A NEW LOOK AT THE SEMIMODULAR LATTICES
A GEOMETRIC APPROACH
(NOT FINISHED MANUSCRIPT)

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Abstract. This paper is an experiment, how it is possible to treat semimodular lattices as geometric shapes. In 2009 I published a paper with *Gábor Czédli*, [4], we proved that every semimodular lattice L can be obtained from a direct power of a chain $G = C^n$ – geometrically a cube – on an easy way. L is the cover-preserving join-homomorphism of G.

We introduce the concept of rectangular semimodular lattices and prove that the building stones of semimodular lattices are special rectangulars, the **pigeonholes**. The building tool is a special S-verklebte sum (introduced by *Christian Herrmann* [21]) this is the **patchwork**.

This manuscript includes parts of the papers [2], [3], [4], [5], [6], [7], [8], [9], [10] and [11].

Contents

Date: January 30, 2016.

PART I

Cover-preserving join-congruences

Part 1. Cover-preserving join-congruences

1. Preliminaries

This paper is a geometric approach to finite semimodular lattices. We consider the diagrams of semimodular lattices as geometric shapes. In this approach the eight element boolean lattice is a cube.

Let L and K be finite lattices. A join-homomorphism $\varphi : L \to K$ is said to be *cover-preserving* iff it preserves the relation $\preceq$. Similarly, a join-congruence Φ of L is called cover-preserving if the natural join-homomorphism $L \to L/\Phi$, $x \mapsto [x]\Phi$ is cover-preserving. $\mathrm{J}(L)$ denotes the order of all nonzero join-irreducible elements of L and $\mathrm{J}_0(L)$ is $\mathrm{J}(L) \cup \{0\}$.

The concept of the *dimension* of a semimodular lattice is a sensitive step in this paper. There are different possibilities. The *width* $\mathrm{w}(P)$ of a (finite) order P is defined to be $\max\{n\colon P$ has an n-element antichain$\}$. If L is a lattice the number $k = \mathrm{w}(\mathrm{J}(L))$ is called the J-*width* of L. This will be denoted by $\mathbf{dim}(L)$. In virtue of Dilworth [12], P is the union of k appropriate chains. This concept is a kind of geometric *dimension*. The Kurosh-Ore dimension $\mathbf{Dim}(L))$ of L is the minimal number of join-irreducibles needed to span the unit element of L.

The dimension of a finite semimodular lattice L is the greatest natural number n such that there is an interval I with the Kuros-Ore dimension $n = \mathbf{Dim}(I)$.

It was proved by G. Czédli and E. T. Schmidt [4], (for planar semimodular lattices see G. Grätzer, E. Knapp [15], [16],[17],[18]):

Theorem 1. *Each finite semimodular lattice L is a cover-preserving join-homomorphic image of a distributive lattice G which is the direct product of* $\mathbf{dim}(L)$ *finite chains.*

Remark. The theorem was proved originally (and independently) in the first edition of M. Stern, [28] (Th6.3.14, p. 240).

G can be interpreted in two different ways:

(1) G is a **grid** (i.e. a coordinate system),

(2) G is a geometric **rectangle** (other names: a n-dimensional cube, a shape or cuboid).

By Theorem 1 we can get every semimodular lattice L from a rectangle, using as tool a cover-preserving join homomorphism. We "carve" the semimodular lattice L from G.

In [4] we proved an other similar theorem:

Theorem 2. *Every finite semimodular lattice L is a cover-preserving join-homomorphic image of the unique distributive lattice D determined by* $\mathrm{J}(D) \cong \mathrm{J}(L)$. *Moreover, the restriction of an appropriate cover-preserving join-homomorphism from D onto L is a* $\mathrm{J}(D) \to \mathrm{J}(L)$ *order isomorphism.*

From the proof of Theorem 2 we can see that $F = \mathrm{H}(\mathrm{J}(L))$, which is the distributive lattice determined by the order $P = \mathrm{J}(L)$ and the poset $\mathrm{H}(P)$ denotes the lattice of all downsets of the order P.

A subset $\{a, b, a \wedge b, a \vee b\}$ of a semimodular lattice is called a *covering square* if $a \wedge b \prec a$ and $a \wedge b \prec b$. The semimodularity implies $a \prec a \vee b$ and $b \prec a \vee b$.

If we have a grid $G = {C_k}^n$ ($C_k = \{1, 2, ..., k\}$ is the chain of natural numbers). then we will denote its elements as vectors $g = (x_1, x_2, ..., x_n)$.

We need the following lemma [4] , which characterize the cover-preserving join-homomorphisms:

Lemma 1. (Covering square lemma.) *Let Φ be a join-congruence of a finite semimodular lattice M. Then Φ is cover-preserving if and only if for any covering square $S = \{a \wedge b, a, b, a \vee b\}$ if $a \wedge b \not\equiv a$ (Φ) and $a \wedge b \not\equiv b$ (Φ) then*

$$a \equiv a \vee b \ (\Phi) \text{ implies } b \equiv a \vee b \ (\Phi).$$

This lemma states: a join-congruence of a finite semimodular lattice M is cover-preserving if and only if its restriction to any covering square is a cover-preserving join-congruence.

$\mathfrak{l}(L)$ denotes the length of L. Let a/b and c/d prime quotients of a lattice L. If $b \vee c = a$, $b \wedge c = d$ we say that a/b is perspective up to c/d and we write $a/b \searrow c/d$.

2. The heuristic approach

2.1. **Distributive lattices.** First, we consider the simplest case. A planar distributive lattice D is the cover-preserving sublattice of a direct product of two finite chains, which is geometrically a rectangle. On Figure 1 you can see a typical example (D is the shaded part), this lattice contains three maximal rectangles $A = [c, 1]$, $B = [a, d]$ and $C = [0, b]$ which are glued together. The intersection (which is a rectangle) $A \cap B$ is a dual ideal of $A1$ and an ideal of B. *Every planar distributive lattice D can be covered by rectangles, that is, in other words it is a "glued system" of rectangles.* This gluing is a special case of the S-verklebte sum, introduced by Christian Herrmann [21] for modular lattices. We can do this in various ways such as with largest rectangular components (on Figure 1 there are three components). If we cut up the lattice defined on Figure 1 into smaller pieces, we may assume that the intersection of two "small" rectangles is either empty or is a part of a side of the rectangular components. The smallest rectangles are obviously the unit squares. This means a planar distributive lattice D looks like the mosaic pavement in the kitchen, the only one difference is that on some places we may have "degenerate" unit squares (blocks) which are unit sections or one element (see Figure 2). On Figure 1. 17 blocks $M_1, ..., M_{17}$. This covering has the following special property: if $M_i \cap M_j \neq \emptyset, i \neq j$, then the union, $M_i \cup M_j$ is the Hall-Dilworth gluing via an edge or corner, i.e. the "dimension" of $M_i \cap M_j$ is smaller then the "dimension" of M_i and M_j. In $G = 3 \times 3$ on Figure 2 we have four blocks $M_1, ..., M_4$. If we take the cover-preserving join-homomorphic image, S_7, then we have again four mosaics $M_1, ..., M_4$, but M_4 is in this case a degenerate mosaic, the edge $\overline{c, 1}$. A *1-narrows* of L is an element $a \in L - \{0, 1\}$ such that $L = \downarrow a \cup \uparrow a$. A *2-narrows* of L is a priminterval $[u, v]$ such that u is meet-irreducible and v is join-irreducible. If the planar distributive lattice D has no n-narrows, $n = 1, 2$ we say that D is *narrows free.*

Every finite distributive lattice L (not only the planar) can be considered as a shape which we obtain from cover-preserving cubes (bricks) applying gluing by faces. We will work with such coverings by semimodular lattices and we will call them *patchwork.* The patchwork was introduced for planar semimodular lattices in [10].

We will define special semimodular lattics the *pigeonholes*. In the class of finite distributive lattices these are the Boolean lattices. Using these terminologies we can say:

Every finite distributive lattice is the patchwork of pigeonholes.

Our goal to extend these trivial statment for all semimodular lattices. As a non-distributive example let us take the lattice (patchwork [10]) on Figure 3.

Let us take again Figure 1. If we start with D (the shaded part) then we glue to this further rectangulars to E, F and G to get a "narrowest big" rectangle. This is a rectangle hull of D. We extend these trivial statement for all semimodular lattices.

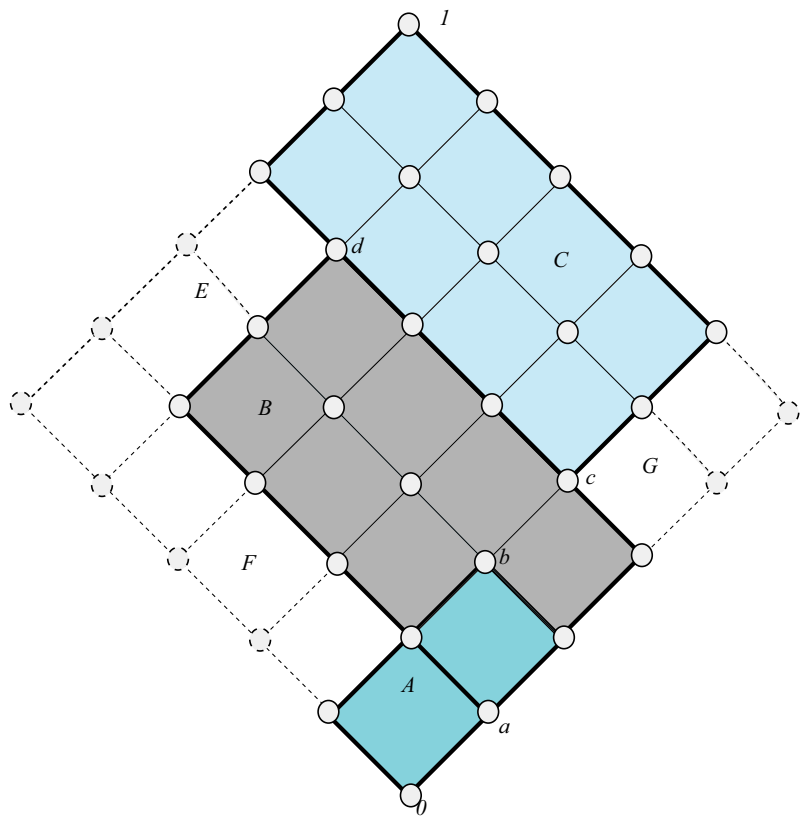

FIGURE 1. A "tipical" planar distributive lattice D (the shadded part)

2.2. **The breakdown procedure.** There is an other way to represent a finite distributive lattice L, this is the *breakdown procedure*: we cut out (carve) from the "big" cuboid $E = \{0, c_5, e_3, 1\}$ (we call *envelop* distributive lattice is the cover-preserving sublattice of a rectangle (cuboid) R such that $\mathrm{Jw}(R) = \mathrm{Jw}(L)$. In the planar case, see Figure 1, we cut out S_1 and S_2. It is easy to define the *corner* and other geometric concepts for arbitrary finite distributive lattice which was defined originally in planar case in [17].

Our goal is to to give "similar" representations for all semimodular lattices. We ask, which properties of special gluings are inherited for semimodular lattices?

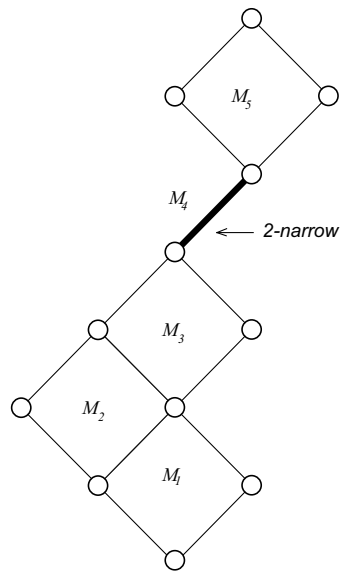

FIGURE 2. A planar distributive lattice with a narrow

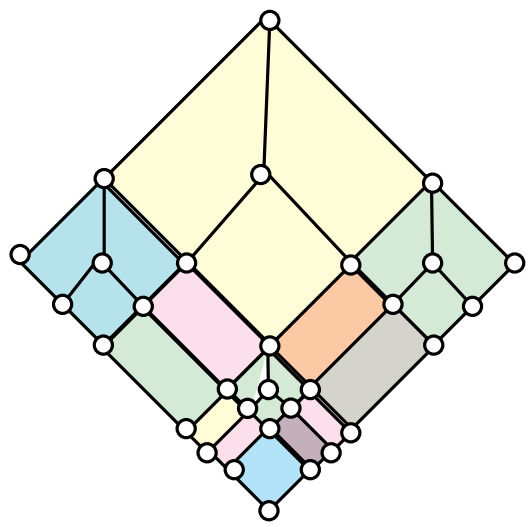

FIGURE 3. A planar semimodular lattice: a patchwork system

First, in section 3 we define distributive lattices which are associated to a semimodular lattice.

2.3. **Minimal cover-preserving join-congruences.** Let s, t be elements of a semimodular lattice L. Then $\mathrm{con}^{\vee\mathrm{cp}}(s,t)$ denotes the smallest cover-preserving join-congruence α of L where $s \equiv t\ (\alpha)$. Let $t \prec s$ be a covering pair. It is clear that for a covering pair $d \prec c$, $c \equiv d\ (\alpha)$ if and only if there is an lower cover u of s such that c/d is perspective up to s/u. That means α is determined by the element

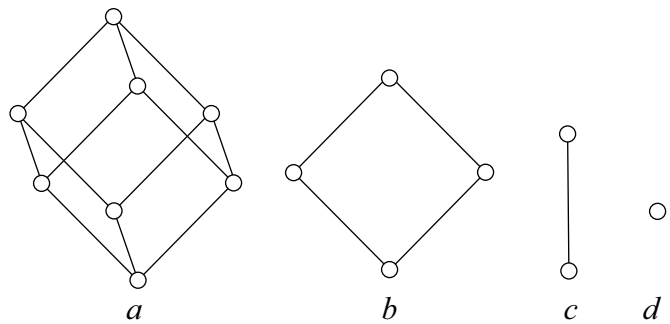

FIGURE 4. The building stones of distributive lattices in the 3-dimensional case)

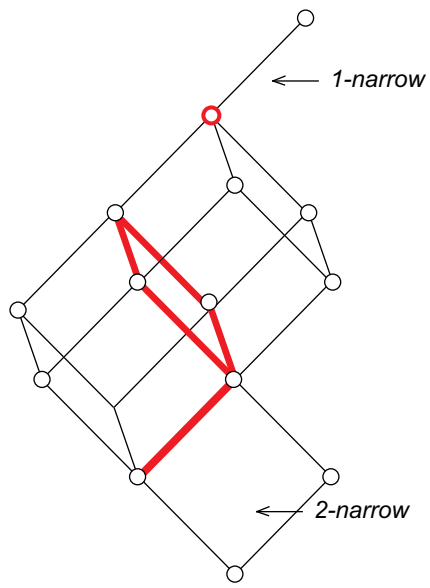

FIGURE 5. A distibutive lattice D, $\mathbf{dim}(D) = 3$

s and we write:

$$\mathrm{con}^{\vee\mathrm{cp}}(s) := \mathrm{con}^{\vee\mathrm{cp}}(s,t)$$

On Figure 7 you can see the planar case. Obviously,

$$\mathrm{con}^{\vee\mathrm{cp}}(s) = \bigvee_{t_i \prec s} (\mathrm{con}^{\vee}(s,t_i)).$$

where $\mathrm{con}^{\vee}(s,t)$ denotes the smallest join-congruence α of L where $s \equiv t \ (\alpha)$.

If we have three chains the $\mathrm{con}^{\vee\mathrm{cp}}(s)$ is given on Figure 8.

3. Distributive lattices associated to a semimodular lattice

Let S be a semimodular lattice. By Theorem 1 and Theorem 2 there are different distributive lattices such that S is the cover-preserving join-homomorphic image of

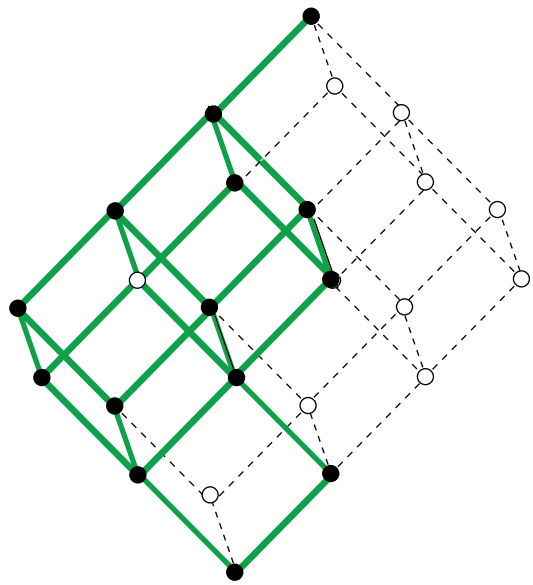

FIGURE 6. L in a rectangular box (envelop) E (Fig5.eps)

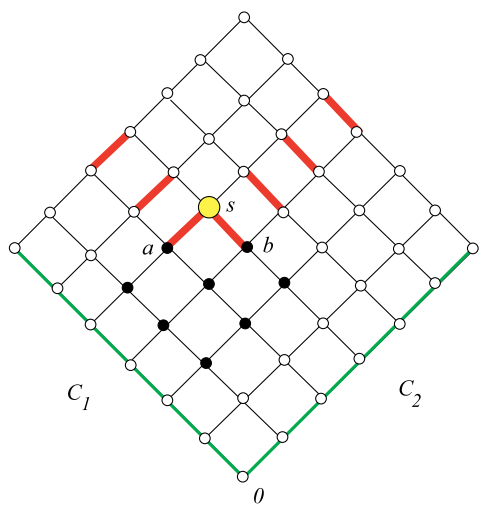

FIGURE 7. $\mathrm{con}^{\vee\mathrm{cp}}(s)$ in a direct product of two chains

these distributive lattices. How can we get these lattices and what is the connection between these lattices?

3.1. **The grids and the frame.** Let S be a finite semimodular lattice and let $C_1, ..., C_n$ be maximal chains of S such that $n = \mathbf{dim}(S)$ and $C_1 \cup C_2 \cup ... \cup C_n \supseteq \mathrm{J}(S)$.

Definition 1. *The direct product $G = C_1 \times ... \times C_n$ is called a* grid *of S.*

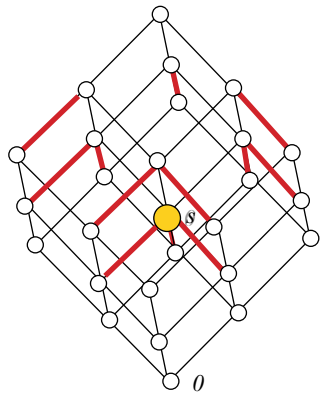

FIGURE 8. $\mathrm{con}^{\vee\mathrm{cp}}(s)$ in a direct product of three chains

The Jordan-Hölder theorem implies that the grid is determined up to isomorphism. Theorem 1 asserts that S is a cover-preserving join-homomorphic image of G.

Let $D_1, ..., D_n$ be pairwise disjoint subchains of of the order $\mathrm{J}_0(S)$ such that $n = \mathrm{w}(\mathrm{J}(S))$ and $D_1 \cup D_2 \cup ... \cup D_n = \mathrm{J}_0(S)$. We may assume that $D_i \subset C_i$,i.e. the D_i is the restriction of C_i to $\mathrm{J}_0(S)$.

Definition 2. *The direct product* $G_{\mathfrak{l}} = D_1 \times ... \times D_n$ *is called a* lower grid *of* S.

$G_{\mathfrak{l}}$ is a sublattice of G. Let us note that the lower grid is not determined uniquely. Obviously, geometrically the grid is a cuboid (rectangular). More results on grids see in G. Czédli, [3]. If G is fixed then $G_{\mathfrak{l}}$ is determined.

It is clear that $\mathfrak{l}(G) = n.\mathfrak{l}(S)$ and $\mathfrak{l}(G_l) = |\,\mathrm{J}(S)|$. The last equality means that every finite distributive lattice D is the cover-preserving sublattice of the direct product of $\mathbf{dim}(D)$ chains, this is the lower grid. The lower grid of M_3 is the 2^3 boolean lattice and the grid is isomorphic to 3^3.

Definition 3. *The* frame *of the semimodular lattice* S *is* $\mathrm{H}(\mathrm{J}(S))$.

It is clear that $\mathrm{J}(\mathrm{H}(\mathrm{J}(S)))$ is order isomorphic to $\mathrm{J}(S)$. The frame is a cover-preserving sublattice of the lower grid, $\mathrm{Frame}(S) \subseteq G_{\mathfrak{l}} \subseteq G$. It is easy to see that $\mathrm{w}(\mathrm{Frame}(S)) = \mathrm{w}(S)$.

3.2. **The skeleton.** The skeleton was introduced by G. Grätzer and R. W. Quackenbush [20] for planar modular lattices (they called them frame). Let S be a planar modular lattice with a planar diagram P, and let $M = [b, t]$ be an interval isomorphic to M_n with exterior atoms $\{a_1, a_2\}$ and interior atoms $\{a_2, ...a_{n-1}\}$ between a_1 and a_2 in the planar diagram P. These are doubly irreducible elements. We say that $a \in S$ is an internal element (with respect to P) if a is an external element. Let $\mathrm{Skeleton}(S)$ be the sublattice of S consisting of all external elements of S, which is unique determined up to isomorphism.

We extend this concept for semimodular lattices.

Definition 4. *A maximal cover-preserving complemented sublattice* C *in a distributive lattice* D *is called a* cell. *If* $C \cong 2^k$ *we say that* C *is a* k*-cell.*

Definition 5. *The* skeleton, Skeleton(S) *of a finite semimodular lattice S is a maximal cover-preserving distributive sublattice D of S such that every element of S is in the interval generated by a cell of D.*

Let us show some examples. We consider the skeleton as a box and the semimodular lattice is packed in this box see Figures 5, 9, 10, 15, 20, 21, 22, 38, 40, 43. (Let us remark that not every semimodular lattice has a skeleton.)

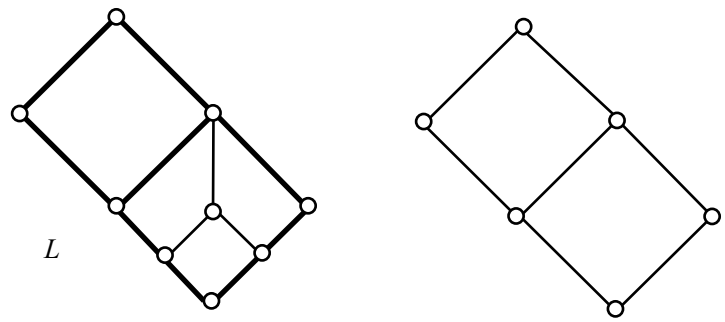

FIGURE 9. The skeleton of a semimodular lattice S

For other examples see Figure 20 and the Fano space is presented on Figure 39.

Definition 6. *A finite distributive lattice D is called* connected *if to arbitrary two cells C_1 and C_n there is a sequence $C_2, ..., C_{n-1}$ such that $|C_i \cap C_{i+1}| > 1$ for every i.*

The following theorem generalize a results of [20]:

Theorem 3. *Every connected finite distributive lattice D is isomorphic to the skeleton of a simple modular lattice.*

Proof. Similar to [20]. □

3.3. **Mappings associated to grids and frames.** $\mathrm{con}^{\vee}(y, x)$ denotes the principal join-congruence, i.e. the smallest join-congruence – under which $y \equiv x$.

Definition 7. *A join-congruence Φ of a distributive lattice D is called* distributive join-congruence *if $\Phi = \bigvee \mathrm{con}^{\vee}(p_i, x_i)$, where $p_i \in D$ are join-irreducible elements.*

This notion was introduced in [23].

Let S be a semimodular lattice, then we have the distributive lattices $G, , G_{\mathfrak{l}}$, and the Frame(S) = H(J(S)). There are several mappings between these lattices:

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & S \\ \downarrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} \\ G_{\mathfrak{l}} & \xrightarrow{\psi} & \mathrm{Frame}(S) \end{array}$$

(I.) φ from G onto S. Theorem 1 states that there is cover-preserving join-homomorphism φ from G onto S:

$$\varphi : G \to S, \quad (x_1, \ldots, x_k) \mapsto x_1 \vee \cdots \vee x_k.$$

(II.) $\alpha : G \longrightarrow G_{\mathfrak{l}}$. We may assume that $D_i \subseteq C_i$ for every $i \in \{1, ..., n\}$. In this case to every $c \in C_i$ there is a largest $c^* \in D_i$ such that $c \geq c^*$.

It is clear that α is a lattice homomorphism of G into G_l (which is determined by the congruence $(c_1, ..., c_n) \equiv \ (c_1{}^*, ...c_n{}^*)(\Theta)$).

(III.) $\psi : G_{\mathfrak{l}} \longrightarrow \mathrm{H}(\mathrm{J}(S)) = \mathrm{Frame}(S)$.. This is a distributive if join-congruence.

Take for simplicity the $n = 2$ case. By the definition of $G_{\mathfrak{l}}$ every $p \in \mathrm{J}(S)$ appears in one of the D_i-s .If $p, q \in D_i$, $p < q$, then $p < q$ in $\mathrm{J}(S)$ too. If $p \in D_i$ and $q \in D_j$, $i \neq j$ but $p < q$ in $\mathrm{J}(S)$ then take the elements $(p, 0), (p, q)$ of $G_{\mathfrak{l}}$ and the join-congruence induced by this pair

$$\mathrm{con}^{\vee}((p, 0), (p, q)) \tag{1}$$

On this way we get a join-homomorphism from $G_{\mathfrak{l}}$ to $\mathrm{Frame}(S)$, i.e. the frame $\mathrm{Frame}(S)$ is distributive join-homomorphic image of the lower grid.

(IV.) $\beta : \mathrm{Frame}(S) = \mathrm{H}(\mathrm{J}(S)) \longrightarrow S$ exists by Theorem 2, β is a cover-preserving join-homomorphism.

Lemma 2. *$\varphi = \beta\psi\alpha$.*

Proof. Easy computation. □

3.4. **Horizontal and vertical edges of an order.** Let us consider $P = \mathrm{J}(D)$, where D is the distributive lattice defined on Figure 1. In the diagram (graph) of P (see Figure 10) we have two type of edges: $\overline{x, y}$, $x \prec y$, $x, y \in D_i$ for some i, these are the *vertical edges* (these are in the two chains) and all others the *horizontal edges*. We omit (cut out) the horizontal edges of $\mathrm{J}(D)$ we get the diagram of $\mathrm{J}(G_{\mathfrak{l}})$, i.e. we have two maximal chains $0 < c_1 < c_2 < c_3 < c_4 < c_5$ and $0 < e_1 < e_2 < e_3$ and the relations, $e_2 > c_3, c_4 > e_1$ which are given by two horizontal edges, $\overline{e_2, c_3}$ and $\overline{c_4, e_1}$ (red lines).

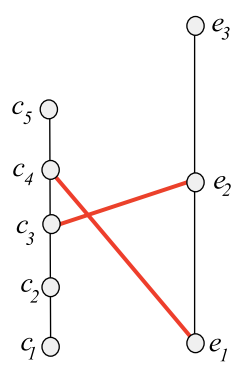

FIGURE 10. $\mathrm{J}(D)$

On $G_{\mathfrak{l}}$ take the corresponding join congruence

$$\Theta = \mathrm{con}^{\vee}((c_4, 0), (c_4, e_1)) \vee \mathrm{con}^{\vee}((0, e_2), (c_2, e_2)) \tag{2}$$

Then $G_{\mathfrak{l}}/\Theta = P$. The natural join-homomorphism $G_{\mathfrak{l}} \to G_l/\Phi$, preserves boundary chains, this observation makes possible to introduce such notions in an easy way

in the J-width > 2 cases. To the horizontal edge $\overline{c_4, e_1}$ we associate the cuboid S_1 and similarly, $\overline{c_4, e_1}$ represents S_2. We have a bijection between the horizontal edges and "corner" cuboids of $G_{\mathfrak{l}}$.

We are going to the $w(P) = 3$ case and take the following poset P see on Figure 11:

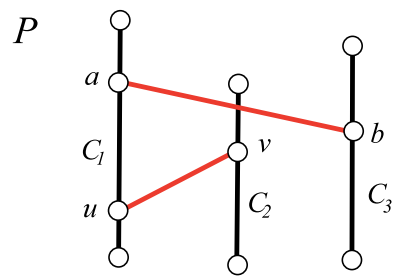

FIGURE 11. Two horizontal edges

We can visualize this as follows, see Figure 12. From the "big" cuboid we cut out two "small" cuboid-s, which are determined by the horizontal edges.

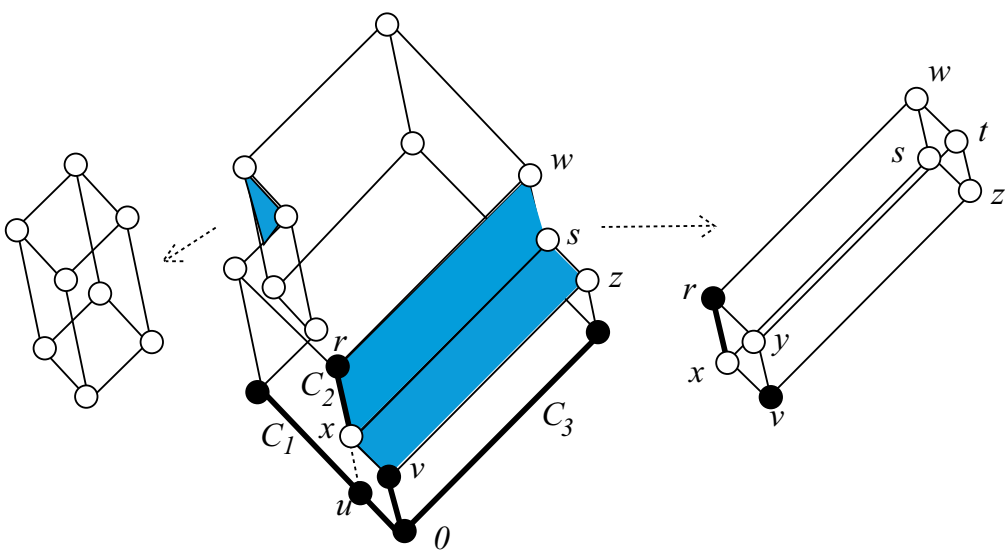

FIGURE 12. Removing corners from a cuboid

It is easy to prove that every finite distributive lattice D of J-width $= n$ is

(1) a cover-preserving sublattice of a rectangular lattice R of $\mathbf{dim}(D) = n$ (envelop),
(2) the S-glued sum of rectangular components of J-width $k \leq n$.

It is easy to define the volume and surface area of an arbitrary finite distributive lattice.

Is something like true for all semimodular lattices?

3.5. **Rectangular lattices.** Rectangular lattices were introduced by G. Grätzer and E. Knapp [15] for planar semimodular lattices: a *left corner* (resp. *right corner*) of a planar lattice K in a double-irreducible element in $K - \{0, 1\}$ on the left (resp., right) boundary of K. A *rectangular* lattice L is a planar semimodular lattice which has exactly one left corner, u_l and exactly one right corner, u_r and they are complementary – that is, $u_l \vee u_r = 1$ and $u_l \wedge u_r = 0$. The direct product of two chains is rectangular.

We introduce this notion for arbitrary finite semimodular lattice. Let X, Y be posets. The *cardinal sum* $X + Y$ of X and Y is the set of all elements in X and Y considered as disjoint. The relation $\leq$ keeps it meaning in X and in Y, while neither $x \geq y$ nor $x \leq y$ for all $x \in X, y \in Y$.

Definition 8. *A* rectangular *lattice L is a finite semimodular lattice in which* $\mathrm{J}(L)$ *is the cardinal sum of chains.*

Geometric lattices are rectangular. In [7] we introduced the *almost geometric lattices* these are lattices in which $\mathrm{J}(L)$ is the cardinal sum of at most two element chains. In the class of finite distributive lattices the rectangular lattices are the the direct products of chains. The lattices $M_3[C_n]$ are modular rectangular lattices (C_n is the chain $\{0, 1, 2, ..., n-1\}$ of integers).

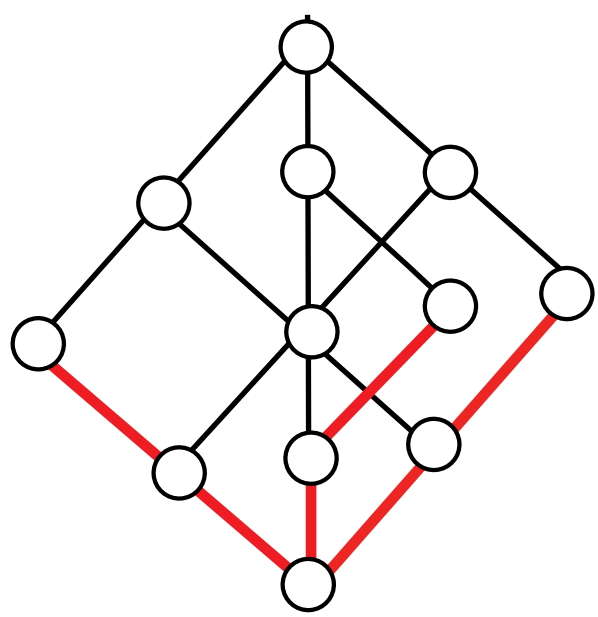

FIGURE 13. A modular non distributive rectangular lattice $M_3[C_3]$

We prove the following:

Lemma 3. *Let G be the direct product of chains and let Θ be a cover-preserving join-congruence of G. Then G/Θ is a rectangular lattice with the property* $\mathrm{J}(G) \cong \mathrm{J}(G/\Theta)$ *if and only if every join-irreducible element of G is a one-element Θ-class.*

Proof. Let g/Θ be a Θ-class containing $g \in G$. Assume that g is a minimal element of these class. If g/Θ is a join-irreducible element of G/Θ then g is a join-irreducible

element of G, i. e. $g = (0, ..., 0, g_i, 0, ..., 0)$ for some i. But $\mathrm{J}(G) \cong \mathrm{J}(G/\Theta)$ and therefore g/Θ must be a one-element class. The converse is trivial, $\mathrm{J}(G/\Theta)$ is order isomorphic to the paset $\{(0, ..., 0, g_i, 0, ..., 0)\}$, i.e. G/Θ is rectangular. □

The modular lattices $\mathcal{M}_n$, $n > 2$ are two dimensional lattices and $\mathrm{w}(\mathrm{J}(\mathcal{M}_n)) = n$, i.e. $\mathbf{dim}(\mathcal{M}_n) < \mathrm{w}(\mathrm{J}(\mathcal{M}_n))$.

4. The source

4.1. **The source cell and the beret (swiss cup).** Let Θ be a cover-preserving join-congruence of a finite distributive lattice D.

Definition 9. *An element $s \in D$ is called a* source element *of Θ if there is a $t, t \prec s$ such that $s \equiv t\ (\Theta)$ and for every prime quotient u/v if $s/t \searrow u/v, s \neq u$ imply $u \not\equiv v\ (\Theta)$. The set S_Θ of all source elements of Θ is the* source *of Θ.*

Let us remark the source element can defined in an arbitrary semimodular lattice. It is clear that Θ is determined by the source $S_\Theta \subset D$.

Lemma 4. *Let x be an arbitrary lower cover of a source element s of Θ. Then $x \equiv s\ (\Theta)$. If $s/x \searrow v/z$, $s \neq v$, then $v \not\equiv z\ (\Theta)$.*

Proof. Let s be a source element of Θ then $s \equiv t\ (\Theta)$ for some t, $t \prec s$. If $x \prec s$ and $x \neq t$ then $\{x \wedge t, x, t, s\}$ form a covering square. Then $x \not\equiv x \wedge t\ (\Theta)$. This implies $x \wedge t \not\equiv t\ (\Theta)$. By Lemma 1 we have $x \equiv s\ (\Theta)$.

To prove that $v \not\equiv z\ (\Theta)$, we may assume that $v \prec s$. Take t, $t \prec s$, then we have three (pairwise different) lower covers of s, namely $.x, v, t$. These generate an eight-element boolean lattice in which $s \equiv t\ (\Theta)$, $s \equiv x\ (\Theta)$ and $s \equiv v\ (\Theta)$. By the choice of t we know that $v \not\equiv v \wedge t\ (\Theta)$, $x \not\equiv x \wedge t\ (\Theta)$ and $z \not\equiv x \wedge t \wedge v\ (\Theta)$. It follows that $x \not\equiv t\ (\Theta)$, otherwise by the transitivity $x \not\equiv v\ (\Theta)$. This implies $t \wedge x \not\equiv t \wedge x \wedge v\ (\Theta)$. Take the covering square $\{x \wedge v \wedge t, z, t \wedge x, x\}$ then by Lemma 1 $z \not\equiv x\ (\Theta)$, which implies $z \not\equiv v\ (\Theta)$. □

Definition 10. *A* source cell *B_s is the boolean lattice generated by the lower covers of a source element $s \in S_\Theta$ and s_* denotes the bottom element of the source-cell. The* beret *T_s of the source cell B_s is the unit element, s and the dual atoms of B_s.*

The set of source cells of Θ is denoted by $\mathrm{Sc}(\Theta)$.

If s is a source element of the cover-preserving join-congruence Θ, then by Lemma 4 its restriction Ψ_s to the source cell has only one nontrivial Ψ_s-class this is the beret. Take the boolean lattice ${C_2}^n$ and the cover-preserving join-congruence Θ_n with the property that the 1-class is the only one non-trivial congruence class. $\mathbb{B}_n$ will be denote the factor lattice ${C_2}^n/\Theta_n$ ($\mathbb{B}_2 \cong C_2$, $\mathbb{B}_3 \cong M_3$ and $\mathbb{B}_4$ on Figure 42. Then we have $B_s/\Psi_s \cong \mathbb{B}_n$.

The meet of the lower covers of the bottom element s_* of a source cell is denoted by s_{**}.

Definition 11. *The* neighborhood *of source cell or a source element s is the interval $N_s = [s_{**}, s]$.*

On Figure 16 and Figure 17 there is the distributive lattice 3^3, with a beret as the top of it. A sours cell has $3^n - 1$ neighborhood cells and N_s is isomorphic to 3^n.

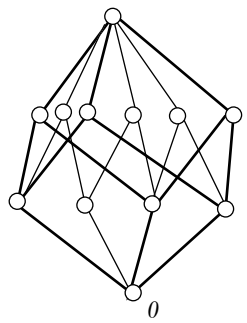

FIGURE 14. Figure 36 $\mathbb{B}_4$

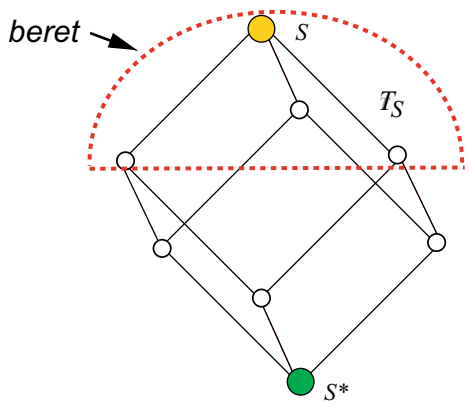

FIGURE 15. The source cell B_s with the beret T_s

Definition 12. *A source element s of Θ is called* bastard *if s itself or at least one of its lower covers t is join-irreducible. The cover-preserving join-congruence Θ is bastard if has a source element which is bastard.*

In the 2-dimensional case this means that s or at least the lower covers of s, a and b is join-irreducible.

Remark. Let $\varphi : G \to L$ be a cover-preserving join-homomorphism. We denote by Θ the cover-preserving join-congruence induced by φ. Take the source S of Θ. A source element $s \in S$ is called *bastard* if s itself or at least one of its lower covers t is join-irreducible. Let $S' \subseteq S$ the set of all non bastard source elements and Θ' denotes the corresponding cover-preserving join-congruence. Then $R = G/\Theta'$ is a rectangular lattice (envelop) and $L = G/\Theta$ is a cover-preserving sublattice of R.

4.2. **Source lattices.** The join-congruence of N_s, where B_s is the only one non-trivial congruence class will be denoted by Φ_s.

Definition 13. *The source lattice is the factor lattice ${C_3}^n/\Phi_n$, where Φ_n is the join-congruence with only one non-trivial congruence class containing the unit element and the dual atoms of ${C_3}^n$.*

$\mathbb{L}_2 \cong S_7$ and $\mathbb{L}_3$ is presented on Figure 11.

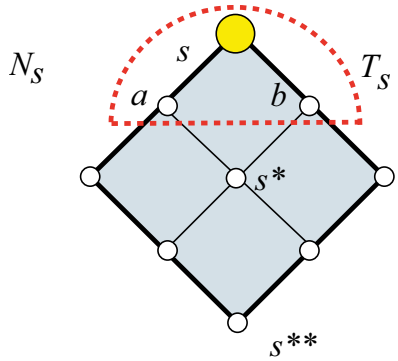

FIGURE 16. A source cell with the neighborhood, N_s and the beret T_s

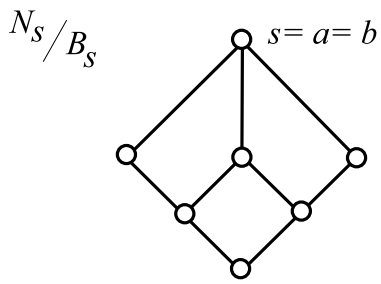

FIGURE 17. The factor of N_s in the $\mathrm{Jw}(D) = 2$ case, $\mathbb{L}_2 = \mathbb{L}^H_{2,1}$

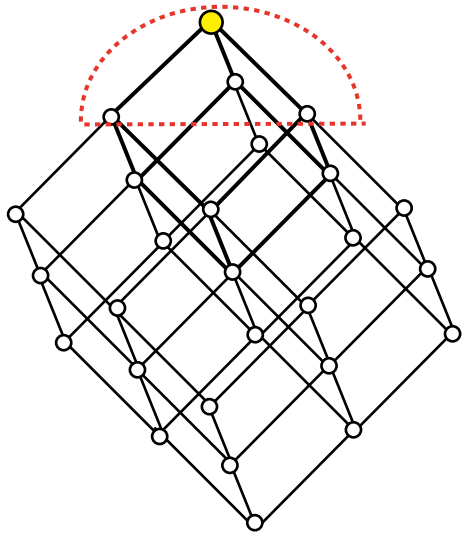

FIGURE 18. A source cell with the neighborhood and the beret

The skeleton of the source lattice $\mathbb{L}_n$ is the 2^n Boole lattice. We defined *Kuros-Ore dimension* of a lattice L is the minimal number of join-irreducibles needed to span the unit element of L. The Kuros-Ore dimension of $\mathbb{L}_n$ is n.

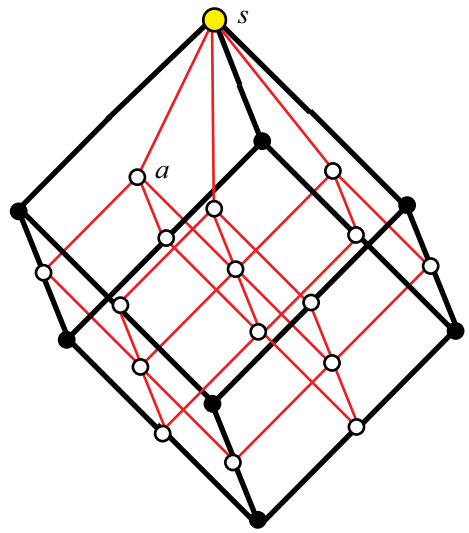

FIGURE 19. The source lattice $\mathbb{L}_3$ and skeleton

Lemma 5. *$\mathbb{L}_n$ is a subdirect irreducible rectangular semimodular lattice.*

Proof. It is an easy exercise, in the J-width = 3 case, Figure 19, $\mathrm{con}(s, a)$ is the smallest non trivial congruence relation. □

4.3. **Independence.** The source elements are independent in the following sense:

Definition 14. *Two elements s_1 and s_2 of a distributive lattice are* s-independent *if $x \prec s_1, y \prec s_2$ then $s_1/x, s_2/y$ are not perspective, $s_1/x \not\sim s_2/y$.*

Lemma 6. *Two elements s_1 and s_2 of a distributive lattice are s-independent if one of the following is satisfied:*

(1) s_1 and s_2 are incomparable,
(2) $s_1 < s_2$ and $t \prec s_2$ implies $t \geq s_1$, i.e. $s_1 \leq {s_2}^{**}$.

Proof. It is clear that for an incomparable pair s_1, s_2 if $u \prec s_1$ and $v \prec s_2$ then s_1/u and s_2/v cannot be projective. On the other case, if $s_1 < s_2$ then $t || s_1$ would imply that s_2/t and $s_1/t \wedge s_1$ are perspective. This means that $t \geq s_1$. □

Lemma 7. *Two source elements of of a cover-preserving join-congruence Θ are s-independent, i.e. if s_1 and s_2 are two different source elements of Θ and $x \prec s_1, y \prec s_2$ then $s_1/x, s_2/y$ cannot be perspective, $s_1/x \not\sim s_2/y$.*

Proof. This is a trivial consequence of Lemma 4 and Lemma 6.

□

$\mathrm{con}^{\vee\mathrm{cp}}(s)$ was defined in 1.3 (see p.7). The source of this cover-preserving join-congruence is $S = \{s\}$. Let $\{s_1, ..., s_n\}$ be an s-independent subset of the distributive lattice D.

Lemma 8. *$\Theta = \bigvee \mathrm{con}^{\vee\mathrm{cp}}(s_i)$ is a cover-preserving join-congruence.*

Proof. Let $\{a \wedge b, a, b, a \vee b\}$ be a covering square and assume that $a \equiv a \vee b\ (\Theta)$. Then there is an i such that $a \equiv a \vee b\ (\mathrm{con}^{\vee\mathrm{cp}}(s_i)$, i.e. an upper cover t of s_i such that$s_i \vee a = a \vee b$ and $s_i \wedge a = t$. Take $s_i \wedge b$. By Lemma 4 $s_i \wedge b \equiv s_i\ (\mathrm{con}^{\vee\mathrm{cp}}(s_i)$, consequently $b \equiv a \vee b\ (\Theta)$.

□

Let $S = \{s_1, ..., s_n\}$ be an s-independent subset of D. Then $\Theta = \bigvee \mathrm{con}^{\vee\mathrm{cp}}(s_i)$ is the smallest cover-preserving join-congruence with the source S.

Lemma 9. *Let $s > 0$ be an element of a finite distributive lattice D. Then* $\mathfrak{l}(D/\mathrm{con}^{\vee\mathrm{cp}}(s_i)) = \mathfrak{l}(D) - 1$.

Proof. Take a maximal chain C of length n through s and an arbitrary $t \in C$, $t \prec s$. By Lemma 4 $t \equiv s\ (\mathrm{con}^{\vee\mathrm{cp}}(s_i)$ and no other prime interval collapses by $\mathrm{con}^{\vee\mathrm{cp}}(s_i)$. Therefore the image of C in $D/\mathrm{con}^{\vee\mathrm{cp}}(s_i)$ has length $n-1$. The lattice $D/\mathrm{con}^{\vee\mathrm{cp}}(s_i)$ is semimodular (see [15]), which proves the lemma. □

Theorem 4. *Every semimodular lattices L can be characterized as a pair (G, S) where G is the direct power of a chain (grid) and S is a s-independent subset of G.*

In a boolean lattice s-independent means that s_1 and s_2 are incomparable, i.e. it is an antichain. As usual, $h(s)$ denotes the hight of s.

Corollary 1. *Every geometric lattices $\mathfrak{G}$ is determined by a pair (B, S), where B is a boolean lattice and S is an antichain of B with the property $h(s) > 2$ for all $s \in S$.*

Not every s-independent set is the source of a cover-preserving join-congruence.

Lemma 10. *Let Φ be a join-congruence of a finite semimodular lattice M. Then Φ is cover-preserving if and only if $a, b, c \in G$ such that $a \prec a \vee b \vee c$, $b \prec a \vee b \vee c$, $c \prec a \vee b \vee c$, $a \equiv a \wedge b\ (\Phi), a \equiv a \wedge c\ (\Phi), c \equiv b \wedge c\ (\Phi)$, $a \wedge b \not\equiv a \wedge b \wedge c\ (\Phi), a \wedge b \not\equiv a \wedge b \wedge c\ (\Phi), a \wedge b \not\equiv a \wedge b \wedge c\ (\Phi)$ are satisfied, then*

$$b \equiv a \wedge b\ (\Phi), b \equiv b \wedge c\ (\Phi).$$

Lemma 11. *Let S be a source of a cover-preserving congruence relation Φ of G then S satisfies the following property:*

(3) if $a, b \in S$ $a \prec a \vee b$, $b \prec a \vee b$ and $c \prec a \vee b$ then $c \in S$.

Problem 1. *Let D be a distributive lattice and $a, b, c \in D$ such that $a \prec a \vee b$, $b \prec a \vee b$, $c \prec a \vee b \vee c$. A subset $S \subseteq D$ is the source of a cover-preserving join-congruence iff S satisfies the conditions:*

(1) S is s-independent,
(2) (3) of Lemma 11.

4.4. **The matrix of a cover-preserving join-congruence.** Let L be a semimodular lattice. By Theorem 1. we have a grid $G = {C_k}^n$ ($C_k = \{1, 2, ..., k\}$ is the chain of natural numbers) and a cover-preserving join-congruence Θ of G such that $G/\Theta \cong L$. On Figure 19 we have a 2-dimensional case a matrix. We write into a cover-preserving squere an "1" emtry if at the bottom we have a source element. Otherwise we write "0". This is a $(0, 1)$-matrix and every row/column contais at most "1" entry. Put 1 into a cell if its top element is in S, otherwise put zero. What we get is an $n \times n$ matrix, $\mathbf{M}_L$, which determines L (if you like you can turn this grid with 45 degrees to see the matrix in the traditional form).

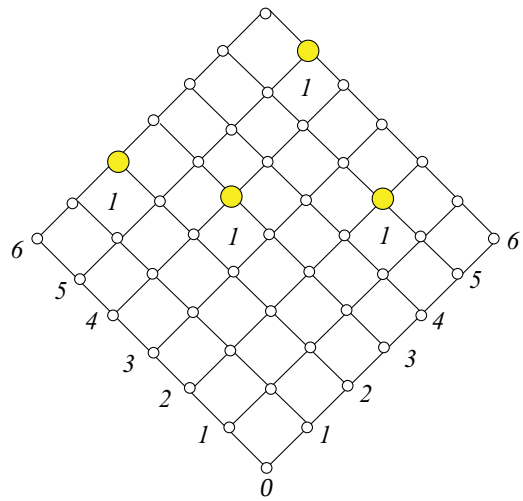

FIGURE 20. A grid and four sours elements

The first matrix is a $n \times n$ $(0,1)$-matrix, where every row/column contains at most 1 entry:

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

There is an other way to define a matrix using the coordinates of the source elements:

$$\begin{vmatrix} 6 & 5 & 4 & 2 \\ 2 & 6 & 3 & 5 \end{vmatrix}$$

For semimodular lattices S with $\mathbf{dim}(S) \geq 3$ we use hypermatrices. The matrix of M_3 is a $(0,1)$-matrix of type $3 \times 3 \times 3$: $[a_{i,j,k}]$, $a_{1,1,1} = 1$ and $a_{i,j,k} = 0$ otherwise.

A *column* $C(3)_{i,j}$ is $\{a_{i,j,k}; k = 1, 2, ..., n\}$ of a $3 \times 3 \times 3$ matrix: $[a_{i,j,k}]$ and similarly,

$C(1)_{i,k}$ is $\{a_{i,j,k}; j = 1, 2, ..., n\}$,

$C(2)_{i,j}$ is $\{a_{j,k}; i = 1, 2, ..., n\}$.

We use $(0,1)$-matrices, where every column contains at most one entry 1.

See Czédli, Schmidt [8], Czédli [2], Czédli, Ozsvárt, Udvari [11].

PART II

Constructions

Part 2. Constructions

5. Constructions of semimodular lattices

5.1. **The n-fork construction (the zipper).** This general construction was introduced in [6] for slim planar semimodular lattices (semimodular lattices of dimension 2), but it was included in same older papers, see in [14] and [24]. Let S be a 4-cell of a slim semimodular lattice L. Then S is a covering square $\{a = b \wedge w$, b, w, $c = b \vee w\}$, see on Figuee 20. We change L to a new lattice L' as follows:

Step1. Firstly, we replace S by a copy of $S_7 \cong \mathbb{L}_2$. This way we get three new 4-cells instead of S.

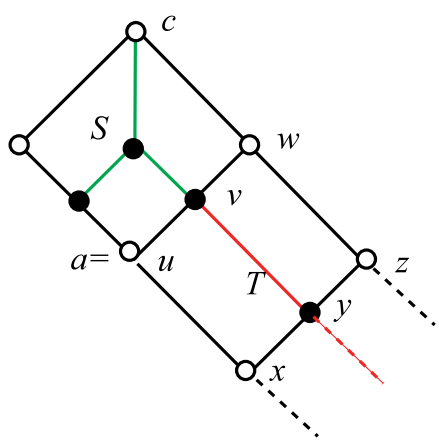

Figure 21. The downward-going procedure and a teeth of a fork

Step 2. Secondly, as long as there is a chain $u \prec v \prec w$ such that v is a new element and $T = \{x = u \wedge z,\ z,\ u,\ w = z \vee u\}$ is a 4-cell in the original lattice L but $x \prec z$ at the present stage, see Figure 22, we insert a new element y such that $x \prec y \prec z$ and $y \prec v$. (This way we get two 4-cells instead of T.) When this "downward-going" procedure terminates, we obtain L'. The collection of all new elements, which is a poset, will be called a *fork*. We say that L' is obtained from L by *adding a fork to L (at the 4-cell S)*, see Figure 21 for an illustration. If we add several forks to L one by one, then we simply speak of adding forks to L.

Theorem 5. [6] *Let L be a slim semimodular lattice consisting of at least three elements. Then L can be obtained from the direct product of two nontrivial finite chains such that*

- *first we add finitely many forks one by one,*
- *and then we remove corners, one by one, finitely many times.*

Remark. The second condition means we have a distributive join-congruence on the grid G.

On Figure 24. you can see a refinement of a grid G, we insert the black filled elements.

Let S be a semimodular lattice and let G be a grid of S with the corresponding cover-preserving join-congruence Θ. If we add a fork to S then this procedure can

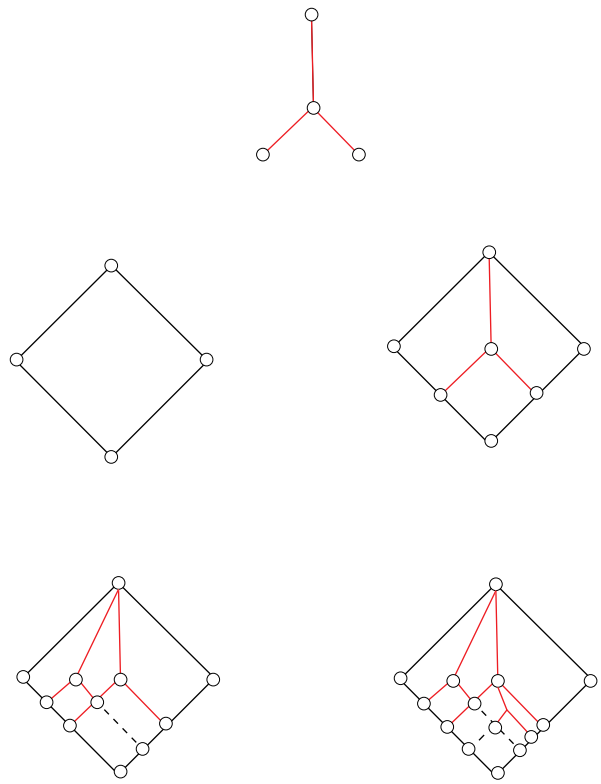

FIGURE 22. 2-fork with the skeleton

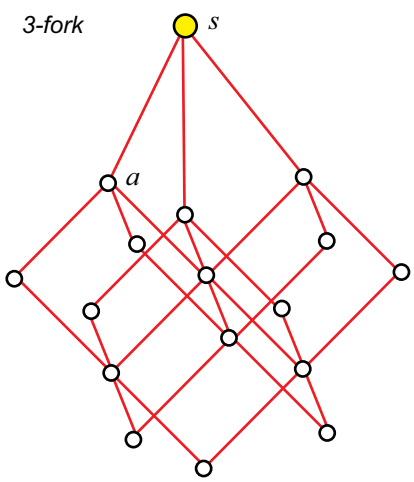

FIGURE 23. 3-fork

be obtains on the following way: we take a refinement of G and select a new source element (which is a new grid element.) See Figure 29-32.

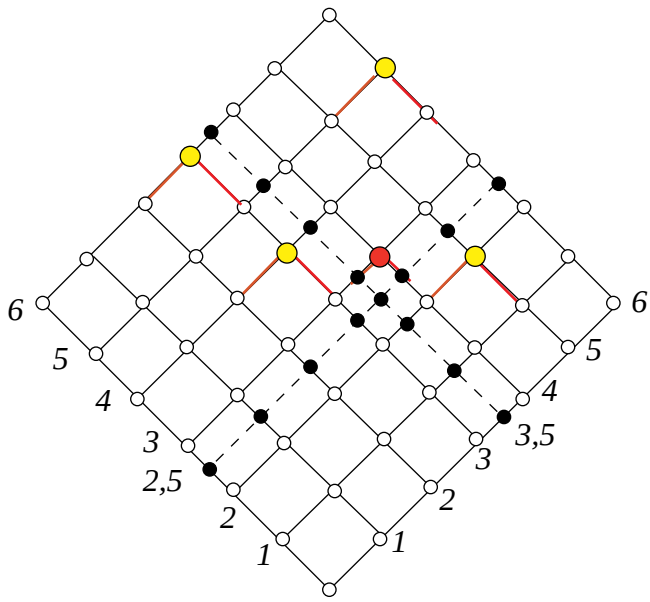

FIGURE 24. Refinement of a grid

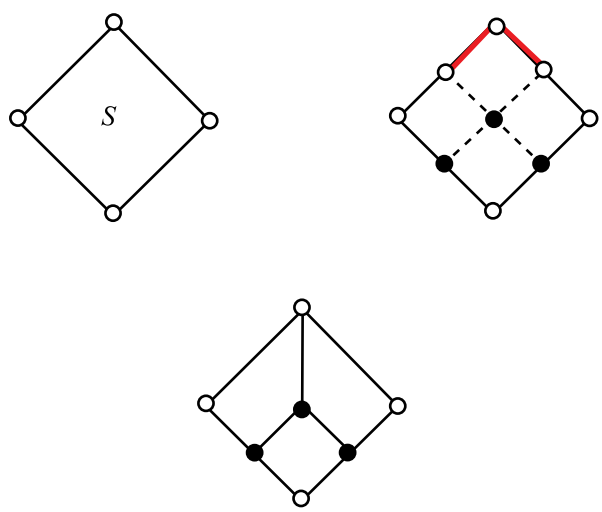

FIGURE 25. Refinement I.

It is easy to generalize this construction for the dimension $n > 2$, we take an n-cell which is isomorphic to 2^n replace by $\mathbb{L}_n$, see Figure 28.

Step1'. Firstly, we replace a covering cube $S \cong 2^3$ by a copy of S_{24}.

Step 2'. Secondly, as long as there is an other neighbor covering cube we insert to the elements x, y, z, u, v the new elements x', y', z', u', v'. When this "downward-going" procedure terminates, we obtain L'. The collection of all new elements, which is a poset, will be called a *3-fork*. We say that L' is obtained from L by

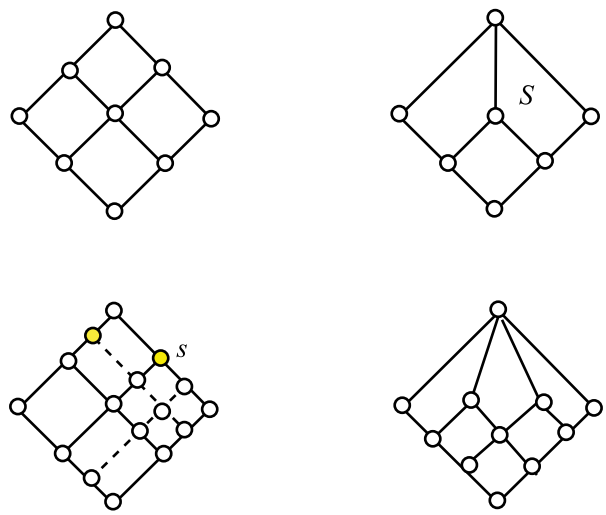

FIGURE 26. Refinement II.

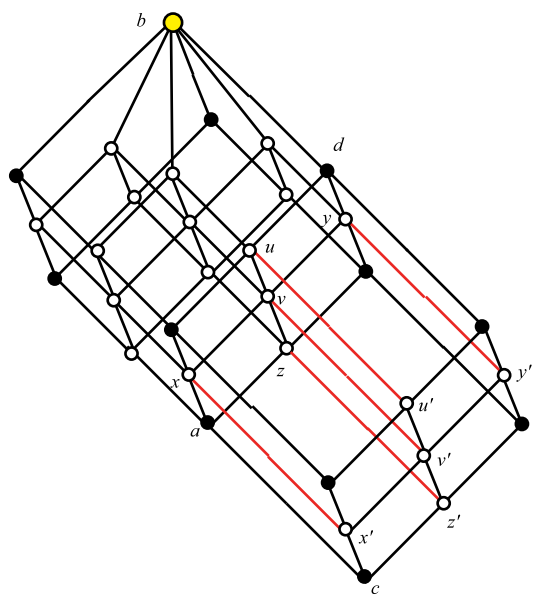

FIGURE 27. The source lattice $\mathbb{L}_3 = S_{24}$ and the "downward-going" procedure

adding a fork to L, see Figure 20 for an illustration. If we add several forks to L one by one, then we simply speak of adding forks to L.

On Figure 24. you can see a 3-fork.

5.2. **The pigeonhole.** To add a fork we can make the following procedure:

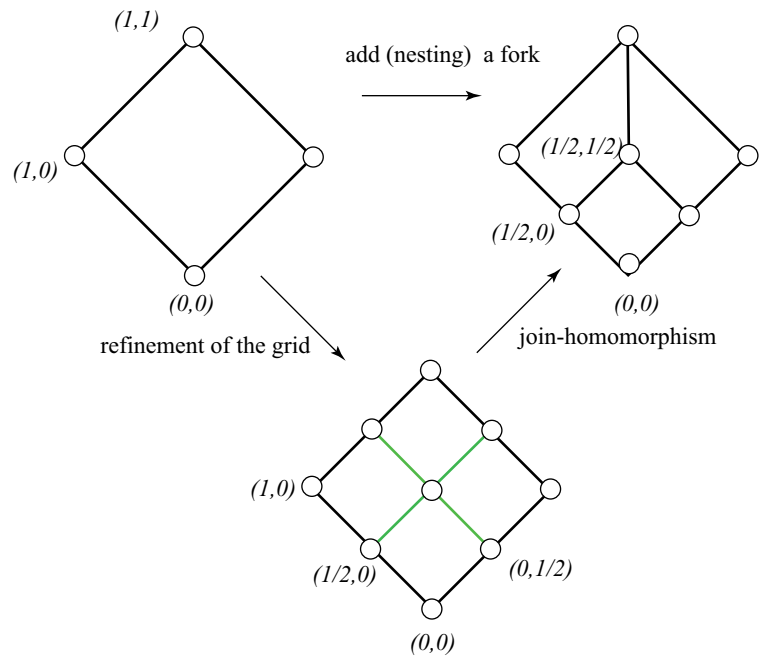

FIGURE 28. Figure 47 The iteractive procedure

5.2.1. *The* $\mathrm{Jw}(L) = 2$ *case.* We define the nested slim lattices inductively, these are lattices where copies of S_7-s are nested into each other:

Definition 15. *A slim semimodular lattice N is called a* 2-nested lattice *if we obtain with the following iteractive procedure*

(1) The four element Boolean lattice. i.e. the 4-cell ${C_2}^2$, this is a 4-cell,
(2) We add a fork to this 4-cell ${C_2}^2$, we obtain the source lattice $S_7 = \mathbb{L}_2$ (on Figure 30 (III) or (V)), and applay the "downward-going" procedure if it is necessery (this two steps gives eight different 2-nested lattices),
(3) S_7 has three 4-cells. To some of these 4-cells we add further forks, i.e. we replace some covering square by a copy of S_7
(4) We continue this procedure in finite many steps (on Figure 30 (II)).

The set of all 2-nested lattices will be denoted by $\mathfrak{N}_1$.

Theorem 6. *Every 2-nested lattice N satisfies the following properties:*

(1) N is rectangular,
(2) N has a complementary pair a, a' ($a \wedge a' = 0$ and $a \vee a' = 1$) such that $a \prec 1$ and $a' \prec 1$,
(3) N contains as ideal two chains C_1 and C_2, such that $C = C_1 \cup C_2$ is congruence-determining.

Proof. . □

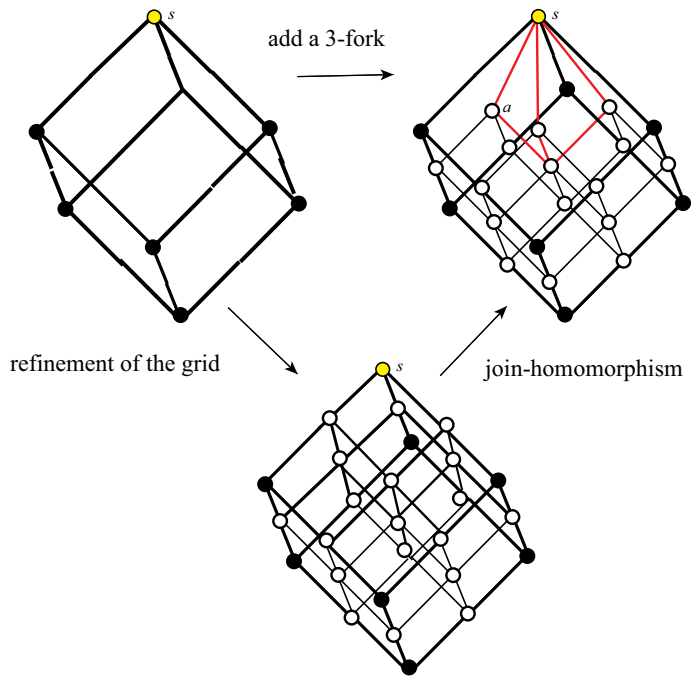

FIGURE 29. Figure 47 The iteractive procedure

On the grid G there is a natural distance, $\delta(a,b)$ of two elements a,b in a grid $G = C^m$ is the smallest natural number n such that there is a sequence $a = c_1, c_2, ..., c_{n-1} = b$, $c_i \prec c_{i+1}$ or $c_i \succ c_{i+1}$ for every i. This defines an equivalence relation $\mathcal{E}$ on S. Two source elements are of the same block iff their are "close to each other".

Definition 16. *Two incomparable elements a and b of a distributive lattice D are* horizontal-adjacent *if $\delta(a,b) = 2$, i.e. if $\{a \wedge b, a, b, a \vee b\}$ is a covering square. Two elements a and b, $b < a$ of a distributive lattice D are* vertical-adjacent *$b = a^{**}$.*

Definition 17. *$a \equiv b$ $(\mathcal{E})$ if and only if there is a sequence $a = s_1, s_2, ..., s_n = b$ of source elements in a grid G such that for every i (s_i, s_{i+1}) is either a horisontal-adjacent pair or a vertical-adjacent pair.*

Definition 18. *$\{a,\ b\}$is a* remote pair *if either (1) a and b are incomparable and $\delta(a,b) > 2$ or (2) $b < a^{**}$.*

Definition 19. *A The element s* isolated *in respect of S if $\delta(s,t) > 2$ for every $t \in S, t \neq s$.*

Lemma 12. *$\mathcal{E}$ is an equvivalence relation on the source S.*

Proof. . □

Let A be a block of $\mathcal{E}$, this is a subset of the source S. Take the interval $\overline{A}$ of G generated by A. We prove that the restriction of S to A defines a nested lattice N_A.

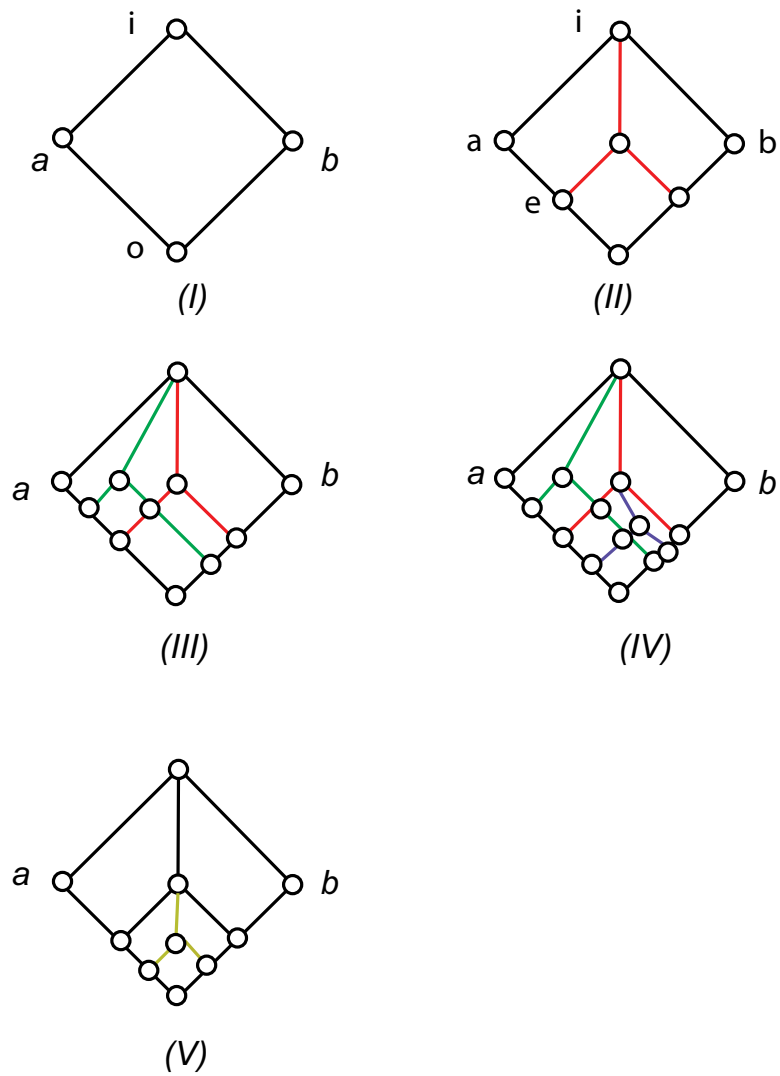

FIGURE 30. The iteractive procedure

5.2.2. *Some nested lattices.* Consider the following semimodular lattices:

Let n and m be natural numbers and let $C = \{0, 1, ..., n-1\}$ be a chain of length $n-1$. Then

$$\mathbb{L}^H_{m,n}$$

is the meet-sublattice of C^m consisting all elements $(x_1, ..., x_m)$ where either $x_1 + x_2 + ... + x_m \leq n-1$ or $x_1 = ... = x_m = n-1$. The elements $s_i = (x_1, ..., x_m)$, $x_1 + x_2 + ... + x_m = n-1$ form an s-independent set S in G, This is a vertically adjacent set. The lattice $\mathbb{L}^H_{m,n}$ is the factor lattice G/Θ_S. Obviously, this is a rectangular lattice of J-width m. See the examples on Figure 45, Figure 47, Figure 49, Figure 54.

$\mathbb{L}^V_{2,1}$ is the lattice S_7. It is easy to see that $\mathbb{L}^V_{2,2}$ is determined by an horizontal-adjacent independent set S, which contains the elements $(x_1, ..., x_m)$ where $x_1 = ..., = x_m \geq 2$ (see Figure 22.) Take again $G = C^m$, where $C = \{0, 1, ..., n-1\}$. We define the vertical crocheted lattice

$$\mathbb{L}^V_{m,n}$$

(see Figure 23). This is the sublattice of C^m which contains all elements in the form $(0, ..., 0, i, 0..., 0)$ and the diagonal elements $(j, j, ..., j)$ for $j > 2$.

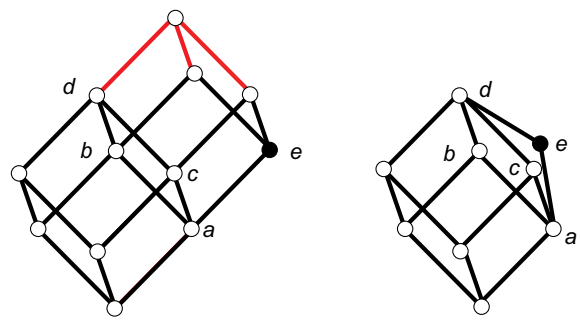

FIGURE 31. The nested lattice $\mathbb{L}^H_{2,7}$ with the skeleton

5.2.3. *The* $\mathrm{Jw}(L) = 3$ *case.* We define the 3-nested lattices similarly as in the the $\mathrm{Jw}(L) = 2$ case. We start with the ${C_2}^3$ Boolean lattice, this is a 3-nested lattice. We add a 3-fork to this 2^3-cell, we obtain the source lattice $\mathbb{L}_3$, see on Figure 27 and applay the "downward-going" procedure if it is necessery see on Figure 24, This is a 3-nested lattice. $\mathbb{L}_3 = S_{24}$ contains 7 2^3-cells. In some of these we insert 3-forks. All these are 3-nested lattices. If we continue this procedure we get all the 3-nested lattices.

5.3. **The patchwork system.** [26],[27] Matching is a special case of S-glued sum. The S-glued sum was introduced by Christian Herrmann [21].

Definition 20. *(S-glued system). Let S and $L_s, s \in S$, be lattices of finite length. The system $L_s, s \in S$ is called an* S-glued system *iff if the following conditions are satisfied:*

(1) For all $s, t \in S$, if $s \leq t$, then either $L_s \cap L_t = \emptyset$ or $L_s \cap L_t$ is a filter in L_s and an ideal in L_t.
(2) For all $s, t \in S$ with $s \leq t$ and for all $a, b \in L_a \cap L_b$, the relation $a \leq b$ holds in L_s iff $a \leq b$ in L_t.
(3) For all $s, t \in S$, the covering $s \prec t$ implies that $L_s \cap L_t =\neq \emptyset$.
(4) If $s, t \in S$, then $L_s \cap L_t \subseteq L_{s \wedge t} \cap L_{s \vee t}$.

Definition 21. *(S-glued sum) Let $L = \bigcup(L_s | s \in S)$, where $L_s, s \in S$ is an S-glued system. Let the partial order $\leq$ in L is defined as follows: for $a, b \in L$, let $a \leq b$ iff there exists a sequence $a = x_0, x_1, ..., x_n = b$ of elements of L and a sequence $s_0, ..., s_n$ of elements of S such that $s_i \leq s_{i+1}$ in $S, i = 1, ..., n-1$, and $x_{i-1} \leq x_i$ in $L_[s_i$, $i = 1, ..., n$. Then L is a lattice,the* S-glued sum *of $L_s, s \in S$.*

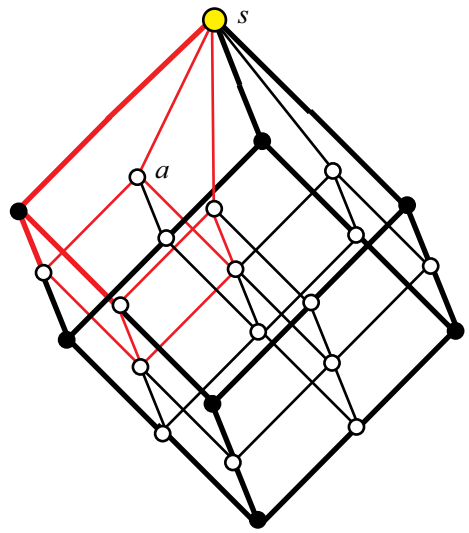

FIGURE 32. The source lattice $\mathbb{L}_3 = S_{24}$ and skeleton

We call the L_s components of the *blocks* of the S-glued sum. Any block is an interval in L. In this paper we will use as bocks special rectangular lattices.

Ch. Herrmann proved that every modular lattice L of finite length is the S-glued sum of its maximal complemented intervals (these are, obviously rectangular lattices).

In section 1.2 there are two examples. In the kitchen the mosaics (which are two dimensional) are glued together by an edge (one dimensional). The bricks (three dimensional) are glued together by a side (two dimensional).

Definition 22. *(Matching) Let S and $L_s, s \in S$, be semimodular lattices of finite length. The S-glued sum $L = \bigcup(L_s | s \in S)$, where $L_s, s \in S$ is called of L_s-s if*

$$\mathrm{Jw}(L_s \cap L_t) < min(\mathrm{Jw}(L_s), \mathrm{Jw}(L_t)).$$

Every planar distributive lattice is the matching of its covering squares and some of the edges (narrows), see Figure 2. For cubes, see Figure 4 and Figure 5.

5.4. **The planar semimodular lattices.** First, we consider the slim planar semimodular lattices (The slim planar semimodular lattice was defined in [15], these are planar lattices where the covering squares are intervals).

Theorem 7. *Every slim planar semimodular lattice is the matching of filters of nested lattices.*

Proof. Induction on the length. Take the equvivales relation $\mathcal{E}$. □

5.5. **The patchwork in higher dimensions.**

Theorem 8. *Let L be a semimodular lattice of J-width $n > 2$. Then L is the matching of filters of nested lattices.*

Proof. Similar to the proof of Theorem 6. □

Revisted: December 13, 2012

PART III

Structure theorems

Part 3. Structure theorems

6. Rectangular lattices as packing boxes and building stones.

6.1. The rectangular hulls of semimodular lattices.

Theorem 9. *Every semimodular lattice L has a rectangular extension R such that*

(1) L and R have the same length, i.e. L is a cover-preserving $(0,1)$-sublattice of R,
(2) L and R have the same dimension, $\mathbf{dim}(L) = \mathbf{dim}(R)$ (i.e. $\mathrm{J}(L)$ and $\mathrm{J}(R)$ have the same width).

R is called a *rectangular hull* of L (or packing box).

Proof. Let L be a semimodular lattice of length n and $\mathrm{Jw}(L) = k$. By Theorem 1, L is the cover-preserving join-homomorphic image of the distributive lattice F. Let Φ be the corresponding cover-preserving join-congruence, S denotes the source of Φ. On the other hand, by Lemma 1, F is the cover-preserving sublattice of the grid G. We extend Φ to G. Let a/b and c/d prime quotients of a lattice L. If $b \vee c = a$, $b \wedge c = d$ we say that a/b is perspective up to c/d and we write $a/b \searrow c/d$. This cover-preserving join-congruence will be denoted by $\overline{\Phi}$ and is defined as follows: for a covering pair $d \prec c$, $c, d \in G$, $c \equiv d$ $(\overline{\Phi})$ if and only if there is a source element $s \in S \subset F$ a lower cover $u \in F$ of s such that c/d is perspective up to s/u. It is easy to prove that the transitive extension is a cover-preserving join-congruence.

Then define R as:

$$R = G/\overline{\Phi}.$$

By the second isomorphism theorem we have the following isomorphisms and join-homomorphisms, presented on the diagram:

$$L \cong F/\Phi \cong (G/\Psi)/((\Psi \vee \overline{\Phi})/\Psi) \cong G/(\Psi \vee \overline{\Phi}) \cong (G/\overline{\Phi})/((\Psi \vee \overline{\Phi})/\overline{\Phi}) \cong R/\Psi,$$

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & L \\ \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \overline{\psi}} \\ G & \xrightarrow{\overline{\varphi}} & R \end{array}$$

Then $c = d \vee s$, which means that c is reducible in G. Assume that d is a join-irreducibble element of G. Every element of the ideal $\downarrow d$ is join-irreducible, i.e. $u \in F$ must be join-irreducible too. This means the join-irreducible element of F is in a non-trivial Φ-class, in contradiction to Lemma 2. This proves that evry join-irreducibble element of G is a one-element $\overline{\Phi}$-class, i.e. R is a rectangular lattice.

Φ and $\overline{\Phi}$ by Lemma 3, F and G have the same length, i.e. L is a cover-preserving sublattice of R.

By the definition of G it is trivial that $\mathrm{J}(L)$ and $\mathrm{J}(R)$ have the same width.

□

6.2. **The patchwork system.** Matching is a special case of S-glued sum. The S-glued sum was introduced by Christian Herrmann [21].

Definition 23. *(S-glued system). Let S and $L_s, s \in S$, be lattices of finite length. The system $L_s, s \in S$ is called an* S-glued system *iff if the following conditions are satisfied:*

(1) For all $s, t \in S$, if $s \leq t$, then either $L_s \cap L_t = \emptyset$ or $L_s \cap L_t$ is a filter in L_s and an ideal in L_t.
(2) For all $s, t \in S$ with $s \leq t$ and for all $a, b \in L_a \cap L_b$, the relation $a \leq b$ holds in L_s iff $a \leq b$ in L_t.
(3) For all $s, t \in S$, the covering $s \prec t$ implies that $L_s \cap L_t =\neq \emptyset$.
(4) If $s, t \in S$, then $L_s \cap L_t \subseteq L_{s \wedge t} \cap L_{s \vee t}$.

Definition 24. *(S-glued sum) Let $L = \bigcup(L_s | s \in S)$, where $L_s, s \in S$ is an S-glued system. Let the partial order $\leq$ in L is defined as follows: for $a, b \in L$, let $a \leq b$ iff there exists a sequence $a = x_0, x_1, ..., x_n = b$ of elements of L and a sequence $s_0, ..., s_n$ of elements of S such that $s_i \leq s_{i+1}$ in $S, i = 1, ..., n-1$, and $x_{i-1} \leq x_i$ in $L_[s_i$, $i = 1, ..., n$. Then L is a lattice,the* S-glued sum *of $L_s, s \in S$.*

We call the L_s components of the *blocks* of the S-glued sum. Any block is an interval in L. In this paper we will use as bocks special rectangular lattices.

Ch. Herrmann proved that every modular lattice L of finite length is the S-glued sum of its maximal complemented intervals (these are, obviously rectangular lattices).

In section 1.2 there are two examples. In the kitchen the mosaics (which are two dimensional) are glued together by an edge (one dimensional). The bricks (three dimensional) are glued together by a side (two dimensional).

Definition 25. *(Matching) Let S and $L_s, s \in S$, be semimodular lattices of finite length. The S-glued sum $L = \bigcup(L_s | s \in S)$, where $L_s, s \in S$ is called of L_s-s if*

$$\mathbf{dim}(L_s \cap L_t) < min(\mathbf{dim}(L_s), \mathbf{dim}(L_t)).$$

Every planar distributive lattice is the matching of its covering squares and some of the edges (narrows), see Figure 2. For cubes, see Figure 4 and Figure 5.

6.3. **The planar semimodular lattices.** First, we consider the slim planar semimodular lattices (The slim planar semimodular lattice was defined in [15], these are planar lattices where the covering squares are intervals).

Theorem 10. *Every slim planar semimodular lattice is the matching of filters of nested lattices.*

Proof. Induction on the length. Take the equvivales relation $\mathcal{E}$. □

6.4. **The patchwork in higher dimensions.**

Theorem 11. *Let L be a semimodular lattice of dimension $n > 2$. Then L is the matching of filters of nested lattices.*

Proof. Similar to the proof of Theorem 6. □

6.5. **Special embeddings.**

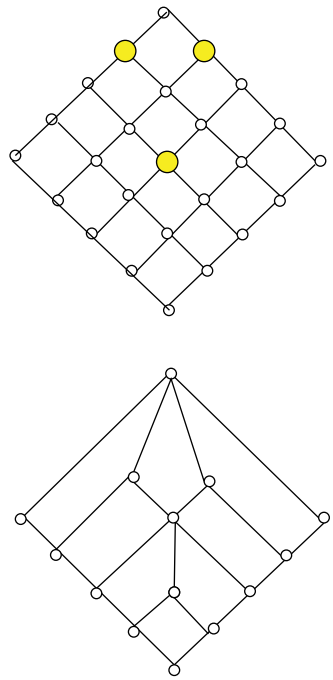

FIGURE 33. $\mathbb{B}[(4,3),(3,4),(2,2)]$

6.5.1. *Congruence-preserving embedding.* Let B be a sublattice of a lattice A. If $0_A, 1_A \in B$, then B is said to be a $(0,1)$-sublattice. If every congruence of A is determined by its restriction to B, then B is called a *congruence-determining* sublattice of A. Ideals that are chains will be called *chain ideals*. Consider the class $\mathcal{K} = \{L : L$ is a finite length semimodular lattice that has a congruence-determining chain ideal$\}$.

Theorem 12. *Let $L \in \mathcal{K}$, and let D be a $(0,1)$-sublattice of $\operatorname{Con} L$. Then there exists an $\overline{L} \in \mathcal{K}$ such that the restriction mapping $\rho : \operatorname{Con}\overline{L} \to \operatorname{Con} L$, $\theta \mapsto \theta\rceil_L$, is actually a $(0,1)$-lattice isomorphism $\operatorname{Con}\overline{L} \to D$; in particular, $\operatorname{Con}\overline{L} \cong D$.*

Theorem 13. *Every planar semimodular lattice L has a congruence-preserving extension K such that*

(1) K is a planar semimodular lattice,
(2) L is an almost-filter of K (it contains 1 and is a cover-preserving sublattice)
(3) K contains as ideal a chain C,
(4) C is congruence-determining.

6.5.2. *Embedding into geometric lattice.* It was proved by G Grätzer and E. W. Kiss [13], (for semimodular lattices of finite length see G. Czédli and E. T. Schmidt [5]):

Theorem 14. *Every finite semimodular lattice L has a cover-preserving embedding into a geometric lattice G.*

7. Modularity

7.1. **Skeleton.** G. Grätzer and R. W. Quackenbush proved in [20] that a planar distributive lattice D with more then two elements is isomorphic to the Skeleton(L), for a subdirectly irreducible planar modular lattice L iff L has no narrows. the results Theorem 4.9, Theorem 5.2 can be generalized for arbitrary finite order-dimension.

We prove that this statement is true for an arbitrary finite distributive lattice D with $\mathrm{Jw}(D) < \aleph_0$ (defined in 1.1).

Lemma 13. *A modular lattice L which contains a prime interval $\mathfrak{p}$ is subdirectly irreducible if and only if L is weakly atomic and any two prime intervals are projective.*

Proof. It is trivial that a modular lattice L which contains a prime interval $\mathfrak{p}$ and satisfies the given conditions is subdirectly irreducible.

Let L be a subdirectly irreducible modular lattice L which contains a prime interval $\mathfrak{p}$. We may assume that for any $a < b$

$$\mathrm{con}(\mathfrak{p}) \leq \mathrm{con}(a, b)$$

. By the weakly modularity ([19], p.194) the interval $[a, b]$ contains a prime interval $\mathfrak{q}$ such that $\mathfrak{p}$ and $\mathfrak{q}$ are projective,i.e.L is weakly atomic. □

Theorem 15. *Let D be a distributive lattice with more than two elements,* $\mathbf{dim}(D) < \aleph_0$*. Then D is isomorphic to* Skeleton(L)*, for a subdirectly irreducible modular lattice L, iff D is narrows free.*

Theorem 16. *There exists a subdirect irreducible modular lattice of order-dimension $\aleph_0$ which does not contain a prime interval.*

Proof. $\mathbb{Q}$ is the $[0, 1]$ chain of rational numbers. Glue together copies of $M_3[\mathbb{Q}]$-s, see E. T. Schmidt [23]. □

7.2. **Diamond-free semimodular lattices.**

7.3. **Modular source.**

Definition 26. *A source S of a distributive lattice D is called* modular *if the lattice D/Θ_S is modular.*

Lemma 14. *Let s be a source element of a grid G. Then G/Θ_s is modular lattice iff s is bastard.*

On the following two figures (Figure 30 and Figure 31) you can see the representation of M_3 with the grid resp. lower grid.

Let D be a planar distributive lattice. The J-width is two. Add doubly-irreducible elements to the interiors of some 4-cells (covering squares) you get a planar modular lattice M. If we start with a distributive lattice D of J-width 3 then we can extend some of the covering-cubes into modular non-distributive lattice. D is the skeleton.

Let Θ be a cover-preserving join-congruence of a grid G. Give a necessary and sufficient condition to be G/Θ modular.

Take the finite field $\mathbf{GF}(p^n)$, $p = 2, n = 1$. The corresponding two-dimensional projective geometry $F = \mathbf{GP_2}$ is the *Fano plane*. The one-dimensional lattice $\mathbf{GP_1}$ is M_3. It is clear that the J-width of F is 7.

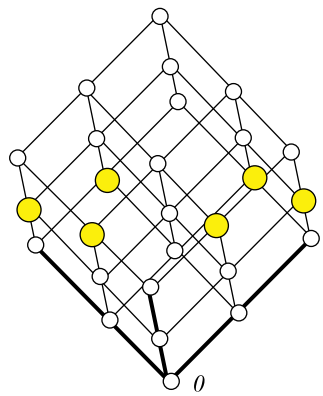

FIGURE 34. The grid of M_3 and the source

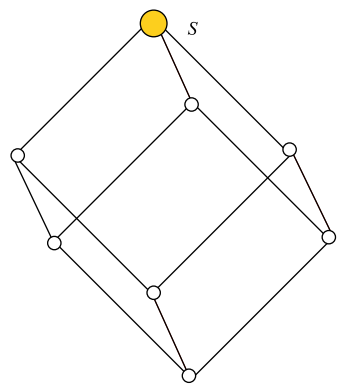

FIGURE 35. The lower grid of M_3 and the source element

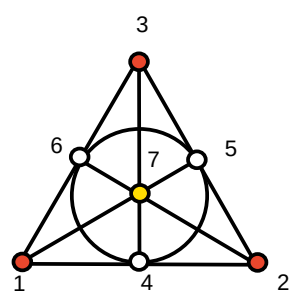

FIGURE 36. The Fano plane given by points and lines

On the next picture you can see the "traditional" presentation of the subspace lattice.

Draw the diagram a little bit others we get the following diagram for the same lattice (the same, but differently).

Here we see a cube (fat lines, the skeleton) and six circles on the faces of this cube and two circles (yellow) are inside the cube. If D is the direct product of three

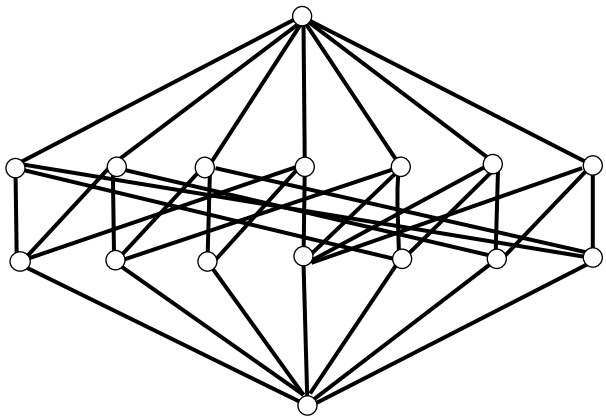

FIGURE 37. The "traditional" diagram of the subspace lattice of the Fano plane

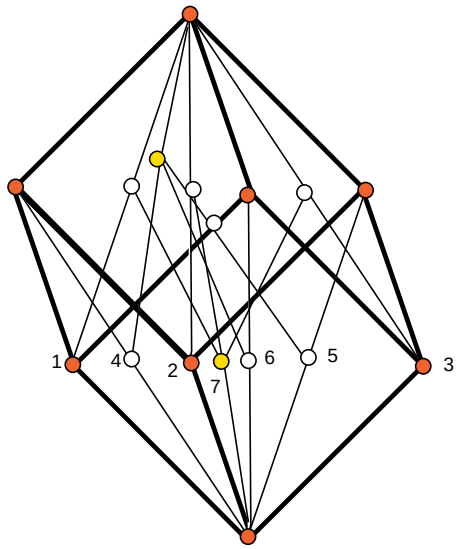

FIGURE 38. The subspace lattice of the Fano plane and the Kuroš-Ore skeleton

chains then this contains unit (covering) cubes. We can extend D if we put the Fano plane into some covering cubes this is the skeleton. (Fano plane "locked" in a cube.) By plain lattices we extend a covering square to an M_3.

8. The Grätzer-Kiss theorem

It was proved by G Grätzer and E. W. Kiss [13]:

Theorem. *Every finite semimodular lattice L has a cover-preserving embedding into a geometric lattice G.*

9. Reduction of J(L)

We delete some edges of a finite poset P, we get the poset Q. The poset P is called an *refinement* of Q or we say Q is an *reduction* (or pruning) of P. Let L be a finite semimodular lattice and take $P = \mathrm{J}(L)$.

Does there exit a semimodular lattice K which satisfies the following properties:

(1) $\mathrm{J}(K) \cong Q$,
(2) L is a cover-preserving sublattice of K?

The answer is no. (See Figure 31.)

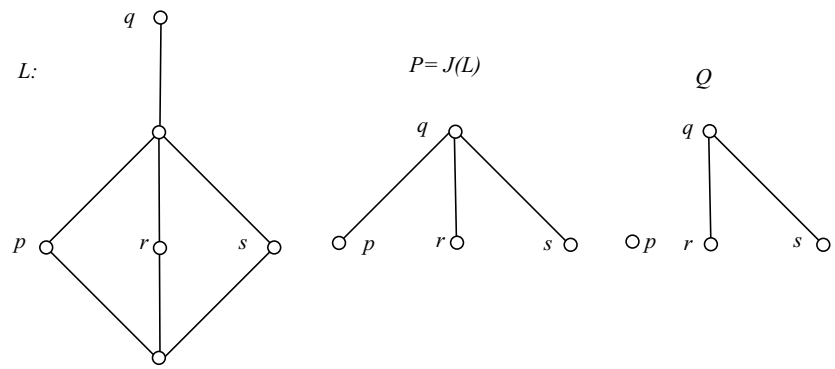

Figure 39. x

Here is a counter example (given by G. Czédli). Let L be a finite semimodular lattice and let Q be a reduction of $P = \mathrm{J}(L)$. For which Q has a cover-preserving embedding into a finite semimodular lattice K with the property $\mathrm{J}(K) \cong Q$.

See L, P and Q given in the Figure 7. Now L is semimodular. $P = \mathrm{J}(L)$. We obtain Q by deleting an edge from the diagram of P, so Q is a refinement of P. But there is no semimodular lattice K such that $\mathrm{J}(K) = Q$ and L is a cover-preserving sublattice of K. To show this, suppose the contrary. We may suppose that $\mathfrak{l}(K) = \mathfrak{l}(L)$ for otherwise K can be replaced with an interval of K. Clearly, r and s (as minimal join-irreducible elements) are atoms in K. By semimodularity the hight $h(r \vee s) = 2$ in K. Since $\mathrm{J}(K) = Q$, we have $r \vee s \leq q$ in K. We cannot have equality here, for $q \in \mathrm{J}(K) = Q$. Hence, $\mathfrak{l}(K) = 3 \leq h(q)$, implying $q = 1_K \geq p$ in K. This contradicts $\mathrm{J}(K) = Q$.

10. Appendix (Picture Gallery)

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