CONGRUENCES OF 2-DIMENSIONAL SEMIMODULAR LATTICES (PROOF-BY-PICTURES VERSION)

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ABSTRACT. In this note we describe the congruences of slim semimodular lattices, i.e. of 2-dimensional semimodular lattices. It was proved by G. Czédli and E. T. Schmidt [1] the following: (for planar semimodular lattices see G. Grätzer, E. Knapp [7] [8]): each finite semimodular lattice L is a cover-preserving join-homomor-phic image of a distributive lattice G which is the direct product of finite chains. We use this theorem to get an overview on the congruences.

1. INTRODUCTION

1.1. **Results.** G. Grätzer, H. Lakser, and E. T. Schmidt [6] proved that every finite distributive lattice is the congruence lattice of a planar semimodular lattice. If a planar semimodular lattice L doesn't contain a M_3 as sublattice, i.e. it is diamond-free then width of $\mathbf{J}(L)$ $(\mathrm{w}(\mathbf{J}(L)))$ is two, we say that the dimension of L is 2. (M_3 has dimension 3.) In this case we can't represent all finite distributive lattices. We characterize the congruence lattices of these lattices, they have an interesting description with special posets: the posets which looks like the poset presented in Figure 2. First we take a poset, see in Figure 1 which we call call *zigzag*. A zigzag is a sequence $a_1, a_2, ..., a_n$ such that for the successive members, a_i, a_{i+1} either $a_i < a_{i+1}$ or $a_i > a_{i+1}$. We extend such a poset defined in Figure 2.

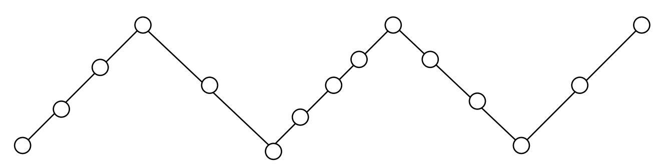

FIGURE 1. A zigzag poset

Theorem 1. *The congruence lattice of a 2-dimensional semimodular lattice L is isomorphic to*

$$(D \dot{+} {C_2}^2) \times C_2^k,$$

where D is the direct product of lattices D_i with the property that

(P) the poset of join-irreducible elements $\mathbf{J}(D_i)$ is an extended zigzag.

In [4] G. Czédli and E. T. Schmidt introduced the concept of patch lattice.

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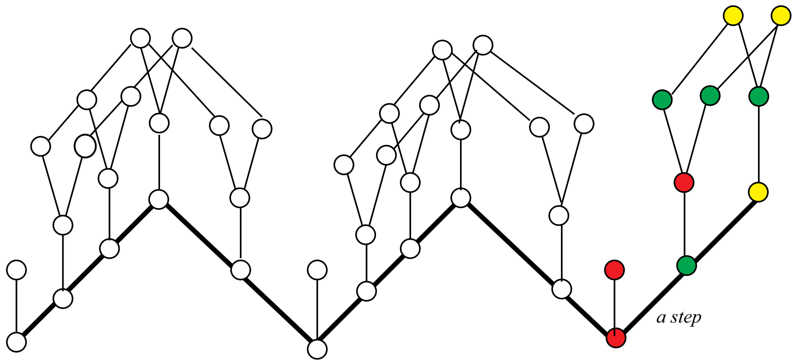

FIGURE 2. An extended zigzag

Definition 1. *A slim semimodular lattice (i.e. two dimensional semimodular lattice) L is a patch lattice if it contains two dual atoms p and q such that their intersection $p \wedge q$ is zero.*

The lattices N_7 and ${C_2}^2$ are patch lattices. $\{0, .p, q, 1\}$ is a sublattice, the *skeleton*, Skeleton(L) of P.

If P is a patch lattice then $k = 0$ in Theorem 1:

Theorem 2. *The congruence lattice of a patch lattice L is isomorphic to*

$$(D\dot{+}{C_2}^2),$$

where D is the direct product of lattices D_i satisfying the property (P).

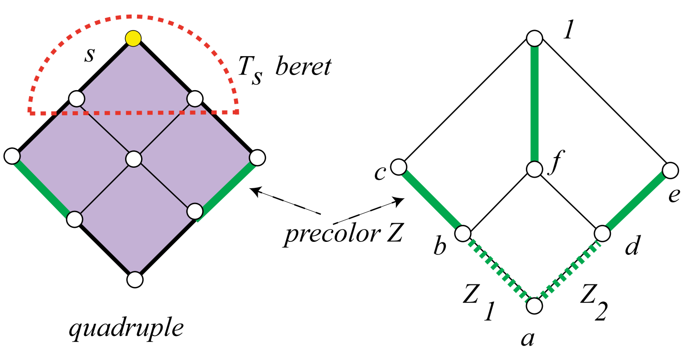

FIGURE 3. The quadruple Q_s is isomorphic to $C_2 \times C_2$ and N_7 as the factor Q_s/T_s

For the notations Z, Z_1, Z_2 in Figure 3. N_7 has three join-irreducible congruence relations, Z, Z_1, Z_2 where Z has the congruence classes $\{c, b\}, \{d, e\}, \{f, 1\}, \{a\}$ the Z_1- classes are: $\{c, b\}, \{d, e\}, \{f, 1\}, \{a\}$ and the Z_2 classes are $\{c, b\}, \{d, e\}, \{f, 1\}, \{a\}$.

As **example** take the lattice F given in Figure 4. This is not a patch lattice.

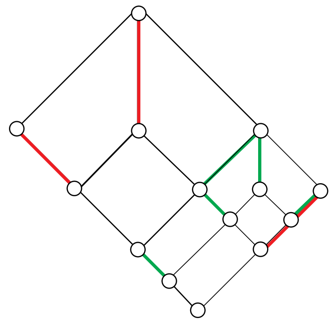

FIGURE 4. A semimodular lattice F with two copies of N_7

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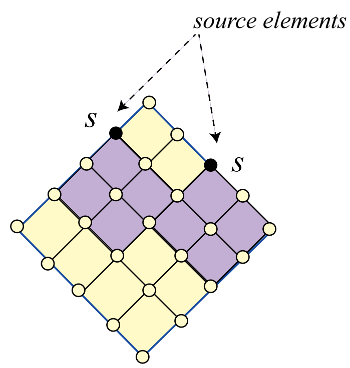

FIGURE 5. The lattice F given with the source elements

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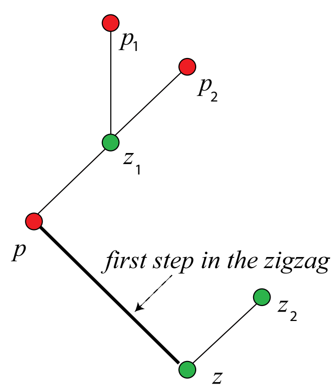

FIGURE 6. Stitched together, $\mathbf{J}(\mathrm{Con}(F))$

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2. PRELIMINARIES

2.1. **cover-preserving join-homomorphism.** The N_7 is a cover-preserving join-homomorphic image of ${C_3}^2$ (Figure 3). If s is a source element of a grid G then the cover-preserving sublattice of G with the top element s is called the *quadruple* with the beret T_s of s, if this is isomorphic to $Q_s = C_3 \times C_3$. Then $N_7 \cong Q_s/T_s$.

G is called a *grid* of L. Let L be a 2-dimensional semimodular lattice. Then we have a grid $G = {C_n}^2$ and a cover-preserving join-congruence Φ such that $L \cong G/\Phi$. This means that L is determined by the pair (G, Φ).

Lemma 1. *Let $\mathcal{L}$ be a planar semimodular lattice. Then the following conditions are equivalent:*

(1) no three join-irreducible elements of L form an antichain, i.e. L is slim,
(2) L is a planar semimodudular lattice, which is M_3-free, i.e. has no eys (the interior element of an interval of length two and has four elements).

Adding a fork to a lattice L at the covering square K, firstly, replace K by a copy of N_7.

Secondly if there is a chain $u \prec v \prec w$ such that the element v has just been added $T = \{x = u \wedge z, z, u, w = z \vee u\}$ is a covering square in the lattice L but $x \prec z$ at the present stage of of the construction, then we insert a new element y such that $x \prec y \prec z$ and $y \prec$. The new elements form an order, called a *fork*. We say that K is obtained from L by adding a fork to L at the covering square S.

The matrix of a slim semimodular lattice is a $(0, 1)$-matrix where every row/column contains at most one non zero entry. As matix we consider G itself (if you want rotete by 45 degree).

Lemma 2. *Let L be a slim semimodular lattice with at least four elements of length n. Then the following conditions are equivalent:*

(1) P is a patch lattice,
(2) P is gluing indecomposable,
(3) P is gluing indecomposable over chains,
(4) P can be obtained from the four element Boolean lattice by a sequence of insertations of forks,
(5) **the matrix of P is invertible $n \times n$-matrix***,* [10].

Supplement. P is determined by a permutation φ of the n-element set $\{1, 2, ..., n\}$. Every 2-dimensional semimodular lattice has a smallest (lattice) homomorphic image which is a patch lattice. Every 2-dimensional semimodular lattice has a maximal sublattice which is a patch lattice.

Definition 2. *(E. T. Schmidt* [9]*) An element $s \in G$ is called a* source element *of Θ if there is a $t, t \prec s$ such that $s \equiv t$ (Θ) and for every prime quotient u/v if $s/t \searrow u/v, s \neq u$ imply $u \not\equiv v$ (Θ). The set $\mathcal{S}_\Theta$ of all source elements of Θ is the* source *of Θ.*

Lemma 3. *Let x be an arbitrary lower cover of a source element s of Θ. Then $x \equiv s$ (Θ). If $s/x \searrow v/z$, $s \neq v$, then $v \not\equiv z$ (Θ).*

It is important to remark that we work not with L but with the (G, S) pair. The structure of the congruences become more transparent. L is determined by the pair (G, S). The (G, S) is an outspread map of L.

By condition (5) of Lemma 2 a semimodular lattice is a patch lattice iff every row/column of the grid contains exactly one source element (except the last row/column).

2.2. **Rectangular semimodular lattices.** L is rectangular if $\mathbf{J}(L)$ is the disjoint sum of two chains. Patch lattice is a rectangular slim lattice (2-dimensional semimodular lattice) which contains the most possible source elements (if the length of L is n then the number of source elements is $n-1$).

2.3. **Coloring.** A colored finite lattice L is a lattice whose prime intervals are labeled so that if the prime intervals p, q are of the same color then $\mathrm{con}(p) = \mathrm{con}(q)$. Coloring helps to describe the congruences. If p has the color c then we define $\mathrm{con}(c) = \mathrm{con}(p)$. Sometimes we don't color all prime intervals of L.

We color the vertices (the elements of L) if L denotes a congruence lattice and some vertices are related to a join-irreducible congruence, see in Figure 2 and Figure 6.

3. Proof-by-Pictures

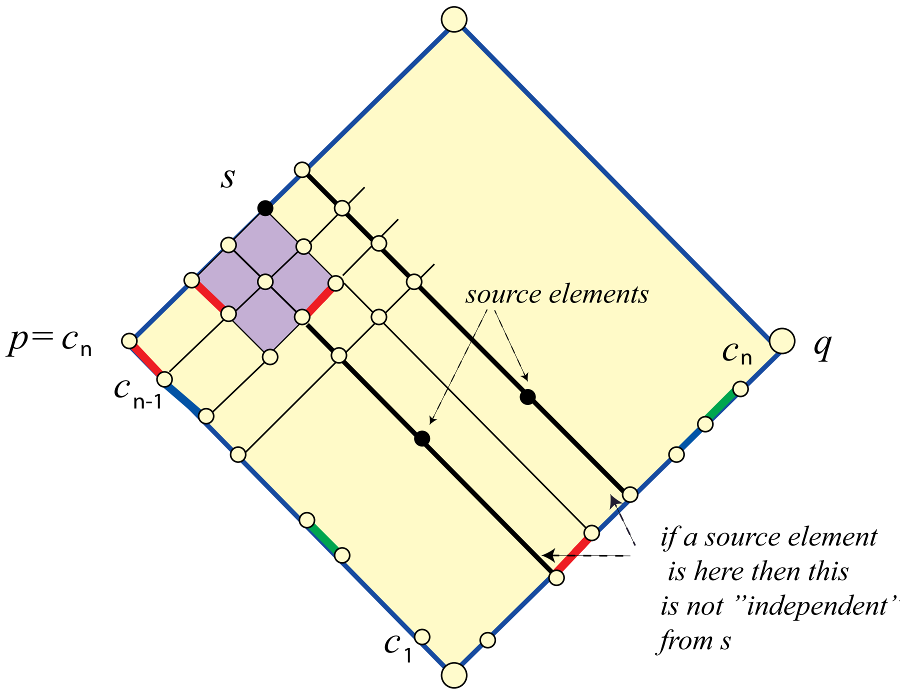

Figure 7. Outspread map of a 2-dimensional semimodular lattice

3.1. **The spreading of congruences.** Let Θ be a congruence relation of P and let a, b, c, d be elements of this lattice If $a \equiv b\ (\Theta)$ congruences on implies that $c \equiv d\ (\Theta)$ then we say that $a \equiv b$ *forces* $c \equiv d$, see Figure xx.

In Figure 5 you can see how does it looks like a join-irreducible congruence of the grid G and the grid with a source element. Every source element determine a source lattice, in the two-dimensional case this is a copy of N_7. Then N_7 has thre join-irreducible congruences, see Figure 4: the green color belongs to Z, the colors of Z_1, Z_2, the "shadows" are dotted green lines.

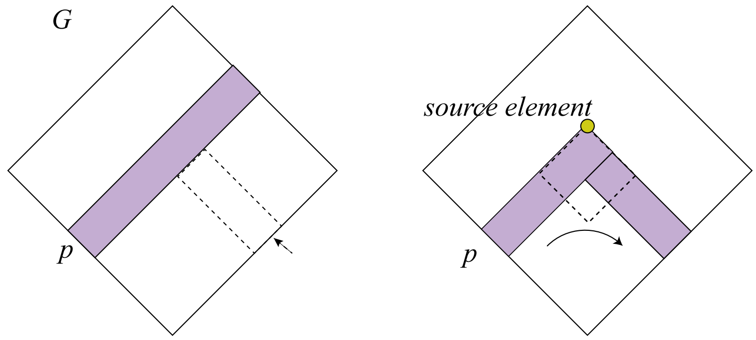

FIGURE 8. con(p) in the grid and with a source element (on the right side)

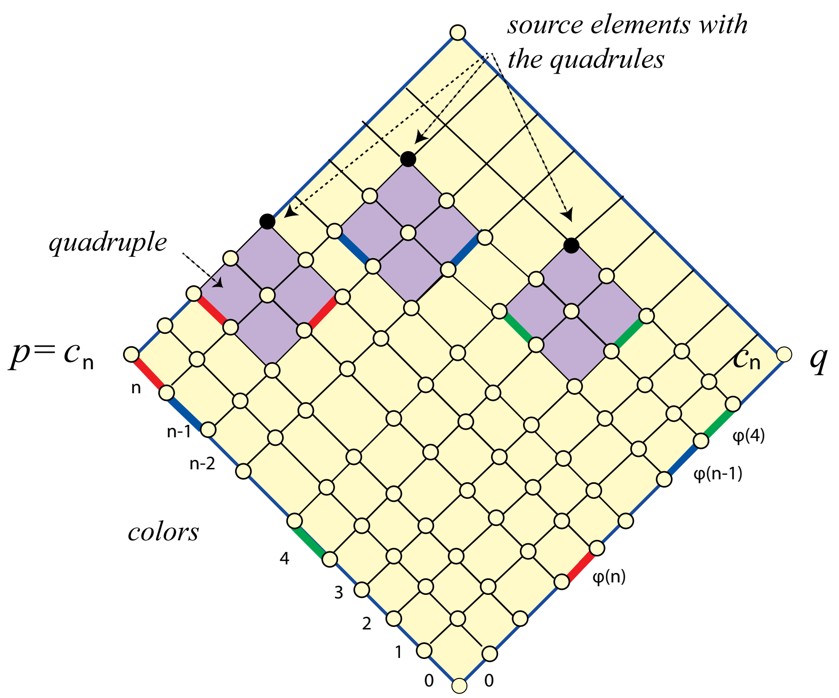

FIGURE 9. Outspread map of a 2-dimensional semimodular lattice

3.2. **Adjacent source elements.** In the graph theory a vertex w is said to be adjacent to other vertex v if the graph contains an edge (v, w). In this case we write $v \sim w$. This is the adjacent relation. The grid G is a graph, therefore we can use $\sim$ in the grid. If s_1 and s_2 are source element then there are not adjacent, i.e. $s_1 \not\sim s_2$. Therefore we use a weaker form of adjacency.

Two source elemens $s_1 = (x, y, {}_2 = (z, u)$ are

(1) *left adjacent* if $z = x + 1, u < y$ and *right adjacent* if $y = u + 1, z < x$.
(2) horosontaly $(1,1)$-*adjacent* source elements if $s_1 \prec s_1 \vee s_2$ and $s_2 \prec s_1 \vee s_2$: the two colors represent the same congruence, we merge the colors.

(3) verticaly $(1,1)$-*adjacent* source elements if $s_1 \prec u$ and $u \prec s_2$ for some u: the two colors represent represent different congruences,
(4) $(1,2)$-*adjacent* source elements if Knight jumpif $s_1 \prec s_1 \vee s_2$ and there is a grid element g such that $s_2 \prec g \prec s_1 \vee s_2$ there is an ordering, red < green, i.e. the same holds for the congruences,
(5) $(2,1)$-*adjacent* source elements ifthere is an ordering, red < green, i.e. the same holds for the congruences,

Two source elements are *remote* if the distans (in the graf G) is more then two. Figure 4 presents the grid and the poset of a 2-dimensional lattice P. To every source element s belongs a quadruple. We color these by different colors (we don't not color all prime intervals, only the most important intervals). As colors we can use the numbers $1, 2, , ..., n$. We consider these colors as *precolors*, later we will change some of them. p and are the q special duual atoms of P with the propertry $p \wedge q = 0$. Then p and q can be considered as elements of G too $(p = (c_n, 0), q = (0, c_n))$. The principal ideals $(p]$ and $(q]$ are chains and prime ideals. The projections $p \wedge x$ resp. $q \wedge y$ define a precoloring of these chains. In Figure 4 we have a source element with quadruples . We color these by red, green and blue.

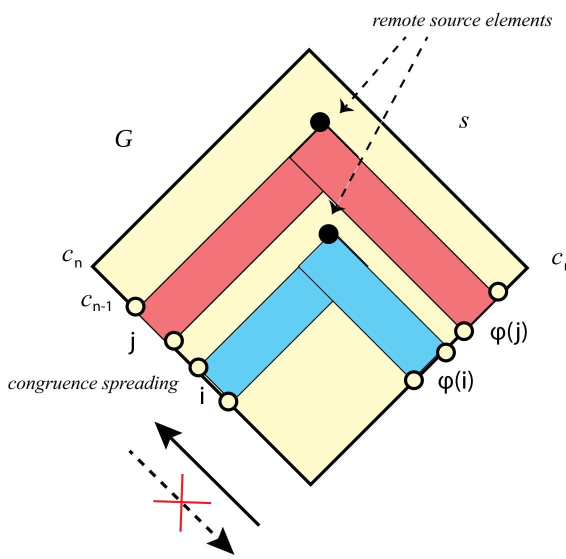

FIGURE 10. The order-preserving (vertical) case

3.3. **Some remarks on coloring.** The conrguence lattice of N_7 is isomorphic to $C_2 \dot{+} {C_2}^2$. In Figure 3 $a \equiv b$ forces $b \equiv c$ (but $b \equiv c$ don't forces $a \equiv b$). We color the congruence $\mathrm{con}(b, c)$, in the given example by green, i. e. we color the intervals $[b, c], [d, e], [f, 1]$.

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3.4. **Congruences.** The patch lattice P has two special congruence relations, α and β. These are dual atoms of the congruence lattice and are generated bt the primeintervals $[0, u], [0, v]$.

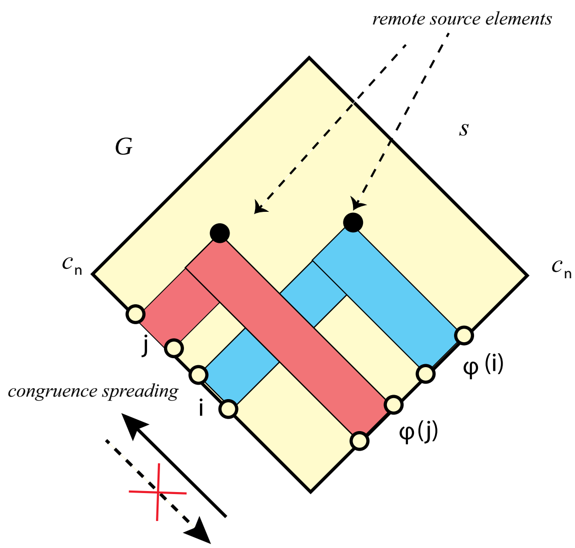

FIGURE 11. The horizontal case: crossroads

To prove the Theorem 1 is enough to see that every congruence relation generated by a prime interval is an atom. Distributive lattice have this property, con(p) is an atom. The proof is easy, we have the situation presented in Figure 4.

If the two source elements are $(1, 1)$-adjacent then the two correpending prime intervals generate the same congruence relation.

$\mathcal{C}$ is the set of the colors, this is a facor of the chain $1 < 2 < ... < n$, factorised by $\sim$.

In Figure 9 we have a semimodular lattice given by a grid and source and then two congruence relations which are atoms of the congruence lattice.

In the colored part of Figure 10 all prim intervals generate the same congruence relation, an atom of the congruence lattice.

3.5. **The proof of the Theorem 1.** Let L be a 2-dimensional semimodular lattice. By a theorem of G. Grätzer, E. Knapp, [8] LN has a congruence-isomorf extension to a rectangular lattice. That means we may assume that L is rectangular. Every congruence relation Θ is determied by its restriction to $\mathbf{J}(L) \cup \{0\}$. Th elements of $\mathbf{J}(L) \cup \{0\}$ are: $(c_i, 0), (0, c_j)$. The prime intervals $\mathbf{p} = (c_{i+1}, 0), (c_i, 0)$ and $(0, c_{i+1}), (0, c_i))$ generate the join-irreducinle congruences con($\mathbf{p}$) of G. We **color** the prime intervals of G on the usual way. The congruence lattice of G is isomorphic to ${C_2}^{2n}$.

L is given by the pair (G, S), we consider G as a colored lattice. We define an equvivalence relation $\sim$ on the source S. Two source elements s_1 and s_2 ae in the same $\sim$ -class if and only if there is knight walk, $(1, k)$-adjacent sequence $k \leq 2$ between these elements (Figure 10). Posets determined by different $\sim$- classes are independet, i.e. we have direct product.

G contains the set S. In every row/column we have at most one source element, if in a row/column there is no source element we say it The last row/columns are always empty !is empty. This means that these rows/columns determine congruences of L, see in Figure 2. The corresponding colors are $p_i, ..., p_k$. Let Φ be the

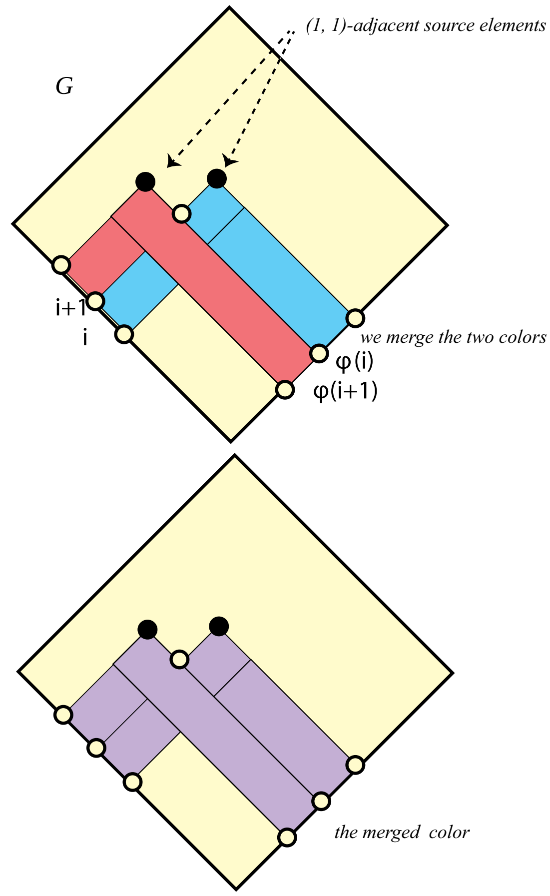

FIGURE 12. 9Merging of colors by $(1, 1)$-adjacent source elements

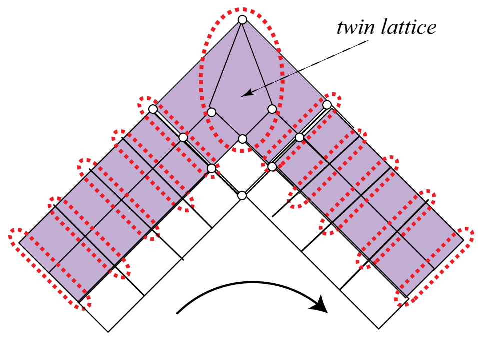

FIGURE 13. Congruence spreading via a twin lattice, change of direction

join of all these congruences. Then $L/|\Phi$ is a patch lattice. Φ is the second direct factor$(D\dot{+}{C_2}^2) \times C_2^k$ of the formula in Theorem 1.

We may assume that L is a patch lattice. Take $S = \{s_1, s_2, ..., s_n\}$ such that s_i is in i-th row. Letk_1 be the smallest natural number such that in the series

$s_1, s_2, ..., s_{k_1}$ the neiburth are adjacent, but $s_{k_i}, s_{k_{i+1}}$ is a remote pair. Then these sequence is the first zigzag Z_1. This determine a distributive Lattice D_1., e Figure 8. we continue this procedure, $s_{k_{i+1}}, ..., s_2, ..., s_{k_1}$

Z_2 gives D_2, and D is the direct product of these lattices.

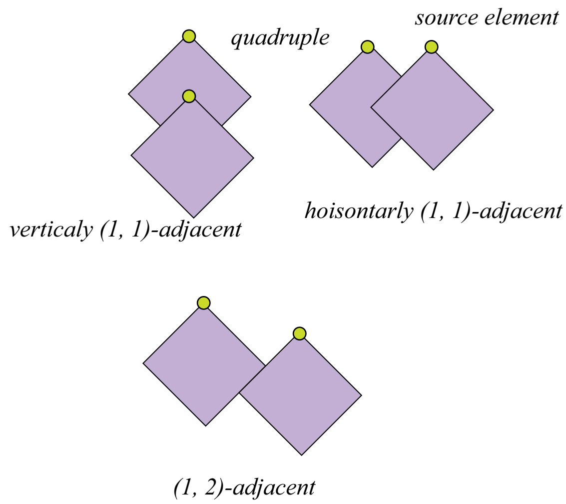

Figure 14. Adjacent pairs

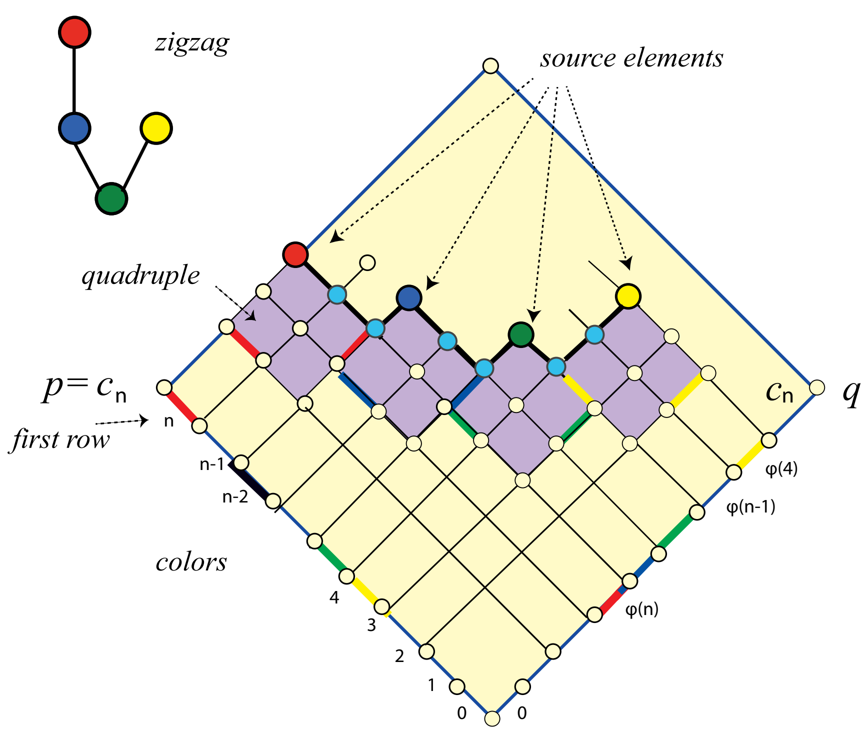

Figure 15. $(1, 2)$-adjacent sequence,

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