RECTANGULAR HULLS OF SEMIMODULAR LATICES (NOT FINISHED MANUSCRIPT)

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Abstract. We prove that every finite semimodular lattice L can be extended to a rectangular lattice $\widehat{L}$ (in which the poset $\widehat{L}$ of of join-irreducible elements is the disjoint sum of chains), such that $\mathrm{J}(L)$ and $\mathrm{J}(\widehat{L})$ have the same width, L and $\widehat{L}$ have the same length.

1. Introduction

In a series of papers Grätzer and E. Knapp [7], [8], [9], [10] investigated structure theorems for finite **planar** semimodular lattices. These papers led to a new research direction where the finite semilattices are treated as geometric shapes, see [3], [4], [5], [11]. The key idea of this approach is presented in G. Czédli, E. T. Schmidt [1] where we proved that *every* finite semimodular lattice can be derived from a distributive lattice in an easy way.

In the present paper we introduce the concept of rectangular lattices in the class $\mathcal{S}$ of finite semimodular lattices. First, we consider the class $\mathcal{D}$ of finite distributive lattices where the rectangular lattices are the direct product of chains. In $\mathcal{D}$ we can use the rectangular lattices in two different roles, see Figure 1:

(I) The lattice $D \in \mathcal{D}$ is in a **smallest box**, smallest rectangular lattice, which contains D. This will be called a *rectangular hull* of D. (This is not uniquely determined.)

(II) The **building stones** of D are n-dimensional cubes (in Figure 1 we have two cubes and one square) which are glued together. These cubes are rectangular distributive lattices. The intersection of two cubes (if this is not empty) are faces of the cubes. This special gluing is the patchwork system (for planar semimodular lattices see [4]; the patchwork lattices are special rectangular lattices).

In the present paper our goal is to extend the concept of rectangular hull for arbitrary semimodular lattice.

Rectangular lattices were introduced by G. Grätzer and E. Knapp [8] for planar semimodular lattices: a *left corner* (resp. *right corner*) of a planar lattice K is a double-irreducible element in $K - \{0, 1\}$ on the left (resp., right) boundary of K. A *rectangular* lattice L is a planar semimodular lattice which has exactly one left corner, u_l and exactly one right corner, u_r and they are complementary – that is, $u_l \vee u_r = 1$ and $u_l \wedge u_r = 0$. The direct product of two chains is rectangular.

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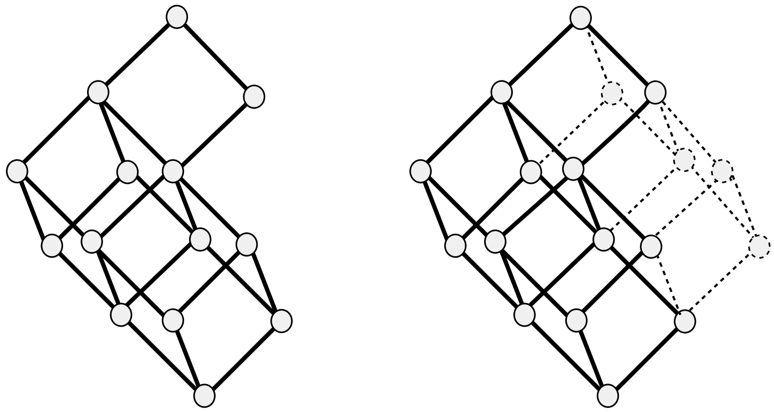

FIGURE 1. A distributive lattice D on the left side and a rectangular hull on the rigth side

Let X, Y be posets. The *disjoint sum* $X+Y$ of X and Y is the set of all elements in X and Y considered as disjoint. The relation $\leq$ keeps its meaning in X and in Y, while neither $x \geq y$ nor $x \leq y$ for all $x \in X, y \in Y$.

$\mathrm{J}(L)$ denotes the ordered set of all nonzero join-irreducible elements of L and $\mathrm{J}_0(L)$ is $\mathrm{J}(L) \cup \{0\}$. A planar semimodular lattice L is called *slim* if no three join-irreducible elements of L form an antichain.

$\mathrm{d}(L)$ denotes the *(order) dimension* of L: it is the smallest cardinal κ such that the order $\leq$ on L is the intersection of κ total orders. The *width* $\mathrm{w}(P)$ of a (finite) order P is defined to be $\max\{n\colon P$ has an n-element antichain$\}$. If P is finite then $\mathrm{d}(P) \leq \mathrm{w}(P)$. It is easy to prove that the dimension of a distributive lattice D is $\mathrm{w}(\mathrm{J}(D))$. On the other hand $2 = \mathrm{d}(M_3) < \mathrm{w}(\mathrm{J}(M_3)) = 3$. If L is a lattice the number $\mathrm{w}(\mathrm{J}(L))$ is called the J-*width* of L. This will be denoted by $\mathrm{Jw}(L)$, it is a kind of dimension.

Definition 1. *A* rectangular *lattice L is a finite semimodular lattice in which* $\mathrm{J}(L)$ *is the disjoint sum of chains.*

Geometric lattices are rectangular. Let us remark that a slim, planar semimodular lattice L is rectangular in sense of Definition 1 iff it is rectangular in sense of [8].

As mentioned earlier a distributive lattice D is rectangular iff D is the direct product of chains. An interesting example of modular non-distributive rectangular lattice is $M_3[C_n]$, see in Figure 2. This is the lattice of balanced triples $(x, y, z) \in {C_n}^3$, which means: $x \wedge y = x \wedge z = y \wedge z$ (C_n denotes the n-element chain). In Figure 2 we have on the left side a modular lattice L and on the right side we see the rectangular hull.

An other example for modular but not distributive lattice rectangular lattice is in Figure 3.

By semimodular lattices in most cases the rectangular lattices looks like a rectangular shape, $M_3[C_3]$ the 3-dimensional rectangular lattice is an exception. In Figure 3 we present an other modular rectangular lattice L. The ideal $(p]$ is M_3 and the filter $[p)$ is a Boolean lattice. It's my conjecture that a modular rectangular lattices R contains an element p such that the ideal $(p]$ is a projective geometry

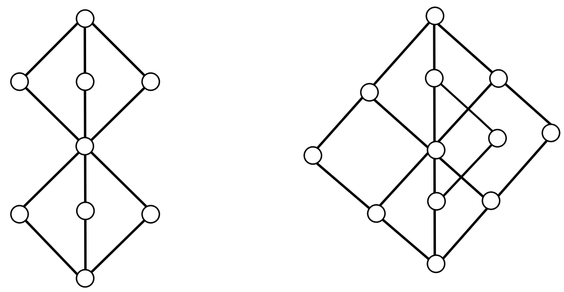

FIGURE 2. On the left side L and on the right side $\widehat{L} = M_3[C_3]$

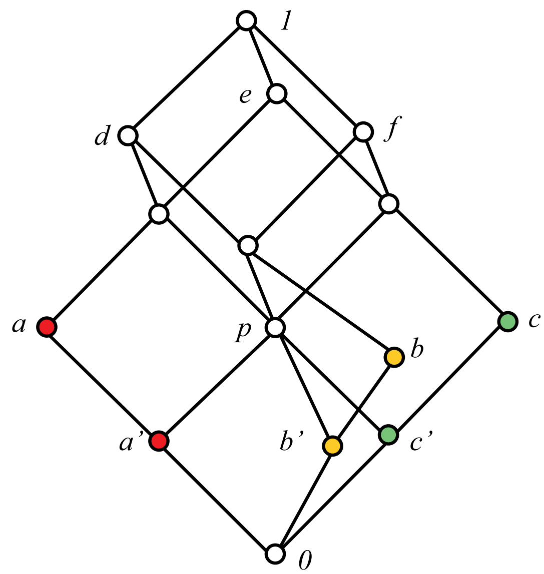

FIGURE 3. Modular rectangular lattice

and the filter $[p)$ is the diret product of chains. We will see that in the geeral case a contains a maximal $(0,1)$-sublattice, which is a boolean lattice $\mathbb{S}_k$. This is the *skeleton* of $\mathbb{S}_3$ and will be denoted by Skeleton$(\mathbb{S}_3)$. In Figure 3 the elements $0, a, b, c, d, e, f, 1$ form a skeleton.

Let L and K be finite lattices. A join-homomorphism $\varphi : L \to K$ is said to be *cover-preserving* iff it preserves the relation $\preceq$. Similarly, a join-congruence Φ of L is called cover-preserving if the natural join-homomorphism $L \to L/\Phi$, $x \mapsto [x]\Phi$ is cover-preserving.

Theorem. *Every finite semimodular lattice L has a rectangular extension $\widehat{L}$ such that*

(1) $\mathrm{J}(L)$ *and* $\mathrm{J}(\widehat{L}\ )$ *have the same width, i.e.* $\mathrm{Jw}(L) = \mathrm{Jw}(\widehat{L})$,
(2) L *and* $\widehat{L}$ *have the same length.*

$\widehat{L}$ is called a *rectangular hull* of L. As we have mentioned, this result was proved for planar semimodular lattices in G. Grätzer, E. Knapp, [8]. Figure 4 shows that this is not uniquely determined (not even in the planar distributive case).

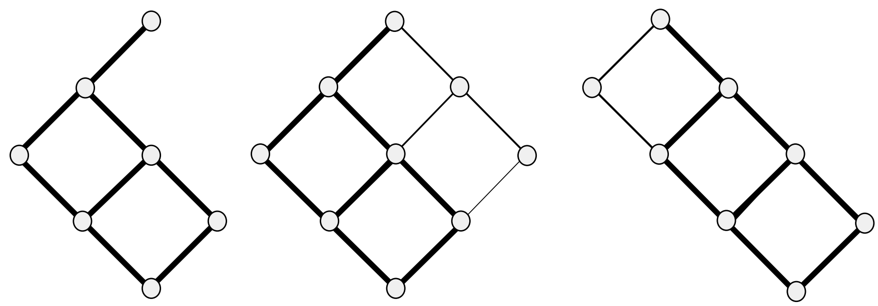

FIGURE 4. A distributive lattice D on the left side and two different rectangular hulls of D

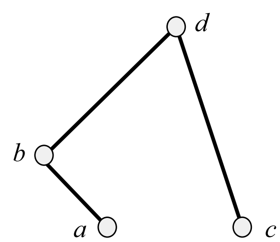

FIGURE 5. $\mathrm{J}(D)$

2. COVER-PRESERVING JOIN-CONGRUENCE

It was proved by G. Czédli and E. T. Schmidt [1]:

Lemma 1. *Every finite semimodular lattice L is a cover-preserving join-homomorphic image of the unique distributive lattice F determined by $\mathrm{J}(F) \cong \mathrm{J}(L)$. Moreover, the restriction of an appropriate cover-preserving join-homomorphism from F onto L is a $\mathrm{J}(F) \to \mathrm{J}(L)$ order isomorphism.*

$F = \mathrm{H}(\mathrm{J}(L))$, the poset of all order ideals of $\mathrm{J}(L)$. In virtue of Dilworth [6], a finite order P is the union of $\mathrm{w}(P)$ appropriate chains. An other result from G. Czédli and E. T. Schmidt [1]:

Lemma 2. *Each finite semimodular lattice L is a cover-preserving join-homomorphic image of a distributive lattice G which is the direct product of $\mathrm{Jw}(L)$ finite chains.*

The distributive lattice in Lemma 1 is the lattice $F = \mathrm{Ord}(\mathrm{J}(L))$, the lattice of all order ideals of $\mathrm{J}(L)$ (an order ideal is a subset of $\mathrm{J}(L)$ that is closed downward) and the join-homomorphism from $F = \mathrm{Ord}(\mathrm{J}(L))$ onto L is $\varphi : \mathrm{Ord}(\mathrm{J}(L)) \to L$, $I \mapsto \bigvee_L I$.

Let $D_1, \ldots\ldots, D_k$ be pairwise disjoint subchains of the order $\mathrm{J}(L)$ such that $k = \mathrm{w}(\mathrm{J}(L))$ and $D_1 \cup D_2 \cup \ldots \cup D_k = \mathrm{J}(L)$. (On Figure 5 you can see $\mathrm{J}(D)$ of the lattice D given on Figure 4.) Then to define the chains D_1 and D_2 we have two possibilities: $\{a, b, d\}, \{c\}$ or $\{a, b\}, \{c, d\}$. Adjoin a new zero 0_i to D_i we get the chains $C_i = D_i \cup \{0_i\}$ $(i = 1, \ldots, k)$ and take their direct product

$$G = C_1 \times \ldots \times C_k. \tag{1}$$

This is called a *grid* of L.

By the definition of G, $|\,\mathrm{J}(G)| = |\,\mathrm{J}(F)|$ which gives:

Lemma 3. *G is a rectangular hull of $F = \mathrm{Ord}(\mathrm{J}(L))$, moreover $\mathrm{Jw}(G) = \mathrm{Jw}(F)$ and G, F have the same length.*

A sublattice $\{a_1 \wedge a_2, a_1, a_2, a_1 \vee a_2\}$ of a lattice is called a *covering square* if $a_1 \wedge a_2 \prec a_i \prec a_1 \vee a_2$ for $i = 1, 2$. In [1] it was proved the following:

Lemma 4. *Let Φ be a join-congruence of a finite semimodular lattice M. Then Φ is cover-preserving if and only if for any covering square $S = \{a \wedge b, a, b, a \vee b\}$ if $a \wedge b \not\equiv a$ (Φ) and $a \wedge b \not\equiv b$ (Φ) then $a \equiv a \vee b$ (Φ) implies $b \equiv a \vee b$ (Φ).*

3. The source

In the following, D denotes a finite distributive lattice and Θ is a cover-preserving join-congruence of D. We prove that Θ is determined by a subset S_Θ of D, which will be called the source of Θ. Let a/b and c/d be prime quotients of a lattice L. If $b \vee c = a$, $b \wedge c = d$ we say that a/b is *perspective down* to c/d and we write $a/b \searrow c/d$. Similarly, $a \vee d = c, a \wedge d = b$ means that a/b is *perspective up* to c/d and we write $a/b \nearrow c/d$. In either case we shall write $a/b \sim c/d$, and say that a/b is perspective to c/d.

Definition 2. *An element s of a distributive lattice D is called a* source element *of Θ if there is a $t, t \prec s$ such that $s \equiv t$ (Θ) and for every prime quotient u/v if $s/t \searrow u/v, s \neq u$ implies $u \not\equiv v$ (Θ). The set S_Θ of all source elements of Θ is the* source *of Θ.*

It is easy to prove (see in [11]) that s_1 and s_2 are two different elements of S_Θ and $x \prec s_1, y \prec s_2$ yields that $s_1/x, s_2/y$ are not perspective, $s_1/x \not\sim s_2/y$, (we say that s_1 and s_2 are *independent*).

Lemma 5. *Let x be an arbitrary lower cover of a source element s of Θ. Then $x \equiv s$ (Θ). If $s/x \searrow v/z$, $s \neq v$, then $v \not\equiv z$ (Θ).*

Proof. Let s be a source element of Θ then $s \equiv t$ (Θ) for some t, $t \prec s$. If $x \prec s$ and $x \neq t$ then $\{x \wedge t, x, t, s\}$ form a covering square. Then $x \not\equiv x \wedge t$ (Θ), which implies $x \wedge t \not\equiv t$ (Θ). By Lemma 4 we have $x \equiv s$ (Θ).

To prove that $v \not\equiv z$ (Θ), we may assume that $v \prec s$. Take the element t, $t \prec s$, then we have three (pairwise different) lower covers of s, namely x, v, t. These generate an eight-element boolean lattice in which $s \equiv t$ (Θ), $s \equiv x$ (Θ) and $s \equiv v$ (Θ). By the choice of t we know that $v \not\equiv v \wedge t$ (Θ), $x \not\equiv x \wedge t$ (Θ) and $z \not\equiv x \wedge t \wedge v$ (Θ).

If $v \equiv z$ (Θ) then take the covering square $\{z \wedge v \wedge t, z, v \wedge t, v\}$ and by Lemma 4 $x \wedge v \wedge t \not\equiv v \wedge t$ (Θ). This implies $x \wedge t \equiv t$ (Θ) and by the transitivity we get $x \wedge t \equiv t$ (Θ), contradiction. □

Let s, t, $s \succ t$ be elements of D. We would like to define $\mathrm{con}^{\vee\mathrm{cp}}(s,t)$ the *principal* cover-preserving join-congruence, where $s \equiv t$. It is easy to see that the intersection of two cover-preserving join-congruences need not to be cover-preserving. $\mathrm{con}^{\vee}(s,t)$ denotes the smallest join-congruence of D such that $s \equiv t$. It is clear that for a covering pair

(2) $\quad y \prec x, x \equiv y \quad (\mathrm{con}^{\vee}(s,t))$ if and only if x/y is perspective up to s/t.

It is easy to prove: $\mathrm{con}^{\vee\mathrm{cp}}(s,t) = \bigvee_{t_i \prec s}(\mathrm{con}^{\vee}(s,t_i)$. By Lemma 4 this is a cover-preserving join-congruence and if $t_1, t_2 \prec s$ then $\mathrm{con}^{\vee\mathrm{cp}}(s,t_1) = \mathrm{con}^{\vee\mathrm{cp}}(s,t_2)$. This allows us to write $\mathrm{con}^{\vee\mathrm{cp}}(s)$ instead of $\mathrm{con}^{\vee\mathrm{cp}}(s,t)$. It is clear that for a covering pair $d \prec c$, $c \equiv d$ $(\mathrm{con}^{\vee\mathrm{cp}}(s))$ if and only if there is a lower cover t of s such that c/d is perspective up to s/t. Let us remark that $\mathrm{con}^{\vee\mathrm{cp}}(s)$ is a lattice congruence iff s is a join-irreducible element.

By Lemma 2, for a grid element, $a \in G$ we define $\overline{a} \in F$ as the smallest element of F for which $a \leq \overline{a}$. Then the mapping $\psi : a \longrightarrow \overline{a}$, which is a closure operator, is a join-homomorphism from G onto F, the image of $\mathrm{J}(G)$ is $\mathrm{J}(F)$ (the task of ψ is to change the ordering of $\mathrm{J}(G)$). The join-congruence induced by this mapping, $\Psi = \mathrm{Ker}\,\psi$ has the following form:

$$\Psi = \bigvee \mathrm{con}^{\vee}(p_i, \overline{p}_i), \tag{3}$$

p_i is join-irreducible element of G. The join-irreducible-elements of G have the form $(0, ...0, c, 0, ..., 0)$, where $c \in C_j, c \neq 0$ for some j. If $C_1 = \{0, c_1, .., c_i, ..., c_n\}$, $0 < c_1 < c_2 < ... < c_n$, then let $p_i = (c_i, 0, 0, ...,)$ for some $i, (0 \leq i \leq n)$.

As you can see on Figure 6 (for the planar case Figure 5) a join-congruence of the form $p_5 \equiv \overline{p}_5$ is determined by a distributive rectangular interval of G: $S(p_5, \overline{p}_5) = [p_4, \overline{p}_5]$. On the right side of Figure 6 is the join congruence induced by $q_3 \equiv \overline{q}_3$ and the corresponding distributive rectangular interval of G: $S(q_3, \overline{q}_3) = [q_2, \overline{q}_3]$, (more on this gluing see in [5]). The Figure 6 presents an example of dimension (J-width) 3. Let L be a nontrivial lattice. If there are a proper ideal I and a proper filter F such that $C = I \cap F$ is a filter of I, C is an ideal of F and $L = I \cup F$, then let us say that L is decomposable with respect to *Hall-Dilworth gluing* over C. We obtain G from F using the Hall-Dilworth gluing. There are several ways to construct the lattice presented on Figure 5, for instance $L_1 = [0, u] \cup S(p_5, q_1)$, $L_2 = D_1 \cup [q_1, v]$, $G = L_3 = L_2 \cup [q_2, 1]$, where $[q_2, 1] = S(q_3, p_3) \cup [w, 1]$. We define: $S(p_i, \overline{p}_i) = [p_{i-1}, \overline{p}_i \vee p_n]$, ($p_n = (c_n, 0, ..., 0)$ and c_n is the greatest element of C_1).

$$G = F \cup \bigcup (S(p_i, \overline{p}_i)) = F \cup \bigcup [p_{i-1}, \overline{p}_i \vee p_n]. \tag{4}$$

The analogous lattice in the three-dimension is on Figure 7. The black shaded circles denote the non-zero join-irreducible elements on Figure 6. resp. Figure 7. F is the lattice of all order-ideals of the poset of the black shaded elements.

Lemma 6. $\mathrm{J}(D) \cong \mathrm{J}(D/\Theta)$ *if and only if every* Θ*-class containing a join-irreducible element* $p \in D$ *is a one-element* Θ*-class.*

Proof. Assume, that for every $p \in \mathrm{J}(D)$ the Θ-class p/Θ is a singleton. p/Θ is a join-irreducible element of D/Θ. Then the set $\{p/\Theta; p \in \mathrm{J}(D)\} \subseteq \mathrm{J}(D/\Theta)$ is order-isomorphic to $\mathrm{J}(D)$. If g/Θ is a join-irreducible element of D/Θ then we may assume that g is a minimal element of this Θ-class. g is a join-irreducible element of D, i.e. g/Θ is a singleton and we have $\mathrm{J}(D) \cong \mathrm{J}(D/\Theta)$.

Conversely, let $\mathrm{J}(D) \cong \mathrm{J}(D/\Theta)$. Take a $p \in \mathrm{J}(D)$. If p/Θ has more then one element and p is a minimal element of these class, then there exists a $q \in \mathrm{J}(D)$ such that $p, q \in \Theta$ are incomparable and $p/\Theta > q/\Theta$, i.e. $\mathrm{J}(D) \ncong \mathrm{J}(D/\Theta)$ $\square$

Corollary 1. *Let G be the direct product of chains and let Θ be a cover-preserving join-congruence of G. Then G/Θ is a rectangular lattice with the property $\mathrm{J}(G) \cong \mathrm{J}(G/\Theta)$ if and only if every join-irreducible element of G is a one element Θ-class.*

Definition 3. *A source element s of Θ is called* bastard *if s itself or at least one of its lower covers, t is join-irreducible. The cover-preserving join-congruence Θ is bastard if it has a source element which is bastard.*

By Corollary 1 G/Θ is a rectangular semimodular lattice iff the source is not bastard.

We denote by $\mathfrak{l}(L)$ the length of the semimodular lattice L.

Lemma 7. *Let Θ be a cover-preserving join-congruence of the finite distributive lattice D with the source S_Θ. Then $\mathfrak{l}(D/\Theta) = \mathfrak{l}(D) - |S_\Theta|$.*

Proof. We have the source S_Θ. Assume that $|S_\Theta| \geq 2, s_1, s_2 \in \Theta$. Take the lower covers $t_1 \prec s_1$ and $t_2 \prec s_2$. Then $s_1 \equiv t_1$ (Θ) and $s_2 \equiv t_2$ (Θ), i.e. $s_1 \equiv t_1$ $(\mathrm{con}^{\vee\mathrm{cp}}(s_1))$ and $s_2 \equiv t_2$ $(\mathrm{con}^{\vee\mathrm{cp}}(s_2))$. Consider a maximal chain C, which contains s_1. Then there is exactly one covering pair $a \prec b, a, b \in C$ such that $s_2/t_2 \nearrow a/b$. By (2) it follows that $a \neq s_1$, i.e. in D/Θ the image of C has length $< l(D)$. $\square$

Finally, a remark to our "hollow" rectangular lattice $M_3[C_3]$, $(C_3 = \{0, 1, 3\})$. We obtain this lattice from the grid $G = {C_3}^3$ with the source: $S = \{(1, 1, 1), (0, 0, 0)\}$.

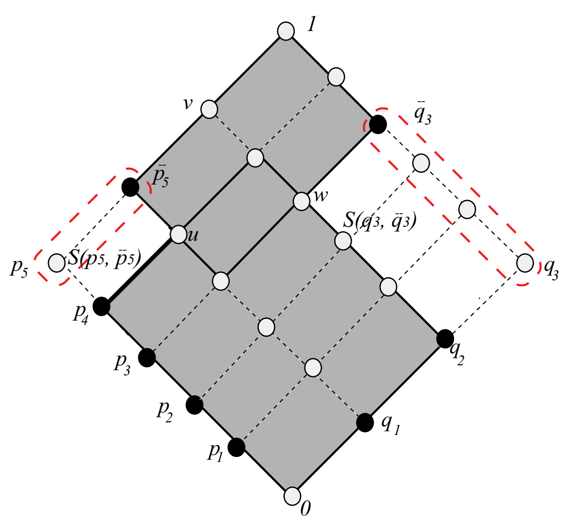

FIGURE 6. A planar distributive lattice D and a rectangular hull G

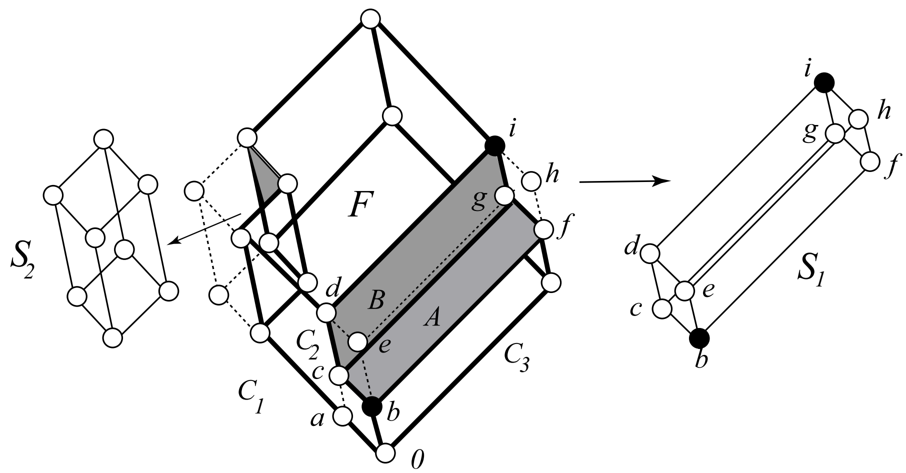

FIGURE 7. The "contour" of a distributive lattice F, $\mathrm{Jw}(F) = 3$ and a rectangular hull $G = C_1 \times C_2 \times C_3$

4. THE PROOF OF THE THEOREM

Let L be a semimodular lattice of length n and $\mathrm{Jw}(L) = k$. By Lemma 1, L is the cover-preserving join-homomorphic image of the distributive lattice $F = \mathrm{H}(\mathrm{J}(L))$. Let Φ be the corresponding cover-preserving join-congruence of F, then

$$L \cong F/\Phi. \tag{5}$$

We denote the source of Φ by $S \subset F$. We extend F to the distributive, rectangular lattice G (see (1)) which is a rectangular hull of F. This cover-preserving join-congruence will be denoted by $\overline{\Phi}$ and we define it as follows: for a covering pair $d \prec c$, $c, d \in G$, $c \equiv d$ $(\overline{\Phi})$ if and only if there is a source element $s \in S \subset F$ a lower cover $u \in F$ of s such that c/d is perspective up to s/u. That means, Φ and $\overline{\Phi}$ have the same source. It is easy to prove that the transitive extension is a cover-preserving join-congruence of G. Finally, define $\widehat{L}$ as:

$$\widehat{L} = G/\overline{\Phi}. \tag{6}$$

(a) From Lemma 16 of [7] it follows that $\widehat{L}$ is a semimodular lattice (the cover-preserving join-homomorphic image of a semimodular lattice is semimodular).

(b) By Lemma 5, F and G have the same length, i.e. F is a cover-preserving sublattice of G. From Lemma 7 it follows that L is a cover-preserving sublattice of $\widehat{L}$.

(c) By the definition of G, $\mathrm{Jw}(F) = \mathrm{Jw}(G)$, i.e. $\mathrm{Jw}(L) = \mathrm{Jw}(\widehat{L})$.

(d) By Lemma 1 we know that $\mathrm{J}(L) \cong \mathrm{J}(F)$. Then by Lemma 6 the source S is not bastard. This implies that S is not bastard in $G \supseteq F$. From Lemma 3 it follows that $\mathrm{J}(G) \cong \mathrm{J}(\widehat{L})$, which means that $\widehat{L}$ is a rectangular lattice.

In [8] the authors proved that the rectangular hulls of planar semimodular lattices are congruence-preserving extensions.

Problem. *Are the rectangular hulls of semimodular lattices congruence-preserving extensions?*

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