A STRUCTURE THEOREM OF SEMIMODULAR LATTICES: THE PATCHWORK REPRESENTATION

E. TAMÁS SCHMIDT

Abstract. In this paper we give a structure theorem of semimodular lattices, which generalize the the results given in [4] for planar semimodular lattices. The main result of [4] asserts, that every planar semimodular lattice is the patchwork of special intervals as show in Figure 1.

The "**building stones**" are special rectangular lattices (in most cases the surface of the diagram is a rectangular shape), we get these from Boolean lattices with a special construction, the *pigeonhole procedure*.

The "**building tool**" is a kind of gluing, the *patchwork construction*. It is related to the Hall-Dilworth gluing, for instance we glue together two cubes (i.e. 2^3 Boole algebras) over faces, see Figure 2. As technical tool we use special *block matrices*, see Figure 21.

Contents

Date: January 15, 2013.

2000 *Mathematics Subject Classification.* Primary: 06C10, Secondary: 06B15.

Key words and phrases. Lattice, semimodular, patchwork, gluing.

This research was supported by the Hungarian National Foundation for Scientific Research (OTKA), grant no. K 77432.

1. Introduction

We present in this paper a structure theorem for finite semimodular lattices. The "**building stones**" are special rectangular lattices (the surface of the diagram in most cases looks like a rectangular shape, the exact definition see below).

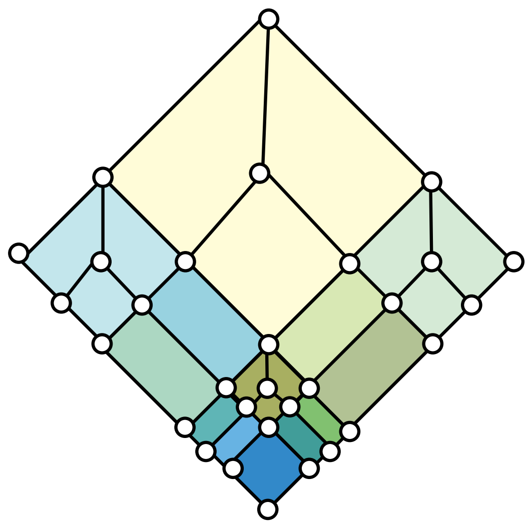

Figure 1. A planar semimodular lattice L as patchwork of intervals

The lattices S_7 contains three sublattices, which are glued together, but this is not a patchwork. $A \cap B$ is not empty, but $A \cup B$ is not a Hall-Dilworth gluing.

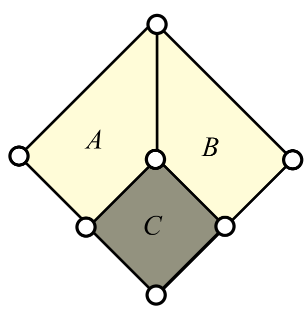

Figure 2. S_7

The "**building method**" is a special gluing, the *patchwork construction*. In the everyday's life: pieces of cloth of various colors and shape (but usually rectangular shapes) sewn together form a so-called patchwork. In Figure 1 you can see a slim semimodular lattice as an example, the building stones are copies of S_7 and covering squares.

This is a *patching system*, which is a special *strong multipasting* (see [7]) and in the class of modular lattice the *S-glued system* which was introduced by Ch. Herrmann, [9]. Here the building stones are *patching-irreducible lattices*, or briefly *patching lattices*. We prove the following structure theorem:

Theorem 1. *Every semimodular lattice is the patchwork of its patchwork-irreducible sublattices.*

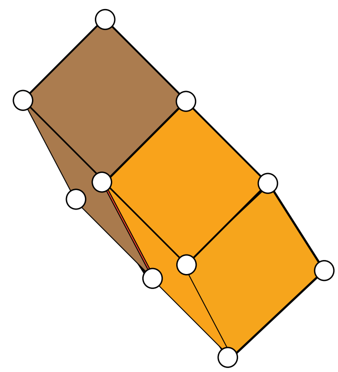

FIGURE 3. Patching of two, colored cubes

Let us see an other example in Figure 3, the 3-dimensional modular lattice $M_3[C_3]$ (see multipastg in [7]). This is a modular but not distributive 3-dimensional "cube" $M_3[C_3]$ as a patchwork of three squares and two M_3-s.

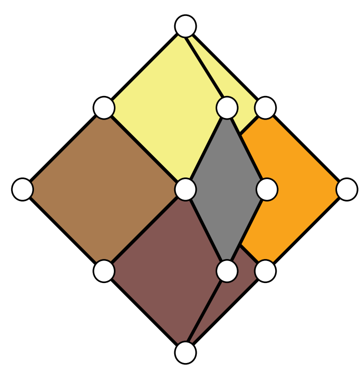

FIGURE 4. $M_3[C_3]$ as a patchwork of three squares and two M_3-s

1.1. **The dimension.** $\mathrm{J}(L)$ denotes the order of all nonzero join-irreducible elements of L and $\mathrm{J}_0(L)$ is $\mathrm{J}(L) \cup \{0\}$. A planar semimodular lattice L is called *slim* if no three join-irreducible elements of L form an antichain.

The *width* $\mathrm{w}(P)$ of a (finite) order P is defined to be $\max\{n\colon P$ has an n-element antichain$\}$. The width of $\mathrm{J}(L)$ is called the J-*width* of L, this will be denoted by

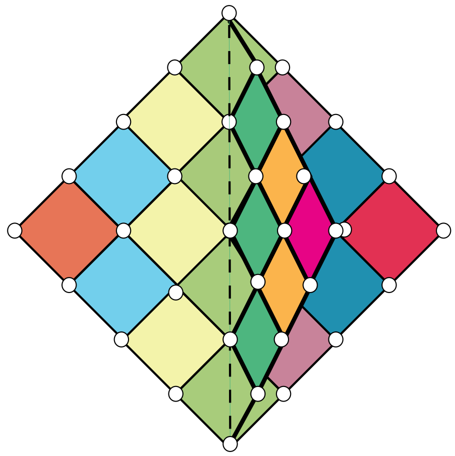

FIGURE 5. $\mathbf{M}_{3,4} = M_3[C_4]$ as a patchwork of squares and M_3-s

Jw(L). The distributive lattice L has dimension $n = \mathbf{Dim}(L)$ if and only if n is the greatest integer such that L contains a sublattice isomorphic to the 2^n-element boolean lattice, this is Jw(L). For non-distributive lattices see paragraph 1.1.

Let X, Y be posets. The *disjoint sum* $X+Y$ of X and Y is the set of all elements in X and Y considered as disjoint. The relation $\leq$ keeps it meaning in X and in Y, while neither $x \geq y$ nor $x \leq y$ for all $x \in X, y \in Y$.

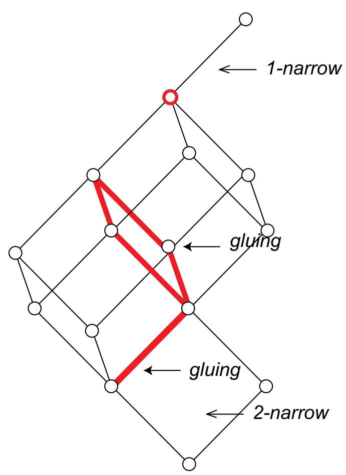

FIGURE 6. A 3-dimensional distributive lattice glued over faces

Rectangular lattices were introduced by Grätzer and Knapp [8] for planar semimodular lattices. In the general case we use the following definition:

Definition 1. *A* rectangular *lattice L is a finite semimodular lattice in which* J(L) *is the disjoint sum of chains.*

Rectangular lattices are building stones of semimodular lattices In [11] we proved that the rectangular lattices can be seen in an opposite role as rectangular hulls (envelops) of semimodular lattices. $M_3[C_3]$ is a modular non distributive rectangular lattice.

To define the dimension $\mathbf{Dim}(L)$ of a semimodular lattice seems to be complicated. The most natural way is to take the size of a maximal independent subsets. In the class of distributive lattices this is a correct definition. But what can we say by M_3? This lattice is a planar lattice, as geometrical shape is two-dimensional, it has no thee-element independent subset. On the other hand we can consider the diagram of M_3 as a three-dimensional shape, see in Figure 2 and Section 7, Definition 11.

Related less known dimension concept: the *Kuroš-Ore dimension* of a lattice L, which is the minimal number of join-irreducibles needed to span the unit element of L. A "good" definition of the dimension we will obtain from Theorem 1: the dimension of the patchwork irreducible lattices Kuroš-Ore dimension and the dimension of L will be the maximum of the Kuroš-Ore dimension of the patching-irreducible components.

We denote the Kuroš-Ore dimension of L by $\mathbf{Dim}(L)$. An onother dimesion concept is $\mathbf{dim}(L)$ the width of $\mathrm{J}(L)$. Obviouly, $\mathbf{Dim}(M_3 2$ and $\mathbf{dim}(M_3) = 3$.

1.2. **The patching.** Let K and L be semimodular lattices, let F be a filter of K, and I be an ideal of L, then we can form the the lattice G, the well-known Hall-Dilworth gluing of K and L over F and I. Assume that $\mathbf{Dim}(K)$, $\mathbf{Dim}(L)$, $\mathbf{Dim}(F)$, $\mathbf{Dim}(I)$ are defined. We call the gluing G the *patching* of K and L if:

$$\text{(Dim)} \qquad \mathbf{Dim}(I) < \min(\mathbf{Dim}(K), \mathbf{Dim}(L)).$$

Let $\{M_i\}$ be a system of intervalls - called blocks if $\{M_i\} = L$ and if $M_i \cap M_j \neq \emptyset, i \neq j$, then the union, $M_i \cup M_j$ is the Hsall-Dilworth gluing satisfying (DID)(i.e. the gluing is via an edge or face. If K_1, K_2 and K_3 are face (2-dimesional) of blocks such that (see in Figure 1) $K_1 \cap K_2$ is an ideal of K_2 and similarly $K_1 \cap K_3$ is an ideal of K_3 and

$$1_2 \wedge 1_3 = 1_1$$

then this is a *patching system*, see Figure 2.

The blocks of the patchwork system are special rectangular lattices. To make the figures more spectacular we can use colors. If we color the blocks, that means we have a set $\mathcal{C}$ − the elements are called *colors* − and a mapping $\psi : \{M_i\} \to \mathcal{C}$ such that if $M_i \cap M_j \neq \emptyset, i \neq j$ M_i and M_j have different colors. The patchwork is considered always in respect a dimension concept.

1.3. **The pigeonhole.** It is necessary to describe the patchwork-irreducible lattices. We presents in this paper a class of patchwork-irreducible lattices, we derive these from Boolean lattices with a procedure which will be called *pigeonhole construction* (or cell division?). This is illustrated for planar semimodular lattices in Figure 14: we take a distributive lattice $L = L_1$ and we insert into some of the covering squares a copy of S_7 getting L_2. By the next step we insert into some of the covering squares of L_1 a copy of S_7 getting L_2, and so on. Accordingly, in the 3-dimensional case we insert into covering cubes the semimodular lattice illustrated

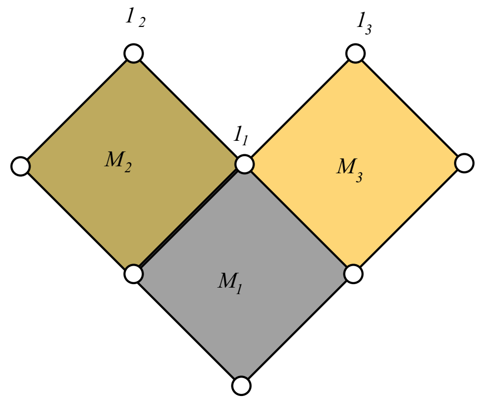

Figure 7. patchwork system

in Figure 12. The pigeonholes are rectangular lattices for which the Kuroš-Ore dimension exist.

2. Distributive lattices

We begin our discussion with distributive lattices.

2.1. **Dimension 2.** A planar distributive lattice D is the cover-preserving sublattice of a direct product of two finite chains, which is geometrically a rectangle. *Every planar distributive lattice D can be covered by rectangles, in other words it is a "glued system" of rectangles.*

If we cut up the lattice into smaller pieces, we may assume that the intersection of two "small" rectangles is either empty or is a part of a side (geometrically) of the rectangular components, i.e. (Did) is satisfied. The smallest rectangle is obviously the covering square B_2 (in [8] is called a 4-cell) and the unit segment C_2, these are the patching-irreducible rectangular lattices. These form a patchwork system of D, obviously **a planar distributive lattice D looks like a patchwork of unit squares and line segments**. B_n is the boolean lattice with n atoms. We represent the distributive lattice D by the patchwork lattices obtained from the patchwork lattices B_1, B_2, see Figure 6:

We are going to the 3-dimensional distributive case.

2.2. **Dimension 3.** In the class of 3-dimensional distributive lattices the patchwork-indecomposabe lattices are ${C_2}^3, {C_2}^2, C_2$, (covering cube, covering square and unit segment).

On the following Figures 4 you can see, the gluing of two cubes over faces: we identify the two faces of these components. This looks like a Rubik cube which contains 27 small unit cubes. This covering satisfies condition (Dim).

2.3. **The grid.** It was proved by G. Czédli and E. T. Schmidt [1] the following:

Theorem 2. *Each finite semimodular lattice L is a cover-preserving join-homomorphic image of a distributive lattice G which is the direct product of $n = \mathrm{Jw}(L)$ finite chains.*

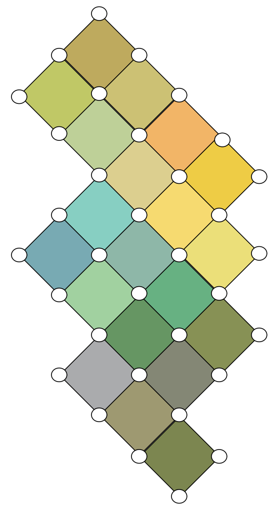

FIGURE 8. The representations of the distributive lattice D with a patchwork system

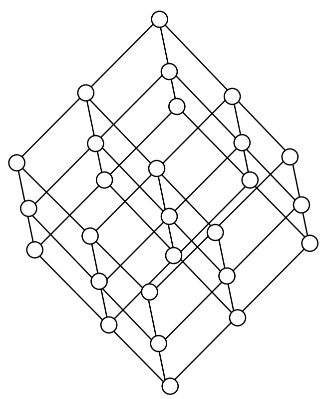

FIGURE 9. The 3-dimensional distributive lattice ${C_3}^3$

Take subchains $D_1, D_2, ..., D_n$ of L such that $\mathrm{J}(L) \subset \bigcup D_i$. $G = D_1 \times D_2 \times ... \times D_n$ is a rectangular distributive lattice. By Theorem 2 we can get every semimodular lattice L from the rectangle G, using as tool a cover-preserving join congruence.

Remark. The theorem was proved originally (and independently) in the first edition of M. Stern, [10] (Th6.3.14, p. 240).

In [1] we proved an other similar theorem:

Theorem 3. *Every finite semimodular lattice L is a cover-preserving join-homomorphic image of the unique distributive lattice F determined by $\mathrm{J}(F) \cong \mathrm{J}(L)$. Moreover, the restriction of an appropriate cover-preserving join-homomorphism from D onto L is a $\mathrm{J}(S) \to \mathrm{J}(L)$ order isomorphism.*

From the proof of Theorem 2 we can see that $F = \mathrm{H}(\mathrm{J}(L))$, which is the distributive lattice determined by the order $P = \mathrm{J}(L)$ (the poset $\mathrm{H}(P)$ denotes the lattice of all downsets of the order P).

G can be interpreted in three different ways:
(1) G is a **grid** (i.e. a coordinate system),
(2) A table, i.e. a **matrix** (an $(n_1 \times n_2 \times ... \times n_k)$-matrix),
(3) G is a geometric **rectangle** (other names: a n-dimensional cube or cuboid).

G is not uniquely determined by L.

By Theorem 3 there is a congruence-preserving join-congruence Θ such that

$$L \cong G/\Theta.$$

We need the following lemma from [1], which characterize the cover-preserving join-congruences:

Lemma 1. *Let Φ be a join-congruence of a finite semimodular lattice M. Then Φ is cover-preserving if and only if for any covering square $S = \{a \wedge b, a, b, a \vee b\}$ if $a \wedge b \not\equiv a$ (Φ) and $a \wedge b \not\equiv b$ (Φ) then $a \equiv a \vee b$ (Φ) implies $b \equiv a \vee b$ (Φ).*

3. The source

Let Θ be a cover-preserving join-congruence of a distributive lattice G (which is not necessarily the grid).

Definition 2. *An element $s \in G$ is called a* source element *of Θ if there is a $t, t \prec s$ such that $s \equiv t$ (Θ) and for every prime quotient u/v if $s/t \searrow u/v, s \neq u$ imply $u \not\equiv v$ (Θ). The set $\mathcal{S}_\Theta$ of all source elements of Θ is the* source *of Θ.*

Lemma 2. *Let x be an arbitrary lower cover of a source element s of Θ. Then $x \equiv s$ (Θ). If $s/x \searrow v/z$, $s \neq v$, then $v \not\equiv z$ (Θ).*

Proof. Let s be a source element of Θ then $s \equiv t$ (Θ) for some t, $t \prec s$. If $x \prec s$ and $x \neq t$ then $\{x \wedge t, x, t, s\}$ form a covering square. Then $x \not\equiv x \wedge t$ (Θ). This implies $x \wedge t \not\equiv t$ (Θ). By Lemma 1 we have $x \equiv s$ (Θ).

To prove that $v \not\equiv z$ (Θ), we may assume that $v \prec s$. Take t, $t \prec s$, then we have three (pairwise different) lower covers of s, namely $.x, v, t$. These generate an eight-element boolean lattice in which $s \equiv t$ (Θ), $s \equiv x$ (Θ) and $s \equiv v$ (Θ). By the choice of t we know that $v \not\equiv v \wedge t$ (Θ), $x \not\equiv x \wedge t$ (Θ) and $z \not\equiv x \wedge t \wedge v$ (Θ). It follows that $x \not\equiv t$ (Θ), otherwise by the transitivity $x \not\equiv v$ (Θ). This implies $t \wedge x \not\equiv t \wedge x \wedge v$ (Θ) . Take the covering square $\{x \wedge v \wedge t, z, t \wedge x, x\}$ then by Lemma 1 $z \not\equiv x$ (Θ), which implies $z \not\equiv v$ (Θ). □

The following results are proved in [12]. The source $\mathcal{S}$ satisfies an independence property:

Definition 3. *Two elements s_1 and s_2 of a distributive lattice are s-*independent *if $x \prec s_1, y \prec s_2$ then $s_1/x, s_2/y$ are not perspective, $s_1/x \not\sim s_2/y$. A subset S is s-independent iff every pair $\{s_1, s_2\}$ is s-independent.*

The meet of the lower covers of the element s is denoted by s^* and $s^{**} = (s^*)^*$. The interval $B_s = [s, s^*]$ is called a *source cell*, which is a Boolean lattice.

Lemma 3. *Two elements s_1 and s_2 of a distributive lattice are s-independent if one of the following is satisfied:*

(1) s_1 *and* s_2 *are incomparable,*
(2) $s_1 < s_2$ *and* $t \prec s_2$ *implies* $t \geq s_1$*, i.e.* $s_1 \leq {s_2}^*$*.*

Proof. It is clear that for an incomparable pair s_1, s_2 if $u \prec s_1$ and $v \prec s_2$ then s_1/u and s_2/v cannot be projective. On the other case, if $s_1 < s_2$ then $t||s_1$ would imply that s_2/t and $s_1/t \wedge s_1$ are perspective. This means that $t \geq s_1$. □

It is easy to prove that every s-independent subset $\mathcal{S}$ generate a cover-preserving join-congruence Θ. The semimodular lattice L is characterized by (G, Θ) or $(G, \mathcal{S})$, where $\mathcal{S}$ is an s-independent subset. We write:

$$L = \mathcal{L}(G, \mathcal{S}).$$

Definition 4. *The* beret $\Delta = \{1, q_i \prec 1\}$ *(*swiss cup*) of a distributive lattice* L *is the unit element and the dual atoms of* L*.*

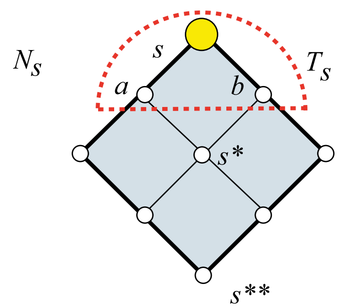

Figure 10. ${C_3}^2$ with the beret

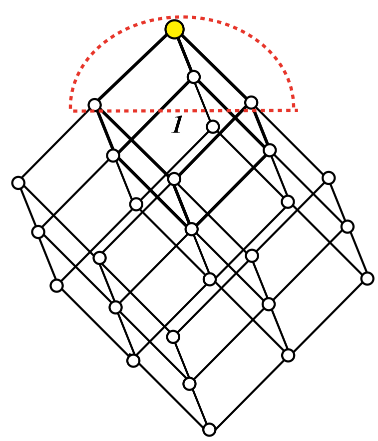

Figure 11. ${C_3}^3$ with the beret

The beret is a join-sublattice (order dual-ideal) of L. Take the cover-preserving join-congruence of L the where the beret is the only one non-trivial congruence class. We denote this cover-preserving join-congruence by the same letter Δ

We introduce the following notations:

$$\mathbb{S}_k = {C_3}^k/\Delta \text{ and } \mathbb{B}_k = {C_2}^k/\Delta.$$

Take the cover-preserving join-congruence of L the where 0 and the atoms are one-element classes and all other elements are in the same class is Ω. Define: $\mathbb{D}_k = {C_3}^k/\Omega$.

Definition 5. *$\mathbb{S}_k$ is called the k-dimensional* source lattice *and $\mathbb{B}_k$ is the k-dimensional* source kernel*.*

The source lattices are the **elementary building stones** of semimodular lattices.

The ${C_2}^k$ boolean lattice is a (0,1)-sublattice of $\mathbb{S}_k$. This is the *skeleton* of $\mathbb{S}_k$ and will be denoted by Skeleton($\mathbb{S}_k$). The dimension of ${C_2}^k$ is k and therefore we define: $\mathbf{Dim}(\mathbb{S}_k) = \mathbf{Dim}(\mathbb{B}_k) = k$. $\mathbb{B}_3$ is M_3 which means that these lattice has dimension 3. The Kuroš-Ore dimension of $\mathbb{S}_k$ in L.

By Theorem 1 L is a glued sum of patchwork irreducible lattices. If the dimension of each component is already defined then the dimension of L is the maximum of the dimensions of the components.

$\mathbb{S}_2 \cong S_7$ and $\mathbb{S}_3$ is in Figure 10. Let us remark that $\mathbb{S}_2$ has three 4-cells and $\mathbb{S}_3$ has seven 2^3-cells.

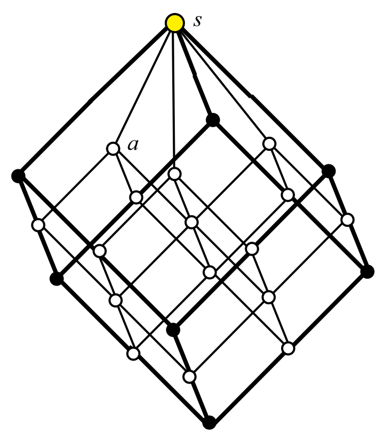

FIGURE 12. The source lattice $\mathbb{S}_3$ with the skeleton

3.1. **Source lattices.** The dual atoms of B are dual atoms of $\mathbb{S}_n$

A *source cell B_s* is the boolean lattice generated by the lower covers of a source element $s \in S_\Theta$ and

The meet of the lower covers of the bottom element s^* of a source cell is denoted by s^{**}.

Definition 6. *The* neighborhood *of source cell or a source element s is the interval* $\mathcal{N}_s = [s^{**}, s]$.

If the neighborhood of a source element exists and s^* is a singleton of Θ then these determine in the factor lattice a source lattice. The top element of this source lattice is s.

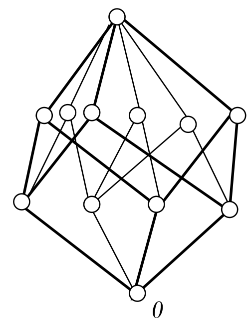

FIGURE 13. $\mathbb{B}_4$

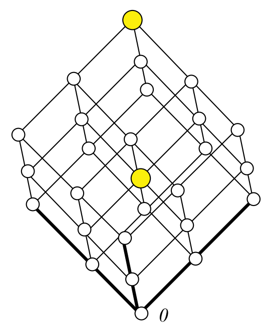

FIGURE 14. The source of $M_3[C_3]$

Definition 7. *A source element s of Θ is called* bastard *if s itself or at least one of its lower covers t is join-irreducible. The cover-preserving join-congruence Θ is bastard if has a source element which is bastard.*

In the two dimensional case this means that s or at least the lower covers of s, a and b is join-irreducible.

Lemma 4. $\mathbb{S}_n$ *is a subdirect irreducible rectangular semimodular lattice. The dimension of* $\mathbb{S}_n$, $\mathbf{Dim}(\mathbb{S}_n) = n$

Proof. It is an easy exercise, in the three-dimensional case, Figure 10, $\mathrm{con}(s, a)$ is the smallest non trivial congruence relation. □

3.2. **The classification of the source elements.** In Figure 13 you can see the classification of the source elements in the 2-dimensional case. We have a grid G and we ask what is G/Θ_s, $s \in G$ depending on its position ?

Case 1. If s is represented by a black filled circle. In this case Θ_s is a **lattice congruence**,

Case 2. If s is represented by a pirple filled circle, then Θ_s changes the ordering of $\mathrm{J}(G)$,

Case 3. If s is represented by a yellow filled circle, then G/Θ_s is a semimodular but not modular lattice. (A slim modular lattice is distributive. Let L be a modular non-distributive lattice then $\mathrm{Jw}(L) \geq 3$.)

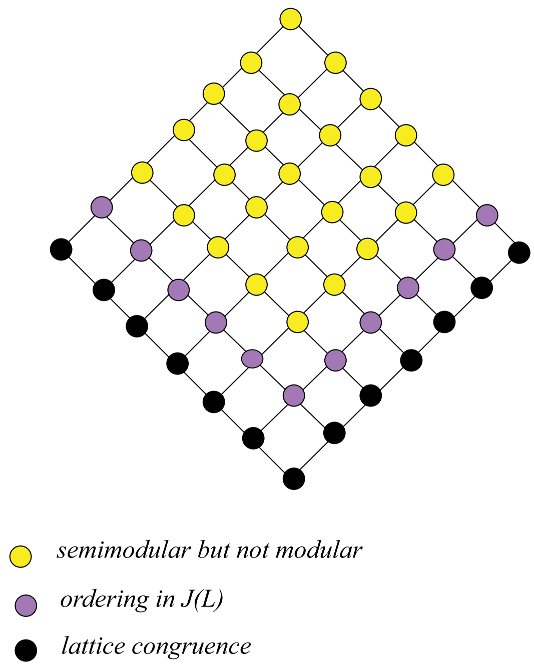

FIGURE 15. The classification of the source elements

4. Pigeonhole

The pigeonhole lattices are the patchwork-irreducible irreducible lattices for diamond-free lattices. In [4] we have given in the class of planar semimodular lattices some characterizations of patchwork-irreducible planar semimodular lattices, these offers six alternative definitions. In this section we give a new characterization with matrices (see [6]).

4.1. **The pigeonhole construction in the planar case.** We prove that every planar semimodular lattice is a patchwork of its maximal patch lattice intervals "sewn together". For a modular planar lattice, our patchwork system coincides with the S-glued system introduced by C. Herrmann in 1973. Among planar semimodular lattices, patch lattices are obtained from the four-element non-chain lattice by adding so called forks, introduced in our preceding paper [3]. Later, we generalize this for the n -dimensional ($n > 2$) case. Therefore we formulate this construction first for planar lattices.

Take $S_7 = \{0, 1, a, b, c, d, e\}$ in Figure 13. The elements $0, 1, a, b$ form the skeleton S of S_7. This is a boolean $(0, 1)$- sublattice, a 4-cell. The elements $c, d, e, 1$ form a poset F, which will be called a *fork*. This is the interior of the skeleton, which means that we insert the poset F, the fork, into the interior of S, and this way we divide S into three new 4-cells. The result is S_7 and therefore we prefer:

we insert S_7 into a 4-cell.

As long as there is a chain $u \prec v \prec w$ such that v is a new element and $T = \{x = u \wedge z,\, z,\, u,\, w = z \vee u\}$ is a 4-cell in the original lattice L but $x \prec z$ at the present stage, we insert a new element y such that $x \prec y \prec z$ and $y \prec v$. (This way

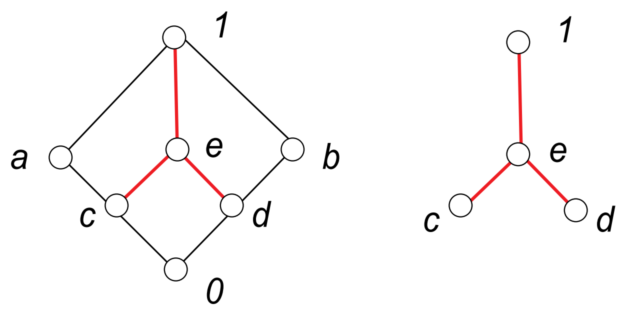

FIGURE 16. S_7 and the 2-fork

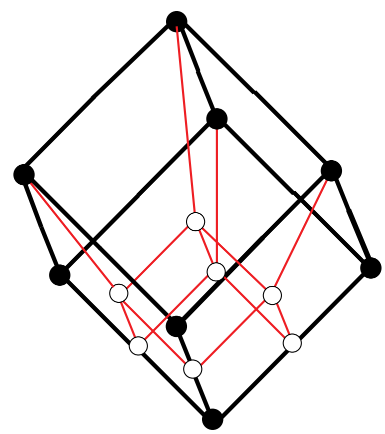

FIGURE 17. A 3-dimensional pigeonhole

we get two 4-cells to replace the cell T.) When this "downward-going" procedure terminates, we obtain L'. The collection of all new elements, which is a poset, will be called an *extended fork*. We say that L' is obtained from L by *adding a fork to* L (at the 4-cell S). Adding forks to L means to add several forks to L one by one.

In Figure 13-14 you can see some patch lattices of length four and five. On the left side are the grids with the sources. The yellow circles denote the source elements.

We recall the following statement [8].

Lemma 5. *Each planar semimodular lattice L is an anti-slimming of one of its slim semimodular (0-1)-sublattices, L'.*

Definition 8. *The planar semimodular lattice L is a pigeonhole lattice (briefly pigeonhole) if it is obtained from the four-element Boolean lattice by adding finitely many forks one by one.*

The main theorem of G. Czédli and E. T. Schmidt [4] asserts:

Theorem 4. *Let L be a planar semimodular lattice. Assume that $|L| \geq 4$. Then the following five conditions are equivalent.*

(i) *L is a patchwork-irreducible lattice;*

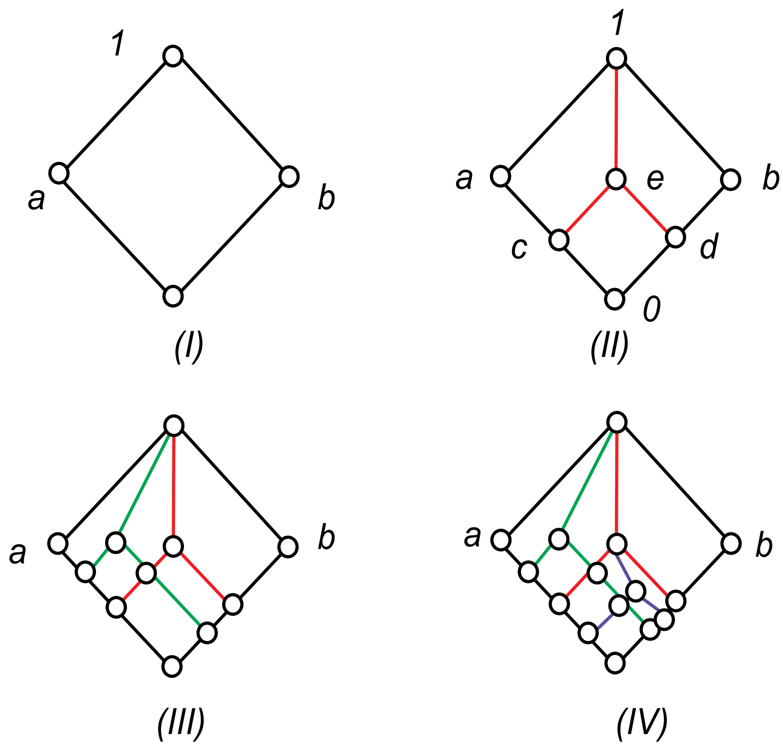

FIGURE 18. The pigeonhole construction

(ii) *L is indecomposable with respect to the Hall-Dilworth gluing;*
(iii) *L is indecomposable with respect to the Hall-Dilworth gluing over chains; corners w_D^ℓ and w_D^r, with respect to some planar diagram D of L, are coatoms);*
(iv) *for each planar diagrams of L, the intersection of the leftmost coatom and the rightmost coatom is 0;*
(v) *L is an anti-slimming of a lattice obtained from a pigeonhole.*

Corollary 1 (Structure theorem I.)**.** *Each slim planar semimodular lattice can be constructed as the last member of a finite list $L_1, L_2, \ldots, L_n$ such that each L_i $(i = 1, \ldots, n)$ is either a patch lattice (constructed according to Theorem 4(v)), or there are $j, k < i$ such that L_i is a Hall-Dilworth gluing of L_j and L_k over a chain. Conversely, every slim, lattice constructed this way is a planar semimodular lattice.*

By condition (v) of Theorem 4 the Corollary 1 can be reformulate as follows;

Corollary 2. *In the planar (2 dimensional) case the patchwork-irreducibles are the anti-sliming of pigeonholes-s.*

4.2. **Matrices of the pigeonhole lattices in the planar case.** We give here a construction of pigeonhole lattices using matrices and we illustrate this by a concrete example. (See the latices in Figures 15 – 16.)

Step 1. We start with an 3×3 invertible (0, 1)-matrix M. In every row and column there is exactly one "1".

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Take the following augmented matrix:

$$\overline{M} = \begin{bmatrix} 0 & 1 & 0 & \mathbf{0} \\ 1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

The entries of the augmented matrix in the last row resp. last column are zeros.

Step 2. Let G be the direct product $\{0 < 1 < 2 < 3 < 4\} \times \{0 < 1 < 2 < 3 < 4\} \cong C_5 \times C_5)$. This will be a grid of our new lattice.

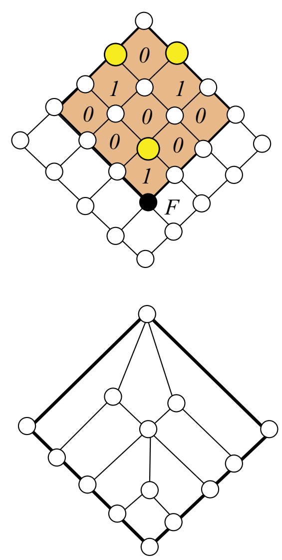

FIGURE 19. A planar, patchwork-irreducible lattices of length 5

F denotes the element $(1, 1) \in G$ and take the filter $\uparrow F = \{1 < .... < 4\} \times \{1 < .... < 4\}$ and can be considered as a 3×3-matrix rotated by $\pi/4$. We have a bijection φ between M and $\uparrow F$. Take the covering squares in $\uparrow F$, we have 9 covering squares (4-cells).

Step 3. Write into these squares 0 or 1 as shown in the given matrix M, see in Figure 15. Let as mark by a yellow circle the top element of the covering square if it contains 1 (using the bijection φ). All these form an s-independent subset $\mathcal{S}$ of the set of grid elements. Consider the join-congruence Θ defined by these set $\mathcal{S}$.

Step 4. Define the semimodular lattice as

$$L_M = G/\Theta.$$

In Figure 16 there are two slim, planar, patchwork-irreducible lattices of length 4 on the right side, on the left side are the matrices. To the lattice S_7 belongs the

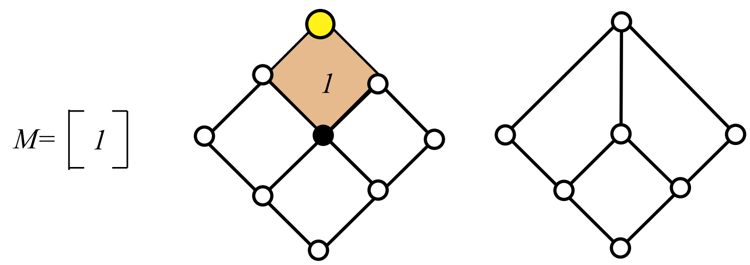

FIGURE 20. S_7 represented by a 1×1-matrix resp. a grid

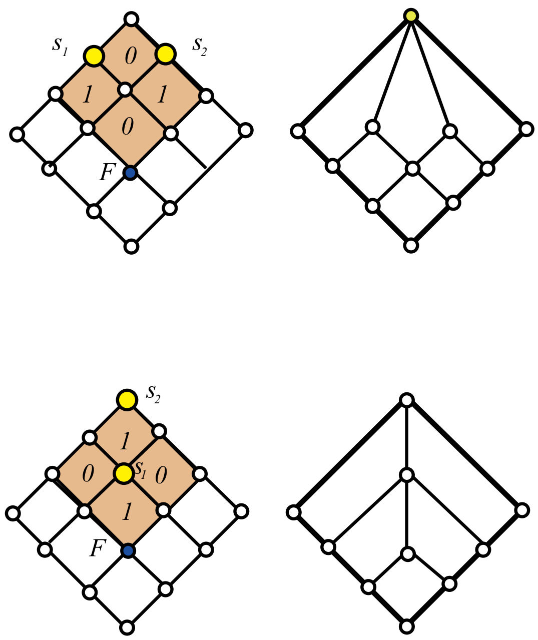

FIGURE 21. Two slim, planar, patchwork-irreducible lattices of length 4

matrix of type 1×1 : $\begin{bmatrix}1\end{bmatrix}$. The matrices of the lattices presented in Figure 14 are: $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ resp. $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

Theorem 5 (Structure theorem II.). *A rectangular slim, planar, semimodular lattice L is a pigeonhole lattice iff:*

(vi) *there is a invertible (0, 1)-matrix M such that $L \cong L_M$.*

Proof. (I.) We prove that condition (vi) implies (iv) of Theorem 4. Take the elements $a = (n, 0)$, $b = (0, n)$ of $G = C_n \times C_n$, then in the $(k+1)$-the row we have somewhere a source element $s_k = (k+1, i)$. and therefore $(k+1, i) \equiv\ (k, i)(\Theta)$. This gives $(k+1, n) \equiv\ (k, n)(\Theta)$ (see Figure 15).

By our assumption in every row – except the first row – we have a source element, which means that $(n, n) \equiv\ (1, n)(\Theta)$. The first row does not contain any source element, which means that $(1, n) \not\equiv\ (0, n)(\Theta)$. In the factor lattice G/Θ the image

$\overline{b}$ of b is a dual atom. Similarly, the image of $a = (n, 0)$, i.e. $\overline{a}$ is a dual atom of L_M. Obviously, $\overline{a}$ is a complement of $\overline{b}$, which proves (v). Finally, by condition (iv) of Theorem 4 implies that L_M is a pigeonhole lattice.

(II.) Let L be a pigeonhole lattice. By Theorem 4 this is equivalent to condition (vii). The four element Boolean lattice B is obviously a lattice in the form L_M with type 0×0. If we insert a fork into B we get S_7. This is a lattice of type L_M with the 1×1 matrix $\begin{bmatrix}1\end{bmatrix}$. We continue this procedure, it is easy to see that every step (adding a fork) we get a lattice of the form L_M. □

This theorem leads to the study of a decomposition of (0,1)-matrices into invertible matrices, where the invertible submatrices – blocks – belong to pigeonhole lattices, see section 4.2.

4.3. **Inserting S_7-s as gluing.** We show in this section that adding (inserting) a fork is a special gluing of the grids. This is presented in Figure 18. Follow the pictures in this figure.

Step 1. We begin with S_7 presented in (A), see in Figure 18. Insert in the left 4-cell a fork, we get (B), this is a lattice, two copies of S_7 crossed together.

Step 2. To each of these S_7 assign a grids: (C) and (D);

Step 3. Take the right covering square G in Figure (C) and the left covering square (G') of (D) and glue together, G=G' (see the shaded area). We get the colored part of (F). Consider the convex hull as a new grid with the given source;

Step 4. These grid and source determines a semimodular lattice given in (F), which is obviously isomorphic to the lattice presented in (B).

More on this construction see in section 4.7.

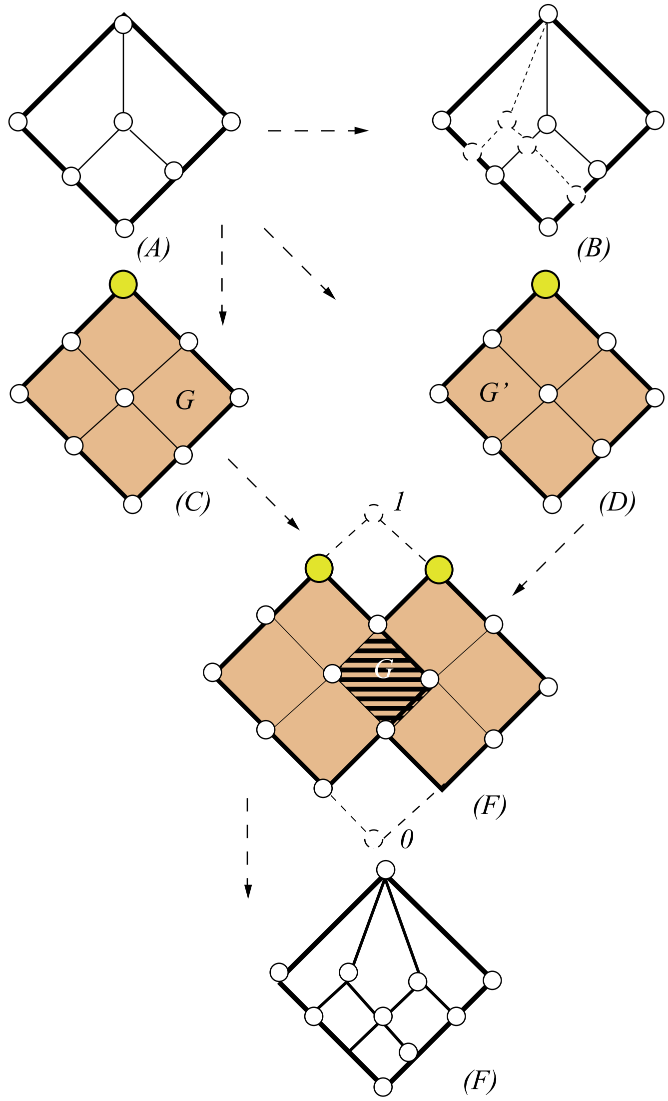

FIGURE 22. Adding a fork as gluing

4.4. **Pigeonholes determined by matrices in the 3D case.** The three-dimensional matrix concept is well known in the literature. Let us take the following $(0, 1)$ -matrix $M = [a_{ijk}]$ of type $2 \times 2 \times 2$.

The matrix of M_3 is a $1 \times 1 \times 1$ matrix with the entry 1. This shows that M_3 in this role is a 3- dimensional lattice.

"Invertible" means that n^3 entry is 1 and in any layer $n \times n$ matrix is invertible in the usual sense and if project the k-th layer onto the l-layer $(k \neq l)$ then the image of an entry 1 can not be 1. It is easy to define a multiplication between $n \times n \times n$ matrices.

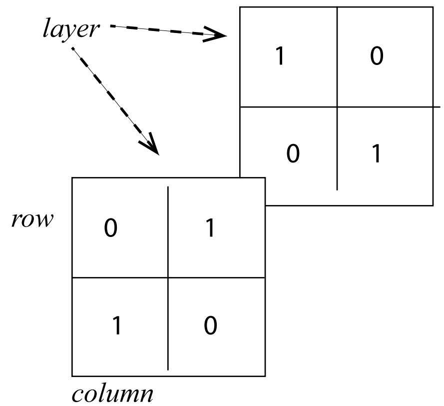

FIGURE 23. A matrix of type $2 \times 2 \times 2$

Take $G = C_3 \times C_3 \times C_3$, the element $F = (1,1,1)$ and the dual ideal $\uparrow F \cong C_2 \times C_2 \times C_2$. Paste the entries of the given matrix into the covering squares of $\uparrow F$. The entries "1" determine $\mathcal{S}$ with G the semimodular lattice L_M.

4.5. **The three-dimensional pigeonhole lattices.** As we have seen in paragraph 3.1 the source lattice $\mathbb{S}_n = {C_3}^n/\Delta$ contains a 2^n-element Boolean $(0,1)$-sublattice the skeleton S_n of $\mathbb{S}_n$. The poset $F_n = \mathbb{S}_n - S_n$ will be called the n-fork. This is the "interior" of $\mathbb{S}_n$, (see F_2 resp. F_3 in Figure 19 resp Figure 20). Let B be a 2^n-element Boolean lattice, and take a mapping $\varphi : S_n \to B$, which is defined on the skeleton and is an onto map. This can be extended to a mapping $\overline{\varphi} : \mathbb{S}_n \to B$, the elements $\overline{\varphi}(f), f \in F_n$. We say that we **inserted** $\mathbb{S}_n$ resp. F_n into B.

The Boolean lattice $\mathcal{P}_0 = {C_2}^n$ and the source lattice are $\mathbb{S}_n = \mathcal{P}_1$ are pigeonhole lattices (we insert $\mathbb{S}_n$ into $\mathcal{P}_0$ and get $\mathcal{P}_1$). The fork divide the lattice into 2^n-cells. If $\mathcal{P}_i$ is a pigeonhole lattice and C is a cover-preserving sublattice isomorphic to ${C_2}^n$ (a 2^n-cell), then we insert $\mathbb{S}_n$ into C such that the image of the skeleton of $\mathbb{S}_n$ is C we get $\mathcal{P}_{i+1}$. This way we obtain from $\mathcal{P}_i$ a new lattice $\mathcal{P}_{i+1}$.

The 3-dimensional, not 2-dimensional case we have so far the following patchwork-irreducible lattices: the pigeonholes: $\mathcal{P}_0 = {C_2}^3$, $\mathcal{P}_1 = \mathbb{S}_3$, $\mathcal{P}_i (i > 1)$, ... and $\mathbf{Dim}(\mathbb{B}_4)$. Let us remark that Kuroš-Ore dimension $3 = \mathbf{Dim}(\mathbb{B}_4) < \mathrm{Jw}(\mathbb{B}_4) = 4$.

We have seen by the planar semimodular lattices that the patchwork-irreducible lattices are the lattices L_M, where M is an invertible matrix.

Theorem 6. *Let L be a semimodular lattice,* $\mathrm{Jw}(L) = k$. *Assume that* $|L| \geq 4$. *Then the following three conditions are equivalent.*

(1) L is a patchwork-irreducible lattice, L_M for an invertible k-dimensional matrix M;
(2) L contains a skeleton $B \cong {C_2}^k$ such that the coatoms of B are coatoms of L;
(3) L is a lattice obtained from the 2^k-element Boolean lattice by adding finitely many forks one by one.

Proof. (3) implies (2), we use induction on n, the number of the inserted forks.

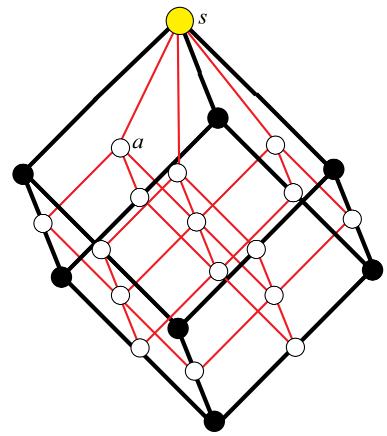

FIGURE 24. $\mathbb{S}_3$

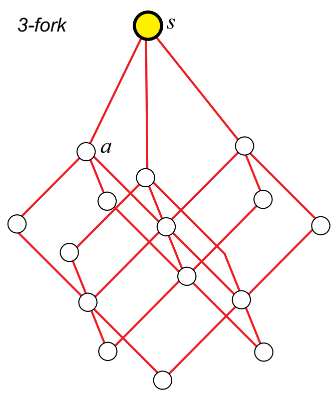

FIGURE 25. The 3-fork

(1) implies (3). Let L_M be determined by a M $n \times n$ - matrix. The corresponding grid is $G \cong C_{n+1} \times C_{n+1} \times .... \times C_{n+1}$. We insert an 2^k-into a 2^{k+1}-cell S of L. Take the grid $G' \cong C_{n+2} \times C_{n+2}$. G a filter of G'. If a covering-square contains "1" then we label the top of the covering-square by s. This gives a source.

(2) implies (1). Let B be a skeleton of L and assume that the coatoms of B are coatoms of L. We prove that there exists a matrix M such that $L = L_M$. (Figure 13 will help to visualize the proof.) Consider the grid G of L and the source $\mathcal{S}$. The grid is n-dimensional matrix. We write into a cover-preserving Boolean sublattice B an "1" iff the its top element is a source element, i.e. B is a source cell. Into all other covering Boolean sublattices we write "0". The coatoms of B are coatoms of L which implies that in the n-dimensional matrix the j-th projection onto the j-the component different 1 entries image is different.

□

4.6. **Properties of pigeonholes.** A pigeonhole lattice is a rectangular lattice, has a skeleton ${C_2}^n$, the Kuroš-Ore dimension is n. It is M_3-free and if $n \geq 2$ it is

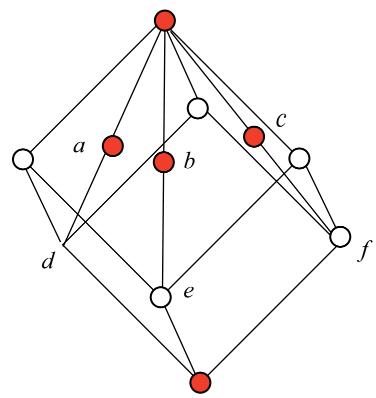

FIGURE 26. A 3D pigeonhole

subdirect irreducible. Its matrix is invertible. The modular patchwork irreducibles are the boolean lattices.

4.7. **The sum of pigeonhole lattices.** We define two operations between pigeonholes P_1 and P_2: the *horizontal sum* $P_1 \Cap P_2$ and *the vertical sum* $P_1 \Cap P_2$. In Figure 21 we present $S_7 \Cap S_7$. In the planar case we have two grids G_1, G_2 and the sources S_1, S_2. Glue together the rightmost covering square A os G_1 and the leftmost covering square B of G_2. Take the grid G generated by $G_1 \cup G_2$ with the source $S = S_1 \cup S_2$ and define:

$$P_1 \Cap P_2 = \mathfrak{L}(G, S).$$

The vertical sum $P_1 \Cup P_2$ of P_1 and P_2 is the following (see $S_7 \Cup S_7$ in Figure 22): we have two grids G_1, G_2 and the sources S_1, S_2. Glue together the top covering square A os G_1 and the bottom covering square B of G_2. Take the grid G generated by $G_1 \cup G_2$ with the source $S = S_1 \cup S_2$ and define:

$$P_1 \Cup P_2 = \mathfrak{L}(G, S).$$

Lemma 6. *The* horizontal sum $P \Cap Q$ *of the pigeonholes* P *and* Q *is a pigeonhole lattice.*

Proof. Condition (2) of Theorem 6 is obviously satisfied. □

Lemma 7. *The* vertical sum $P \Cup Q$ *of the pigeonholes* P *and* Q *is a pigeonhole lattice.*

Proof. Condition (2) of Theorem 6 is obviously satisfied. □

5. BLOCK MATRICES AND PIGEONHOLES

Although we defined the sum of pigeonholes with help of the corresponding grids and sources which are determined by (augmented) matrices we study in this section these matrices.

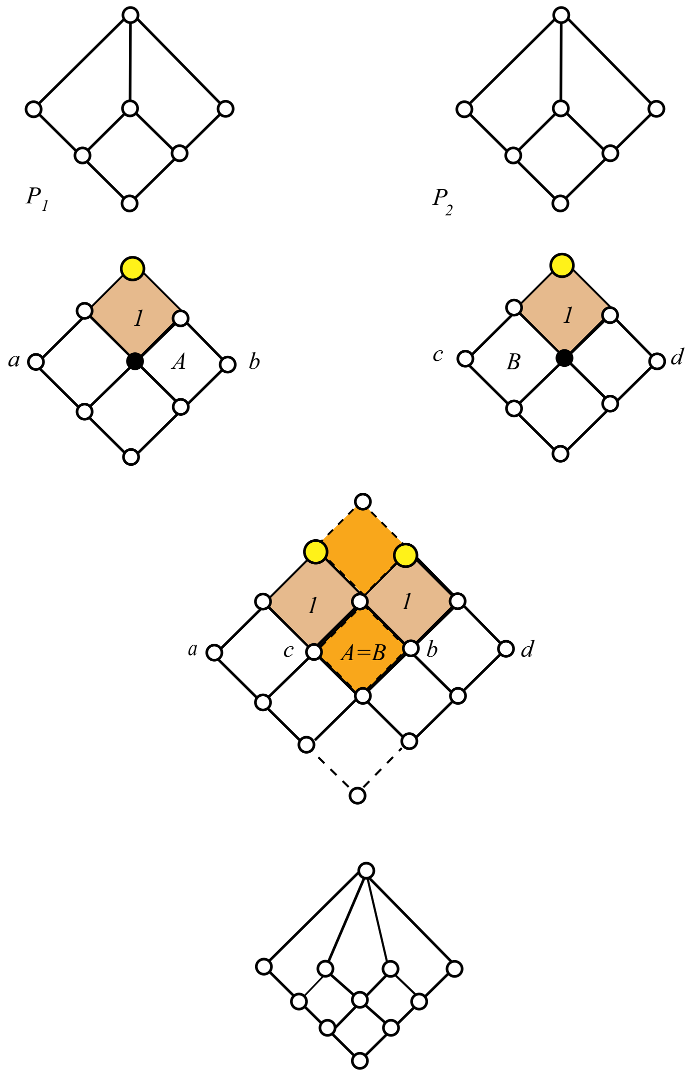

FIGURE 27. The horizontal sum

A *matrix* is a matrix broken into sections called blocks, see in Figure 23 (in the "empty squares" imagine 0).

Every semimodular lattice L can be characterized with a $(0,1)$-matrix, see in [5]. We have seen the patchwork-irreducible lattices are determined by invertible matrices, these are blocks of the matrix of L, we call these *pi-blocks*

The vertical and horizontal sum with matrices:

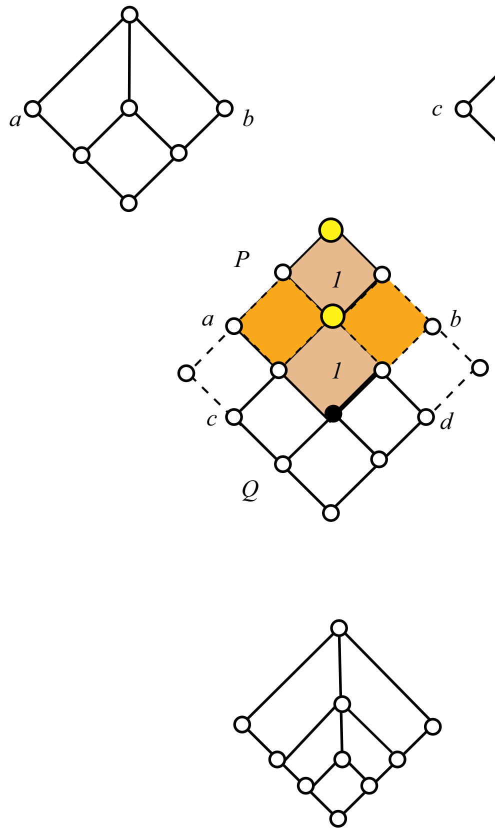

FIGURE 28. The vertical sum

						0	1	0	0		
						0	0	1	0		
						1	0	0	0		
1	0					0	0	0	0		
0	0	0	0	1	0						
		1	0	0	0						
		0	1	0	0						
		0	0	0	0						

FIGURE 29. A augmented block matrix

Here is an example of the horizontal sum of two augmented augmented invertible M_1, M_2 matrices. The result $M_1 \cup M_2$ is an augmented invertible matrix.

The 4×4 matrix is represented in Figure 15 and a 2×2 represents S_7.

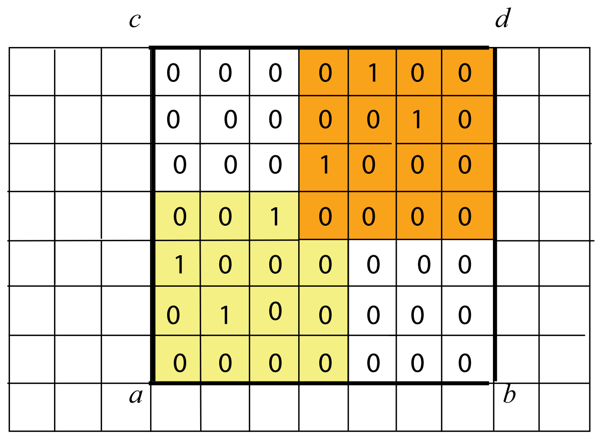

FIGURE 30. horizontal sum of matrices

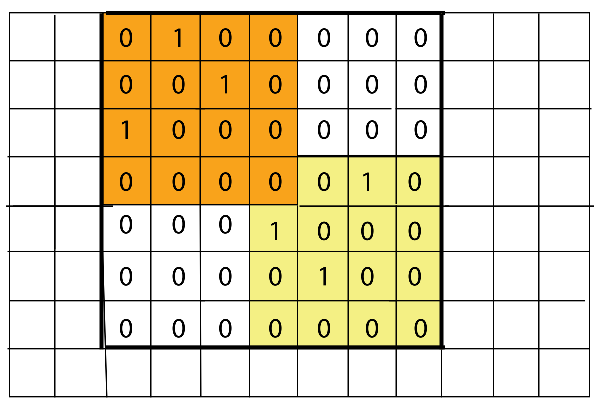

FIGURE 31. Vertical sum of matrices

Lemma 8.
$$L_M \Cap L_N = L_{M \Cap N}, L_M \Cup L_N = L_{M \Cup N}.$$

Lemma 9.
$$M(P) \Cap M(Q) = M(P \Cap Q), M(P) \Cup M(Q) = M(P \Cup Q).$$

Using the introduced concepts we reformulate Theorem 1 as follows:

Theorem. *Every finite semimodular lattice is the patchwork of patchwork irreducible lattices.*

Theorem. *Every finite M_3 less semimodular lattice is the patchwork of patchwork irreducible lattices.*

Proof. Let L be a semimodular lattice of dimension 3. We may assume that L does not contains any narrow. Take the souce representation of L:
$$L = \mathcal{L}(G, \mathcal{S}).$$

Consider G as a matrix. We choose an arbitrary source element $s \in \mathcal{S}$ and the corresponding 1×1 submatrix [1]. Let M_s be a block which contains this 1×1 matrix and is a maximal augmented matrix.

Let N be an other block which is an augmented, invertible submatrix of G. Then $M_s \cap N \neq \emptyset$ implies $\mathbf{Dim}(M_s \cap N) \geq 1$.

If $\mathbf{Dim}(M_s \cap N) > 1$ then $M_s \Cap N$ or $M_s \Cup N$ exists this contradicts the maximality of M_s, i.e. $M_s \cup N = 1$. □

Theorem 7. *Every diamond-free semimodular lattice L is the patchwork of pigeon-holes.*

6. Modular patchwork irreducible lattices

Let S be a 4-cell of a planar lattice L. Replace this 4-cell by a copy of M_3, the five-element non-distributive modular lattice (with a fixed diagram). This means that we insert a new element, which is called an *eye*, into the interior of S, and this way we divide S into two new 4-cells. This way we obtain a new lattice. If $L^{\bullet}$ is obtained from L by inserting eyes one by one, then $L^{\bullet}$ is called an *anti-slimming* of L (or fattening ?). This construction works only in the planar case.

Take the four-element Boolean lattice $B \cong {C_2}^2$ and insert M_3, we get M_3. We have two 4-cells. Insert into one of this again M_3 we obtain M_4. The lattices M_n are the *modular patchwork irreducibles* in the 2-dimensional case.

In The class of 3-dimensional modular lattices we have more: $C_2 \times M_n$ and the two-dimensional projective planes (among them the Fano plane is the smallest).

Lemma 10. *Let $\mathcal{S}$ be a source on a grid G and. If $\mathcal{L}(G, \mathcal{S})$ is a modular lattice*

(1) let $s \in \mathcal{S}$ be a minimal source element then the hight of s, $h(s) = 3$,
(2) if $s' \in \mathcal{S}$ is an upper cover of s in $\mathcal{S}$ then $h(s') - h(s) = 3$

Proof. The lattices $\mathbb{S}_k$
$\mathbb{S}_k = {C_3}^k / \Delta$ and $\mathbb{B}_k = {C_2}^k / \Delta$. □

6.1. **The horisontal case: the projective geometries.** Take the finite field $\mathbf{GF}(p^n)$, $p = 2, n = 1$ the corresponding two-dimensional projective geometry $\mathbf{GP_2}$ is the *Fano-plane*. The one-dimensional lattice $\mathbf{GP_1}$ is M_3.

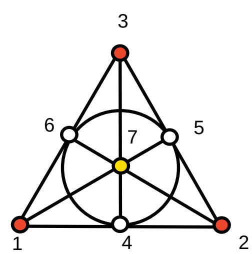

Figure 32. The Fano plane given by points and lines

On the next picture you can see the "traditional" presentation of the subspace lattice.

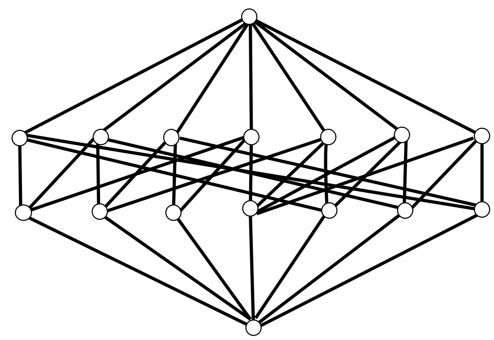

FIGURE 33. The "traditional" diagram of the subspace lattice of the Fano plane

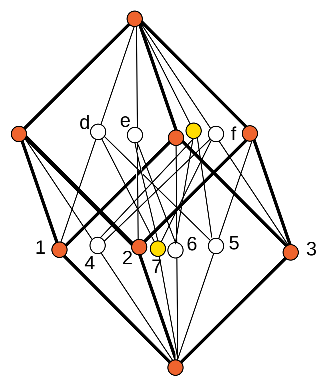

FIGURE 34. The subspace lattice of the Fano plane

Draw the diagram a little bit others we get the following diagram for the same lattice (the same, but differently).

Here we see a cube (fat lines, the skeleton) and six circles on the faces of this cube and two circles (yellow) are inside the cube. If D is the direct product of three chains then this contains unit (covering) cubes. We can extend D if we put the Fano-plane into some covering cubes. (Fano plane "locked" in a cube.) The patching of covering Fano planes in the three-dimensional case in Figure 31.

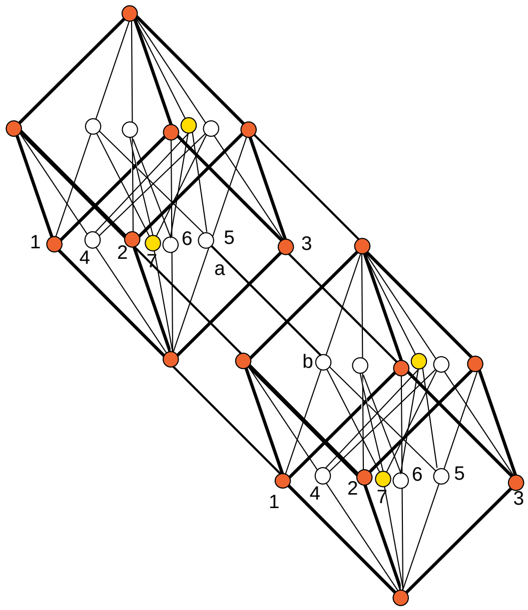

FIGURE 35. Patching of two Fano lattices and a $M_3 \times C_2$

6.2. **The vertical case: the lattices $M_3[C_n]$.**

7. PATCHWORK-IRREDUCIBLE LATTICES (SUMMARY)

dimension	name	skeleton	Jw(L)	Kuroš-Ore
2D diamond-free	$\mathcal{P}$ pigeonholes	${C_2}^2$	2	2
	$\mathcal{P}$ anti-slimmed pigeonholes	${C_2}^2$	k	2
2D modular	${C_2}^2$, M_k	${C_2}^2$	k	2
3D	$\mathcal{P}_k$ pigeonholes	${C_2}^3$	3	3
3D modular	projective geometries	${C_2}^3$	7	3
	${C_2}^3$, $M_k[C_n]$	${C_2}^3$	$k+1$	3

REFERENCES

[1] G. Czédli, E. T. Schmidt, *How to derive finite semimodular lattices from distributive lattices?*, Acta Math. Acad. Sci. Hungar.,**121** (2008), 277–282.

[2] G. Czédli, E. T. Schmidt, *Intersections of composition series in groups and slim semimodular lattices by permutations,* submitted

[3] G. Czédli and E. T. Schmidt, *Slim semimodular lattices. I. Visual approach*, Order, **29** (2012), —(DOI: 10.1007/s1083-011-9215-3)

[4] G. Czédli, E. T. Schmidt, *Slim semimodular lattices. II. A description by patchwork systems*—, manuscript

[5] G. Czédli and E. T. Schmidt, *Compositions series in groups and the structure of slim semimodular*, Advances in Mathematics, (May 14, 2011)

[6] G. Czédli, *The matrix of blim semimodolar lattices*, Order, **29** (2012), 85–103.

[7] E. Fried, G. Grätzer and E. T. Schmidt, *Multipasting of lattices*, Algebra Universalis **30** (1993), 241–261.

[8] G. Grätzer, E. Knapp, *Notes on planar semimodular lattices. I. Constructions,* Acta Sci. Math. (Szeged) **73** (2007), 445–462.

[9] C. Herrmann, *S-verklebte Summen von Verbänden.* Math. Z. **130** (1973), 255-274.

[10] M. Stern, *Semimodular Lattices. Theory and Applications.* Encyclopedia of Mathematics and its Applications, **73**. Cambridge University Press (1999)
[11] E. T. Schmidt, *Rectangular hulls of semimodular lattices,* manuscript.
[12] E. T. Schmidt, *A new look at the semimodular lattices. A collection of ideas and conjectures*, manuscript.
[13] E. T. Schmidt, *Play with matrices get a structure theorem for semimodular lattices,* manuscript.
[14] E. T. Schmidt, *A structure theorem of semimodular lattices: the patchwork representation,* manuscript.

Mathematical Institute of the Budapest University of Technology and Economics, Műegyetem rkp. 3, H-1521 Budapest, Hungary
E-mail address, E. T. Schmidt: schmidt@math.bme.hu
URL: http://www.math.bme.hu/∼schmidt/