iterative procedure

6.2.2. *Some nested lattices.* Consider the following semimodular lattices:

Let  $n$  and  $m$  be natural numbers and let  $C = \{0, 1, \dots, n-1\}$  be a chain of length  $n-1$ . Then

$$\mathbb{L}_{m,n}^H$$

is the meet-sublattice of  $C^m$  consisting all elements  $(x_1, \dots, x_m)$  where either  $x_1 + x_2 + \dots + x_m \leq n-1$  or  $x_1 = \dots = x_m = n-1$ . The elements  $s_i = (x_1, \dots, x_m)$ ,  $x_1 + x_2 + \dots + x_m = n-1$  form an  $s$ -independent set  $S$  in  $G$ . This is a vertically adjacent set. The lattice  $\mathbb{L}_{m,n}^H$  is the factor lattice  $G/\Theta_S$ . Obviously, this is a rectangular lattice of  $J$ -width  $m$ . See the examples on Figure 45, Figure 47, Figure 49, Figure 54.

$\mathbb{L}_{2,1}^V$  is the lattice  $S_7$ . It is easy to see that  $\mathbb{L}_{2,2}^V$  is determined by an horizontal-adjacent independent set  $S$ , which contains the elements  $(x_1, \dots, x_m)$  where  $x_1 = \dots, = x_m \geq 2$  (see Figure 22.) Take again  $G = C^m$ , where  $C = \{0, 1, \dots, n-1\}$ . We define the vertical crocheted lattice

$$\mathbb{L}_{m,n}^V$$

(see Figure 23). This is the sublattice of  $C^m$  which contains all elements in the form  $(0, \dots, 0, i, 0, \dots, 0)$  and the diagonal elements  $(j, j, \dots, j)$  for  $j > 2$ .

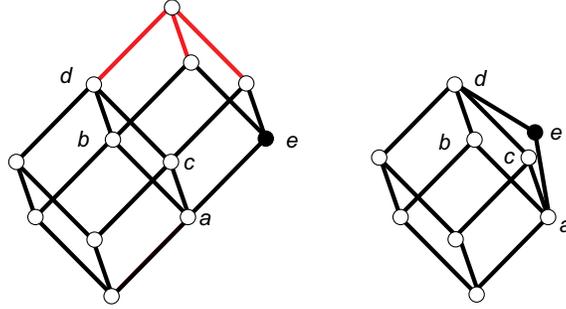


FIGURE 33. The nested lattice  $\mathbb{L}_{2,7}^H$  with the skeleton

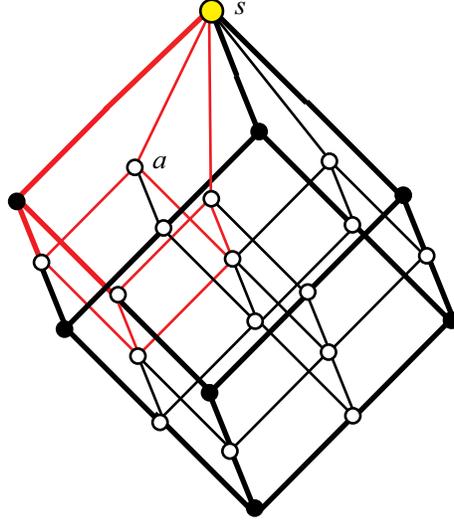
6.2.3. *The  $\text{Jw}(L) = 3$  case.* We define the 3-nested lattices similarly as in the the  $\text{Jw}(L) = 2$  case. We start with the  $C_2^3$  Boolean lattice, this is a 3-nested lattice. We add a 3-fork to this  $2^3$ -cell, we obtain the source lattice  $\mathbb{L}_3$ , see on Figure 27 and apply the "downward-going" procedure if it is necessary see on Figure 24, This is a 3-nested lattice.  $\mathbb{L}_3 = S_{24}$  contains 7  $2^3$ -cells. In some of these we insert 3-forks. All these are 3-nested lattices. If we continue this procedure we get all the 3-nested lattices.

6.3. **The patchwork system.** [26],[27] Matching is a special case of S-glued sum. The S-glued sum was introduced by Christian Herrmann [21].

**Definition 16.** (*S-glued system*). Let  $S$  and  $L_s, s \in S$ , be lattices of finite length. The system  $L_s, s \in S$  is called an S-glued system iff if the following conditions are satisfied:

- (1) For all  $s, t \in S$ , if  $s \leq t$ , then either  $L_s \cap L_t = \emptyset$  or  $L_s \cap L_t$  is a filter in  $L_s$  and an ideal in  $L_t$ .
- (2) For all  $s, t \in S$  with  $s \leq t$  and for all  $a, b \in L_a \cap L_b$ , the relation  $a \leq b$  holds in  $L_s$  iff  $a \leq b$  in  $L_t$ .
- (3) For all  $s, t \in S$ , the covering  $s \prec t$  implies that  $L_s \cap L_t \neq \emptyset$ .
- (4) If  $s, t \in S$ , then  $L_s \cap L_t \subseteq L_{s \wedge t} \cap L_{s \vee t}$ .

**Definition 17.** (*S-glued sum*) Let  $L = \bigcup(L_s | s \in S)$ , where  $L_s, s \in S$  is an S-glued system. Let the partial order  $\leq$  in  $L$  is defined as follows: for  $a, b \in L$ , let  $a \leq b$  iff there exists a sequence  $a = x_0, x_1, \dots, x_n = b$  of elements of  $L$  and a sequence  $s_0, \dots, s_n$  of elements of  $S$  such that  $s_i \leq s_{i+1}$  in  $S, i = 1, \dots, n-1$ , and  $x_{i-1} \leq x_i$  in  $L_{[s_i, i = 1, \dots, n}$ . Then  $L$  is a lattice, the S-glued sum of  $L_s, s \in S$ .

FIGURE 34. The source lattice  $\mathbb{L}_3 = S_{24}$  and skeleton

We call the  $L_s$  components of the *blocks* of the S-glued sum. Any block is an interval in  $L$ . In this paper we will use as blocks special rectangular lattices.

Ch. Herrmann proved that every modular lattice  $L$  of finite length is the S-glued sum of its maximal complemented intervals (these are, obviously rectangular lattices).

In section 1.2 there are two examples. In the kitchen the mosaics (which are two dimensional) are glued together by an edge (one dimensional). The bricks (three dimensional) are glued together by a side (two dimensional).

**Definition 18.** (*Matching*) Let  $S$  and  $L_s, s \in S$ , be semimodular lattices of finite length. The S-glued sum  $L = \bigcup(L_s | s \in S)$ , where  $L_s, s \in S$  is called of  $L_s$ - $s$  if

$$\text{Jw}(L_s \cap L_t) < \min(\text{Jw}(L_s), \text{Jw}(L_t)).$$

Every planar distributive lattice is the matching of its covering squares and some of the edges (narrows), see Figure 2. For cubes, see Figure 4 and Figure 5.

**6.4. The planar semimodular lattices.** First, we consider the slim planar semimodular lattices (The slim planar semimodular lattice was defined in [15], these are planar lattices where the covering squares are intervals).

**Theorem 6.** *Every slim planar semimodular lattice is the matching of filters of nested lattices.*

*Proof.* Induction on the length. Take the equivalence relation  $\mathcal{E}$ . □

**6.5. The patchwork in higher dimensions.**

**Theorem 7.** *Let  $L$  be a semimodular lattice of  $J$ -width  $n > 2$ . Then  $L$  is the matching of filters of nested lattices.*

*Proof.* Similar to the proof of Theorem 6. □

### Part 3. Structure theorems

#### 7. THE MAIN THEOREM

We prove the following structure theorem:

**Theorem 8.** *Every finite semimodular lattice  $L$  is the patchwork of patch lattices.*

**7.1. Block matrices.** We prove a special decomposition theorem for hypermatrices.

#### 8. RECTANGULAR LATTICES AS PACKING BOXES AND BUILDING STONES.

##### 8.1. The rectangular hulls of semimodular lattices.

**Theorem 9.** *Every semimodular lattice  $L$  has a rectangular extension  $R$  such that*

- (1)  *$L$  and  $R$  have the same length, i.e.  $L$  is a cover-preserving  $(0,1)$ -sublattice of  $R$ ,*
- (2)  *$L$  and  $R$  have the same dimension,  $\mathbf{dim}(L) = \mathbf{dim}(R)$  (i.e.  $J(L)$  and  $J(R)$  have the same width).*

$R$  is called a *rectangular hull* of  $L$  (or packing box).

*Proof.* Let  $L$  be a semimodular lattice of length  $n$  and  $Jw(L) = k$ . By Theorem 1,  $L$  is the cover-preserving join-homomorphic image of the distributive lattice  $F$ . Let  $\Phi$  be the corresponding cover-preserving join-congruence,  $S$  denotes the source of  $\Phi$ . On the other hand, by Lemma 1,  $F$  is the cover-preserving sublattice of the grid  $G$ . We extend  $\Phi$  to  $G$ . Let  $a/b$  and  $c/d$  prime quotients of a lattice  $L$ . If  $b \vee c = a$ ,  $b \wedge c = d$  we say that  $a/b$  is perspective up to  $c/d$  and we write  $a/b \searrow c/d$ . This cover-preserving join-congruence will be denoted by  $\bar{\Phi}$  and is defined as follows: for a covering pair  $d \prec c$ ,  $c, d \in G$ ,  $c \equiv d \pmod{\bar{\Phi}}$  if and only if there is a source element  $s \in S \subset F$  a lower cover  $u \in F$  of  $s$  such that  $c/d$  is perspective up to  $s/u$ . It is easy to prove that the transitive extension is a cover-preserving join-congruence.

Then define  $R$  as:

$$R = G/\bar{\Phi}.$$

By the second isomorphism theorem we have the following isomorphisms and join-homomorphisms, presented on the diagram:

$$L \cong F/\Phi \cong (G/\Psi)/((\Psi \vee \bar{\Phi})/\Psi) \cong G/(\Psi \vee \bar{\Phi}) \cong (G/\bar{\Phi})/((\Psi \vee \bar{\Phi})/\bar{\Phi}) \cong R/\bar{\Psi},$$

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & L \\ \uparrow \psi & & \uparrow \bar{\psi} \\ G & \xrightarrow{\bar{\varphi}} & R \end{array}$$

Then  $c = d \vee s$ , which means that  $\bar{c}$  is reducible in  $G$ . Assume that  $d$  is a join-irreducible element of  $G$ . Every element of the ideal  $\downarrow d$  is join-irreducible, i.e.  $u \in F$  must be join-irreducible too. This means the join-irreducible element of  $F$  is in a non-trivial  $\Phi$ -class, in contradiction to Lemma 2. This proves that every join-irreducible element of  $G$  is a one-element  $\bar{\Phi}$ -class, i.e.  $R$  is a rectangular lattice.

$\Phi$  and  $\bar{\Phi}$  by Lemma 3,  $F$  and  $G$  have the same length, i.e.  $L$  is a cover-preserving sublattice of  $R$ .

By the definition of  $G$  it is trivial that  $J(L)$  and  $J(R)$  have the same width.  $\square$

Matching is a special case of S-glued sum. The S-glued sum was introduced by Christian Herrmann [21].

**Definition 19.** (*S-glued system*). Let  $S$  and  $L_s, s \in S$ , be lattices of finite length. The system  $L_s, s \in S$  is called an S-glued system iff if the following conditions are satisfied:

- (1) For all  $s, t \in S$ , if  $s \leq t$ , then either  $L_s \cap L_t = \emptyset$  or  $L_s \cap L_t$  is a filter in  $L_s$  and an ideal in  $L_t$ .
- (2) For all  $s, t \in S$  with  $s \leq t$  and for all  $a, b \in L_s \cap L_t$ , the relation  $a \leq b$  holds in  $L_s$  iff  $a \leq b$  in  $L_t$ .
- (3) For all  $s, t \in S$ , the covering  $s \prec t$  implies that  $L_s \cap L_t \neq \emptyset$ .
- (4) If  $s, t \in S$ , then  $L_s \cap L_t \subseteq L_{s \wedge t} \cap L_{s \vee t}$ .

**Definition 20.** (*S-glued sum*) Let  $L = \bigcup(L_s | s \in S)$ , where  $L_s, s \in S$  is an S-glued system. Let the partial order  $\leq$  in  $L$  is defined as follows: for  $a, b \in L$ , let  $a \leq b$  iff there exists a sequence  $a = x_0, x_1, \dots, x_n = b$  of elements of  $L$  and a sequence  $s_0, \dots, s_n$  of elements of  $S$  such that  $s_i \leq s_{i+1}$  in  $S, i = 1, \dots, n-1$ , and  $x_{i-1} \leq x_i$  in  $L_{s_i}, i = 1, \dots, n$ . Then  $L$  is a lattice, the S-glued sum of  $L_s, s \in S$ .

We call the  $L_s$  components of the blocks of the S-glued sum. Any block is an interval in  $L$ . In this paper we will use as blocks special rectangular lattices.

Ch. Herrmann proved that every modular lattice  $L$  of finite length is the S-glued sum of its maximal complemented intervals (these are, obviously rectangular lattices).

In section 1.2 there are two examples. In the kitchen the mosaics (which are two dimensional) are glued together by an edge (one dimensional). The bricks (three dimensional) are glued together by a side (two dimensional).

**Definition 21.** (*Matching*) Let  $S$  and  $L_s, s \in S$ , be semimodular lattices of finite length. The S-glued sum  $L = \bigcup(L_s | s \in S)$ , where  $L_s, s \in S$  is called of  $L_s$ - $s$  if

$$\mathbf{dim}(L_s \cap L_t) < \min(\mathbf{dim}(L_s), \mathbf{dim}(L_t)).$$

Every planar distributive lattice is the matching of its covering squares and some of the edges (narrows), see Figure 2. For cubes, see Figure 4 and Figure 5.

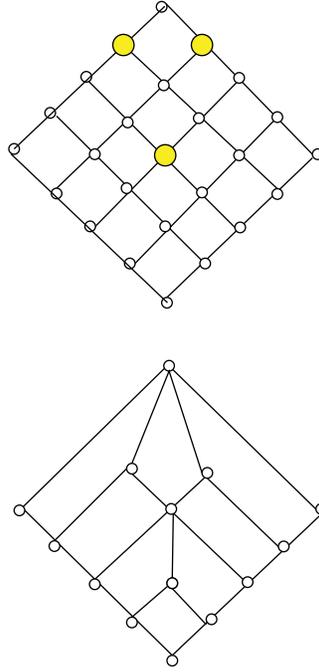
**Theorem 10.** *Every slim planar semimodular lattice is the matching of filters of nested lattices.*

*Proof.* Induction on the length. Take the equivalence relation  $\mathcal{E}$ .  $\square$

**Theorem 11.** *Let  $L$  be a semimodular lattice of dimension  $n > 2$ . Then  $L$  is the matching of filters of nested lattices.*

*Proof.* Similar to the proof of Theorem 6.  $\square$

## 8.2. Special embeddings.

FIGURE 35.  $\mathbb{B}[(4, 3), (3, 4), (2, 2)]$ 

8.2.1. *Congruence-preserving embedding.* Let  $B$  be a sublattice of a lattice  $A$ . If  $0_A, 1_A \in B$ , then  $B$  is said to be a  $(0, 1)$ -sublattice. If every congruence of  $A$  is determined by its restriction to  $B$ , then  $B$  is called a *congruence-determining* sublattice of  $A$ . Ideals that are chains will be called *chain ideals*. Consider the class  $\mathcal{K} = \{L : L \text{ is a finite length semimodular lattice that has a congruence-determining chain ideal}\}$ .

**Theorem 12.** *Let  $L \in \mathcal{K}$ , and let  $D$  be a  $(0, 1)$ -sublattice of  $\text{Con } L$ . Then there exists an  $\bar{L} \in \mathcal{K}$  such that the restriction mapping  $\rho : \text{Con } \bar{L} \rightarrow \text{Con } L, \theta \mapsto \theta|_L$ , is actually a  $(0, 1)$ -lattice isomorphism  $\text{Con } \bar{L} \rightarrow D$ ; in particular,  $\text{Con } \bar{L} \cong D$ .*

**Theorem 13.** *Every planar semimodular lattice  $L$  has a congruence-preserving extension  $K$  such that*

- (1)  $K$  is a planar semimodular lattice,
- (2)  $L$  is an almost-filter of  $K$  (it contains 1 and is a cover-preserving sublattice)
- (3)  $K$  contains as ideal a chain  $C$ ,
- (4)  $C$  is congruence-determining.

8.2.2. *Embedding into geometric lattice.* It was proved by G Grätzer and E. W. Kiss [13], (for semimodular lattices of finite length see G. Czédli and E. T. Schmidt [5]):

**Theorem 14.** *Every finite semimodular lattice  $L$  has a cover-preserving embedding into a geometric lattice  $G$ .*

## 9. MODULARARITY

**9.1. Skeleton.** G. Grätzer and R. W. Quackenbush proved in [20] that a planar distributive lattice  $D$  with more than two elements is isomorphic to the  $\text{Skeleton}(L)$ , for a subdirectly irreducible planar modular lattice  $L$  iff  $L$  has no narrows. The results Theorem 4.9, Theorem 5.2 can be generalized for arbitrary finite order-dimension.

We prove that this statement is true for an arbitrary finite distributive lattice  $D$  with  $\text{Jw}(D) < \aleph_0$  (defined in 1.1).

**Lemma 10.** *A modular lattice  $L$  which contains a prime interval  $\mathfrak{p}$  is subdirectly irreducible if and only if  $L$  is weakly atomic and any two prime intervals are projective.*

*Proof.* It is trivial that a modular lattice  $L$  which contains a prime interval  $\mathfrak{p}$  and satisfies the given conditions is subdirectly irreducible.

Let  $L$  be a subdirectly irreducible modular lattice  $L$  which contains a prime interval  $\mathfrak{p}$ . We may assume that for any  $a < b$

$$\text{con}(\mathfrak{p}) \leq \text{con}(a, b)$$

. By the weakly modularity ([19], p.194) the interval  $[a, b]$  contains a prime interval  $\mathfrak{q}$  such that  $\mathfrak{p}$  and  $\mathfrak{q}$  are projective, i.e.  $L$  is weakly atomic.  $\square$

**Theorem 15.** *Let  $D$  be a distributive lattice with more than two elements,  $\text{dim}_3(D) < \aleph_0$ . Then  $D$  is isomorphic to  $\text{Skeleton}(L)$ , for a subdirectly irreducible modular lattice  $L$ , iff  $D$  is narrows free.*

**Theorem 16.** *There exists a subdirect irreducible modular lattice of order-dimension  $\aleph_0$  which does not contain a prime interval.*

*Proof.*  $\mathbb{Q}$  is the  $[0, 1]$  chain of rational numbers. Glue together copies of  $M_3[\mathbb{Q}]$ -s, see E. T. Schmidt [23].  $\square$

## 9.2. Diamond-free semimodular lattices.

## 9.3. Modular source.

**Definition 22.** *A source  $S$  of a distributive lattice  $D$  is called modular if the lattice  $D/\Theta_S$  is modular.*

**Lemma 11.** *Let  $s$  be a source element of a grid  $G$ . Then  $G/\Theta_s$  is modular lattice iff  $s$  is bastard.*

On the following two figures (Figure 30 and Figure 31) you can see the representation of  $M_3$  with the grid resp. lower grid.

Let  $D$  be a planar distributive lattice. The J-width is two. Add doubly-irreducible elements to the interiors of some 4-cells (covering squares) you get a planar modular lattice  $M$ . If we start with a distributive lattice  $D$  of J-width 3 then we can extend some of the covering-cubes into modular non-distributive lattice.  $D$  is the skeleton.

Let  $\Theta$  be a cover-preserving join-congruence of a grid  $G$ . Give a necessary and sufficient condition to be  $G/\Theta$  modular.

Take the finite field  $\mathbf{GF}(p^n)$ ,  $p = 2, n = 1$ . The corresponding two-dimensional projective geometry  $F = \mathbf{GP}_2$  is the *Fano plane*. The one-dimensional lattice  $\mathbf{GP}_1$  is  $M_3$ . It is clear that the J-width of  $F$  is 7.

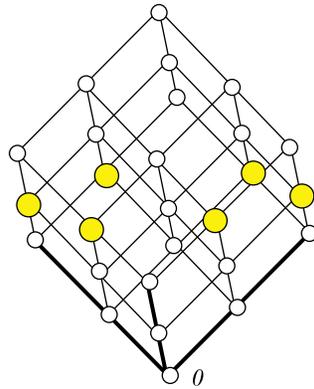
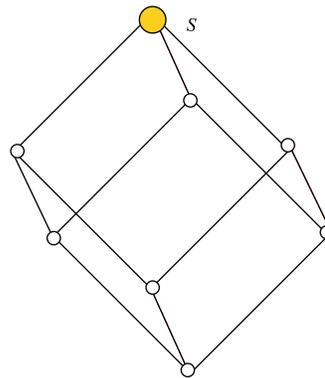
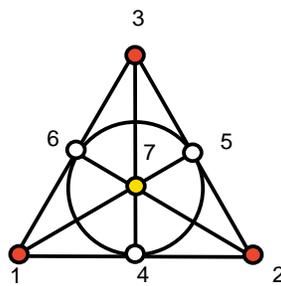
FIGURE 36. The grid of  $M_3$  and the sourceFIGURE 37. The lower grid of  $M_3$  and the source element

FIGURE 38. The Fano plane given by points and lines

On the next picture you can see the "traditional" presentation of the subspace lattice.

Draw the diagram a little bit others we get the following diagram for the same lattice (the same, but differently).

Here we see a cube (fat lines, the skeleton) and six circles on the faces of this cube and two circles (yellow) are inside the cube. If  $D$  is the direct product of three

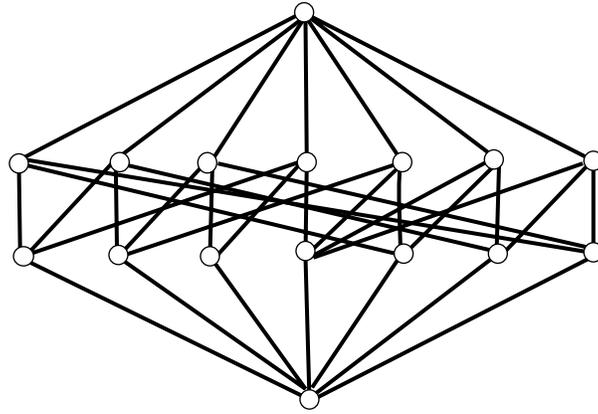


FIGURE 39. The "traditional" diagram of the subspace lattice of the Fano plane

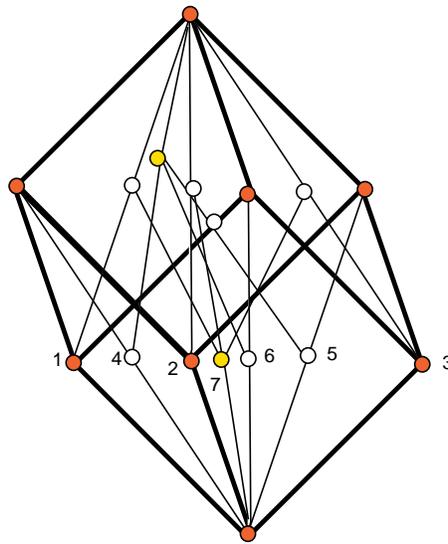


FIGURE 40. The subspace lattice of the Fano plane and the Kuroš-Ore skeleton

chains then this contains unit (covering) cubes. We can extend  $D$  if we put the Fano plane into some covering cubes this is the skeleton. (Fano plane "locked" in a cube.) By plain lattices we extend a covering square to an  $M_3$ .

#### 10. THE GRÄTZER-KISS THEOREM

It was proved by G Grätzer and E. W. Kiss [13]:

**Theorem.** *Every finite semimodular lattice  $L$  has a cover-preserving embedding into a geometric lattice  $G$ .*

11. REDUCTION OF  $J(L)$ 

We delete some edges of a finite poset  $P$ , we get the poset  $Q$ . The poset  $P$  is called a *refinement* of  $Q$  or we say  $Q$  is a *reduction* (or pruning) of  $P$ . Let  $L$  be a finite semimodular lattice and take  $P = J(L)$ .

Does there exist a semimodular lattice  $K$  which satisfies the following properties:

- (1)  $J(K) \cong Q$ ,
- (2)  $L$  is a cover-preserving sublattice of  $K$ ?

The answer is no. (See Figure 31.)

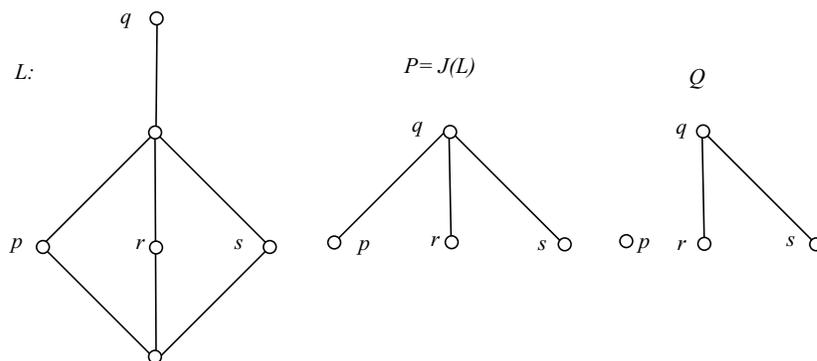


FIGURE 41. x

Here is a counter example (given by G. Czédli). Let  $L$  be a finite semimodular lattice and let  $Q$  be a reduction of  $P = J(L)$ . For which  $Q$  has a cover-preserving embedding into a finite semimodular lattice  $K$  with the property  $J(K) \cong Q$ .

See  $L, P$  and  $Q$  given in the Figure 7. Now  $L$  is semimodular.  $P = J(L)$ . We obtain  $Q$  by deleting an edge from the diagram of  $P$ , so  $Q$  is a refinement of  $P$ . But there is no semimodular lattice  $K$  such that  $J(K) = Q$  and  $L$  is a cover-preserving sublattice of  $K$ . To show this, suppose the contrary. We may suppose that  $\ell(K) = \ell(L)$  for otherwise  $K$  can be replaced with an interval of  $K$ . Clearly,  $r$  and  $s$  (as minimal join-irreducible elements) are atoms in  $K$ . By semimodularity the height  $h(r \vee s) = 2$  in  $K$ . Since  $J(K) = Q$ , we have  $r \vee s \leq q$  in  $K$ . We cannot have equality here, for  $q \in J(K) = Q$ . Hence,  $\ell(K) = 3 \leq h(q)$ , implying  $q = 1_K \geq p$  in  $K$ . This contradicts  $J(K) = Q$ .

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