A short proof of the congruence representation theorem for semimodular lattices

G. Grätzer and E. T. Schmidt

Abstract. In a 1998 paper with H. Lakser, the authors proved that every finite distributive lattice D can be represented as the congruence lattice of a finite *semimodular lattice*.

Some ten years later, the first author and E. Knapp proved a much stronger result, proving the representation theorem for *rectangular lattices*.

In this note we present a short proof of these results.

1. Introduction

In [5], the authors with H. Lakser proved the following result:

Theorem 1. *Let D be a finite distributive lattice. Then there is a* planar semimodular lattice K *such that*

$$D \cong \operatorname{Con} K.$$

A stronger result was proved some 10 years later. To state it, we need a few concepts.

Let A be a planar lattice. A *left corner* (resp., *right corner*) of the lattice A is a doubly-irreducible element in $A - \{0, 1\}$ on the left (resp., right) boundary of A.

We define a *rectangular lattice* L, as in G. Grätzer and E. Knapp [4], as a planar semimodular lattice that has exactly one left corner, u_l, and exactly one right corner, u_r, and they are complementary—that is, $u_l \vee u_r = 1$ and $u_l \wedge u_r = 0$.

The first author and E. Knapp [4] proved the following much stronger form of Theorem 1:

Theorem 2. *Let D be a finite distributive lattice. Then there is a* rectangular lattice K *such that*

$$D \cong \operatorname{Con} K.$$

In this note we present a short proof of this result.

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2. Notation

We use the standard notation, see [3].

For a rectangular lattice L, we use the notation $C_{\mathrm{ll}} = \mathrm{id}(u_l)$, $C_{\mathrm{ul}} = \mathrm{fil}(u_l)$, $C_{\mathrm{lr}} = \mathrm{id}(u_r)$, $C_{\mathrm{ur}} = \mathrm{fil}(u_r)$ for the four boundary chains; if we have to specify the lattice L, we write $C_{\mathrm{ll}}(L)$, and so on. (See G. Czédli and G. Grätzer [1] for a survey of semimodular lattices, in general, and rectangular lattices, in particular.)

3. Proof

Let D be the finite distributive lattice of Theorem 2. Let $P = \mathrm{Ji}\, D$. Let n be the number of elements in P and e the number of coverings in P.

We shall construct a rectangular lattice K representing D by induction on e. Let $m_i \prec n_i$, for $1 \leq i \leq e$, list all coverings of P. Let P_j, for $0 \leq j \leq e$, be the order we get from P by removing the coverings $m_i \prec n_i$ for $j < i \leq e$. Then P_0 is an antichain and $P_e = P$.

For all $0 \leq i \leq e$, we construct a rectangular lattice K_i inductively. Let $K_0 = \mathsf{C}_{n+1}^2$ be a grid, in which we replace the covering squares of the main diagonal by covering M_3-s; see Figure 1 for $n = 3$. Clearly, this lattice is rectangular and $\mathrm{Con}\, K_0$ is the boolean lattice with n atoms.

Now assume that K_{i-1} has been constructed. Let the three-element chain $0 \prec m_i \prec n_i$ be represented by the lattice S_8, see Figure 1.

Take the four lattices

$$\mathsf{S}_8,\ K_{i-1},\ \mathsf{C}_3 \times C_{\mathrm{ul}}(K_{i-1}), C_{\mathrm{ur}}(K_{i-1}) \times \mathsf{C}_3$$

and put them together as in Figure 2, where we sketch K_{i-1} for $n \geq 3$ and $3 < i \leq e$. We add two more elements to turn two covering squares into covering M_3-s, see Figure 2, so that the prime interval of S_8 marked by m defines the same congruence as the prime interval of K_{i-1} marked by m; and the same for n. Let K_i be the lattice we obtain. The reader should have no trouble to directly verify that K_i is a rectangular lattice. (See G. Czédli and G. Grätzer [1] for general techniques that could be employed.)

The lattice K for Theorem 2 is the lattice K_e.

See G. Grätzer [2] for a comparison how this short proof compares to the proofs in G. Grätzer, H. Lakser, and E. T. Schmidt [5] and in G. Grätzer and E. Knapp [4].

References

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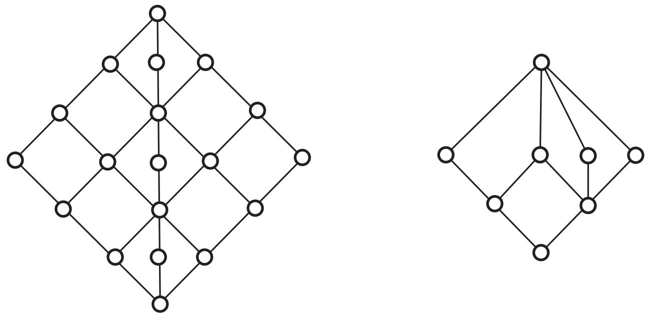

Figure 1. The M_3-grid for $n = 3$ and the lattice S_8

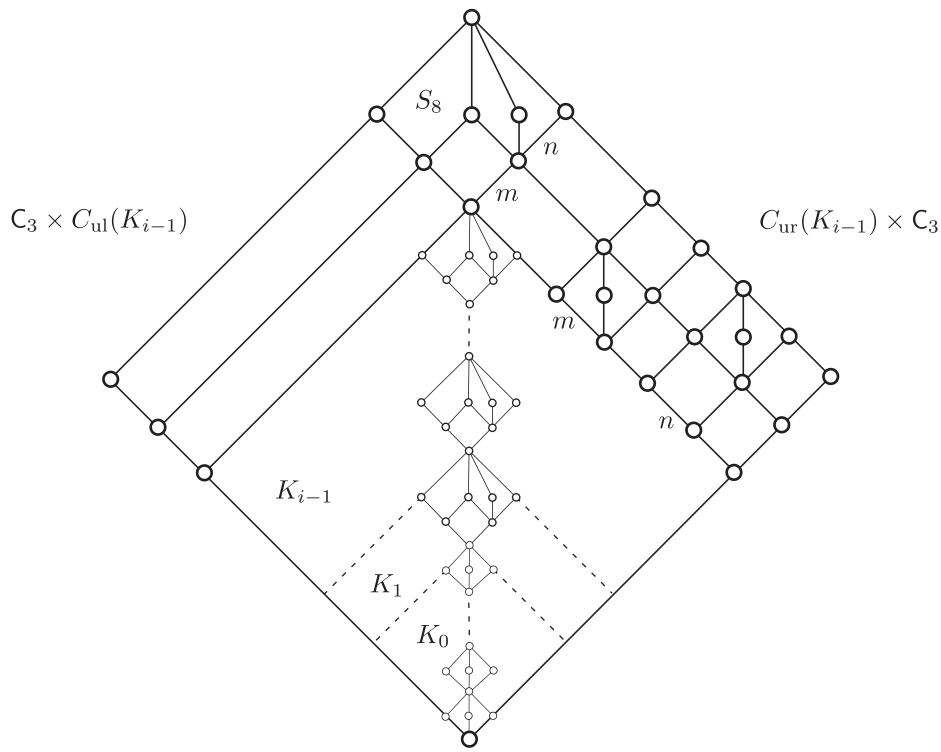

Figure 2. A sketch of the lattice K_i for $n \geq 3$ and $3 < i \leq e$

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