AN EXTENSION THEOREM
FOR PLANAR SEMIMODULAR LATTICES

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To Lászlo Fuchs,
our teacher,
on his 90th birthday

Abstract. We prove that every finite distributive lattice D can be represented as the congruence lattice of a rectangular lattice K in which all congruences are principal. We verify this result in a stronger form as an extension theorem.

1. Introduction

In G. Grätzer and E. T. Schmidt [16], we proved that every finite distributive lattice D can be represented as the congruence lattice of a sectionally complemented finite lattice K. In such a lattice, of course, all congruences are principal, using the notation of G. Grätzer [11], $\mathrm{Con}\, K = \mathrm{Princ}\, K$.

Since every finite distributive lattice D can be represented as the congruence lattice of a planar semimodular lattice K (see G. Grätzer, H. Lakser, and E. T. Schmidt [15]), it is reasonable to ask whether instead of the sectional complemented lattice of the previous paragraph, we can construct a planar semimodular lattice K.

G. Grätzer and E. Knapp [13] proved a result stronger than the Grätzer–Lakser–Schmidt result: every finite distributive lattice D can be represented as the congruence lattice of a rectangular lattice K—see Section 2.1 for the definition. (For a new proof of this result, see G. Grätzer and E. T. Schmidt [19].) Keeping this in mind, we prove:

Theorem 1. *Every finite distributive lattice D can be represented as the congruence lattice of a rectangular lattice K with the property that all congruences are principal.*

We prove this representation result in a much stronger form, as an extension theorem.

Theorem 2. *Let L be a planar semimodular lattice. Then L has an extension K satisfying the following conditions:*

(i) *K is a rectangular lattice;*
(ii) *K is a congruence-preserving extension of L;*
(iii) *K is a cover-preserving extension of L;*
(iv) *every congruence relation of K is principal.*

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Observe that we only have to prove Theorem 2. Indeed, let Theorem 2 hold and let D be a finite distributive lattice. By G. Grätzer and E. Knapp [13], there is a planar semimodular lattice K_1 whose congruence lattice is isomorphic to D. By Theorem 2, the lattice K_1 has a congruence-preserving extension K in which every congruence relation is principal. This lattice K satisfies the conditions of Theorem 1.

We will use the notations and concepts of lattice theory as in [8]. See [7] for a deeper coverage of finite congruence lattices. See G. Czédli and G. Grätzer [4] and G. Grätzer [9] for an overview of semimodular lattices, structure and congruences.

2. Background

We need some concepts and results from the literature to prove Theorem 2.

2.1. **Rectangular lattices.** Let L be a planar lattice. A *left corner* (resp., *right corner*) of the lattice L is a doubly-irreducible element in $L - \{0, 1\}$ on the left (resp., right) boundary of L. A *corner* of L is an element in L that is either a left or a right corner of L. G. Grätzer and E. Knapp [13] define a *rectangular lattice* L as a planar semimodular lattice which has exactly one left corner, $\mathrm{lc(L)}$, and exactly one right corner, $\mathrm{rc(L)}$, and they are complementary—that is, $\mathrm{lc(L)} \vee \mathrm{rc(L)} = 1$ and $\mathrm{lc(L)} \wedge \mathrm{rc(L)} = 0$. In a rectangular lattice L, there are four boundary chains: the lower left, the lower right, the upper left, and the upper right, denoted by $C_{\mathrm{ll}}(L)$, $C_{\mathrm{lr}}(L)$, $C_{\mathrm{ul}}(L)$, and $C_{\mathrm{ur}}(L)$, respectively.

Let A and B be rectangular lattices. We define the *rectangular gluing* of A and B as the gluing of A and B over the ideal I and filter J, where I is the lower left boundary chain of A and J is the upper right boundary chain of B (or symmetrically).

We recap some basic facts about rectangular lattices (G. Grätzer and E. Knapp [13] and [14], G. Czédli and E. T. Schmidt [5] and [6]).

Theorem 3. *Let L be a rectangular lattice.*

(i) *The ideal $\downarrow \mathrm{lc(L)}$ is the chain $C_{\mathrm{ll}}(L)$, and symmetrically.*
(ii) *The filter $\uparrow \mathrm{lc(L)}$ is the chain $C_{\mathrm{ul}}(L)$, and symmetrically.*
(iii) *For every $a \leq \mathrm{lc(L)}$, the interval $[a, \mathrm{rc(L)} \vee a]$ is a chain, and symmetrically.*
(iv) *For every $a \leq \mathrm{lc(L)}$, L is a rectangular gluing of the filter $\uparrow a$ and the ideal $\downarrow \mathrm{rc(L)} \vee a$.*
(v) *For every prime interval $\mathfrak{p}$ of the chain $[a, \mathrm{rc(L)} \vee a]$, there is a prime interval $\mathfrak{q}$ of the chain C_{lr} so that $\mathfrak{p}$ and $\mathfrak{q}$ are perspective.*

Note that it follows from (v) that

$$\mathrm{con}(C_{\mathrm{ul}}) = \mathrm{con}(a, \mathrm{rc(L)} \vee a) = \mathrm{con}(C_{\mathrm{lr}}).$$

2.2. **Eyes.** Let L be a planar lattice. An interior element of an interval of length two is called an *eye* of L. We will *insert* and *remove* eyes in the obvious sense. A planar semimodular lattice L is *slim* if it has no eyes.

2.3. **Forks.** We need from G. Czédli and E. T. Schmidt [6] the fork construction.

Let L be a planar semimodular lattice. Let L be slim. *Inserting a fork* into L at the covering square S, firstly, replaces S by a copy of S_7. We get three new covering squares replacing S of L. We will name the elements of the inserted S_7 as in Figure 1.

Secondly, if there is a chain $u \prec v \prec w$ such that the element v has just been inserted (the element a or b in S_7 in the first step) and $T = \{x = u \wedge z, z, u, w = z \vee u\}$ is a covering square in the lattice L (and so $u \prec v \prec w$ is not on the boundary of L) but $x \prec z$ at the present stage of the construction, then we insert a new element y into the interval $[x, z]$ such that $x \prec y \prec z$ and $y \prec v$, see Figure 2. We get two covering squares to replace the covering square T.

Let K denote the lattice we obtain when the procedure terminates (that is, when the new element is on the boundary); see Figure 3 for an example.

The new elements form an order, called a *fork* (the black filled elements in Figure 3). We say that K is obtained from L by *inserting a fork into* L at the covering square S.

Here are some basic facts, based on G. Czédli and E. T. Schmidt [6], about this construction.

Lemma 4. *Let L be a planar semimodular lattice and let S be a covering square in L. If L is slim, then inserting a fork into L at S we obtain a slim planar semimodular lattice K. If L is rectangular, so is K.*

If y is an element of the fork outside of S, then $[y_*, y]$ is up-perspective to $[o, a]$ or $[o, b]$, where y_* is the lower cover of y in $K - L$.

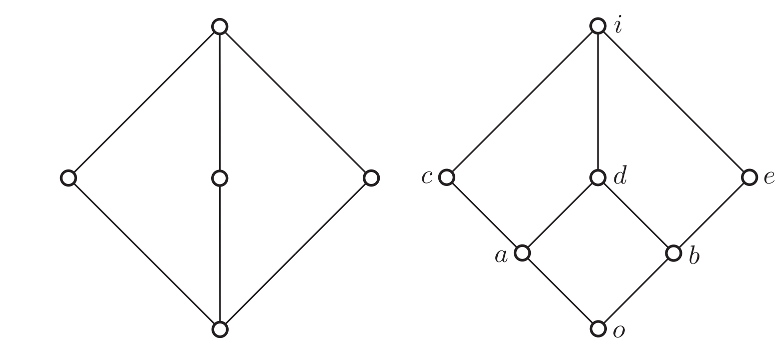

FIGURE 1. The lattices M_3 and S_7

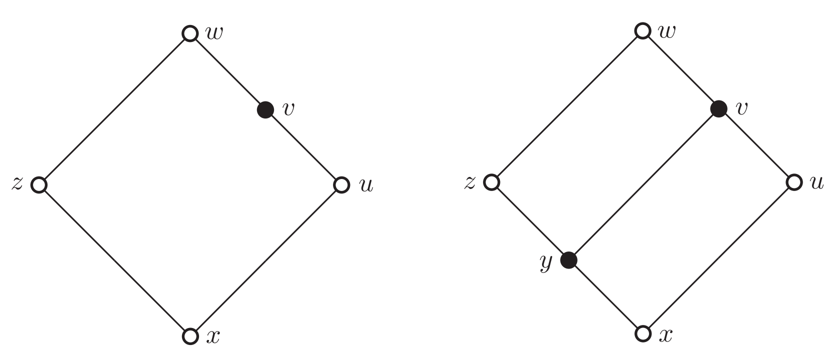

FIGURE 2. A step in inserting a fork

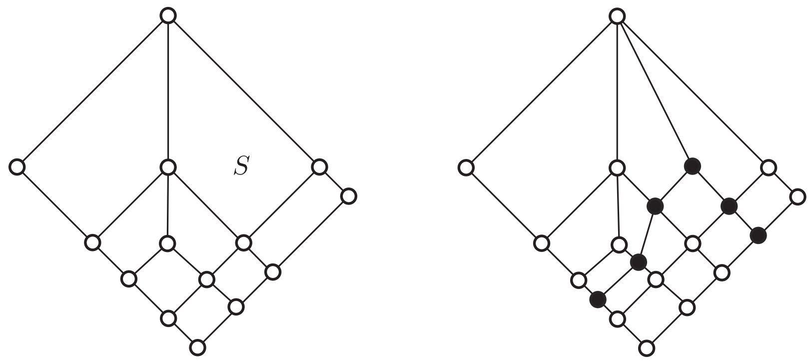

FIGURE 3. Inserting a fork at S

2.4. **Patch lattices.** Let us call a rectangular lattice L a *patch lattice* if lc(A) and rc(A) are dual atoms; Figure 1 has two examples. The next lemma is a trivial application of Lemma 4.

Lemma 5. *Let L be a slim patch lattice and let S be a covering square in L. Inserting a fork into L at S, we obtain a slim patch lattice K.*

2.5. **The structure theorems.** Now we state the structure theorems for patch lattices and rectangular lattices of G. Czédli and E. T. Schmidt [6].

Theorem 6. *Let L be a patch lattice. Then we can obtain L from the four-element Boolean lattice C_2^2 by first inserting forks, then inserting eyes.*

Theorem 7. *Let L be a rectangular lattice. Then there is a sequence of lattices*

$$K_1, K_2, \ldots, K_n = L$$

such that each K_i, for $i = 1, 2, \ldots, n$, is either a patch lattice or it is the rectangular gluing of the lattices K_j and K_k for $j, k < i$.

See also G. Grätzer and E. Knapp [14] and G. Grätzer [10].

2.6. **A congruence-preserving extension.** Finally, we need the following result of G. Grätzer and E. Knapp [13].

Theorem 8. *Let L be a planar semimodular lattice. Then there exists a rectangular, cover-preserving, and congruence-preserving extension K of L.*

3. CONGRUENCES OF RECTANGULAR LATTICES

To prove Theorem 2, we need a "coordinatization" of the congruences of rectangular lattices.

Theorem 9. *Let L be a rectangular lattice and let $\boldsymbol{\alpha}$ be a congruence of L. Let $\boldsymbol{\alpha}^l$ denote the restriction of $\boldsymbol{\alpha}$ to C_{ll}. Let $\boldsymbol{\alpha}^r$ denote the restriction of $\boldsymbol{\alpha}$ to C_{lr}. Then the congruence $\boldsymbol{\alpha}$ is determined by the pair $(\boldsymbol{\alpha}^l, \boldsymbol{\alpha}^r)$. In fact,*

$$\boldsymbol{\alpha} = \mathrm{con}(\boldsymbol{\alpha}^l \cup \boldsymbol{\alpha}^r).$$

Proof. Since $\boldsymbol{\alpha} \geq \mathrm{con}(\boldsymbol{\alpha}^l \cup \boldsymbol{\alpha}^r)$, it is sufficient to prove that

(P) if the prime interval $\mathfrak{p}$ of A is collapsed by the congruence $\boldsymbol{\alpha}$, then it is collapsed by the congruence $\mathrm{con}(\boldsymbol{\alpha}^l \cup \boldsymbol{\alpha}^r)$.

First, let L be a slim patch lattice. By Theorem 6, we obtain L from the square, C_2^2, with a sequence of n fork insertions. We induct on n.

If $n = 0$, then $L = \mathsf{C}_2^2$, and the statement is trivial.

Let the statement hold for $n - 1$ and let K be the patch lattice we obtain by $n - 1$ fork insertions into C_2^2, so that we obtain L from K by one fork insertion at the covering square S. We have three cases to consider.

Case 1. $\mathfrak{p}$ is a prime interval of K. Then the statement holds for $\mathfrak{p}$ and $\boldsymbol{\alpha}\rceil_K$, the restriction of $\boldsymbol{\alpha}$ to K by induction. So $\mathfrak{p}$ is collapsed by $\operatorname{con}((\boldsymbol{\alpha}\rceil_K)^l \cup (\boldsymbol{\alpha}\rceil_K)^r)$ in K. Therefore, (P) holds in L.

In the next two cases, we assume that $\mathfrak{p}$ is not in K.

Case 2. $\mathfrak{p}$ is perspective to a prime interval of K. Same proof as in Case 1. This case includes $\mathfrak{p} = [o, a]$ and any one of the new intervals up-perspective with $[o, a]$.

Case 3. $\mathfrak{p} = [a, c]$ and any one of the new intervals is up-perspective with $[a, c]$. Then the fork extension defines the terminating prime interval $\mathfrak{q} = [y, z]$ on the boundary of L which is up-perspective with $\mathfrak{p}$, verifying (P).

Secondly, let L be a patch lattice, not necessarily slim. This case is obvious because (P) is preserved when inserting an eye.

Finally, if L is not a patch lattice, we induct on $|L|$. By Theorem 7, L is the rectangular gluing of the rectangular lattices A and B over the ideal I and filter J. Let $\mathfrak{p}$ be a prime interval of L. Then $\mathfrak{p}$ is a prime interval of A or B, say, of A. (If $\mathfrak{p}$ is a prime interval of B, then the argument is easier.) By induction, $\mathfrak{p}$ is collapsed by $\operatorname{con}(\,\boldsymbol{\alpha}\rceil_{C_{\mathrm{ll}}(A)} \cup \boldsymbol{\alpha}\rceil_{C_{\mathrm{lr}}(A)})$, so it is collapsed by $\operatorname{con}(\,\boldsymbol{\alpha}\rceil_{C_{\mathrm{ll}}(L)} \cup \boldsymbol{\alpha}\rceil_{C_{\mathrm{lr}}(L)}\,) = \operatorname{con}(\boldsymbol{\alpha}^l \cup \boldsymbol{\alpha}^r)$. □

4. Construction

Now we proceed with the construction for the planar semimodular lattice L for Theorem 2.

Step 1. We apply Theorem 8 to get a rectangular, cover-preserving, and congruence-preserving extension K_1 of K.

Step 2. Let $D = C_{\mathrm{lr}}(K_1)$. We form the lattice D^2, and insert eyes into the covering squares of the main diagonal, obtaining the lattice $\widehat{D}$, see Figure 4.

Now we do a rectangular gluing of K_1 and $\widehat{D}$, obtaining the lattice K_2. Here is the crucial statement:

Lemma 10. *K_2 is a rectangular, cover-preserving, and congruence-preserving extension of L. For every join-irreducible congruence $\boldsymbol{\alpha}$ of L, there is a prime interval $\mathfrak{p}_{\boldsymbol{\alpha}}$ of $C = C_{\mathrm{ll}}(K_2)$ such that $\operatorname{con}(\mathfrak{p}_{\boldsymbol{\alpha}})$ in K_2 is the unique extension of $\boldsymbol{\alpha}$ to K_2.*

Proof. Indeed, by Theorem 9, there is a prime interval $\mathfrak{q}_{\boldsymbol{\alpha}}^l$ of $C_{\mathrm{ll}}(K_1)$ or a prime interval $\mathfrak{q}_{\boldsymbol{\alpha}}^r$ of $C_{\mathrm{lr}}(K_1)$ such that $\operatorname{con}(\mathfrak{q}_{\boldsymbol{\alpha}}^l)$ or $\operatorname{con}(\mathfrak{q}_{\boldsymbol{\alpha}}^r)$ in K_1 is the unique extension of $\boldsymbol{\alpha}$ to K_1. If we have $\mathfrak{q}_{\boldsymbol{\alpha}}^l \subseteq C_{\mathrm{ll}}(K_1) \subseteq C$, set $\mathfrak{q}_{\boldsymbol{\alpha}}^l = \mathfrak{p}_{\boldsymbol{\alpha}}$ and we are done.

If we have $\mathfrak{q}_{\boldsymbol{\alpha}}^r \subseteq C_{\mathrm{lr}}(K_1)$ with $\operatorname{con}(\mathfrak{q}_{\boldsymbol{\alpha}}^r)$ the unique extension of $\boldsymbol{\alpha}$ to K_1, then in K_2 there is a unique $\mathfrak{q} \subseteq C_{\mathrm{ll}}(\widehat{D}) \subseteq C_{\mathrm{ll}}(K_2)$ such that in $\widehat{D}$, the prime intervals $\mathfrak{q}_{\boldsymbol{\alpha}}^r$ and $\mathfrak{q}$ are connected by an M_3 on the main diagonal; see Figure 5 for an illustration.

Now clearly, we can set $\mathfrak{p}_{\boldsymbol{\alpha}} = \mathfrak{q}$. □

Note: Lemma 10 is a variant of several published results. Maybe G. Czédli [1, Lemma 7.2] is its closest predecessor.

Step 3. For the final step of the construction, take the chain $C = C_{\mathrm{ll}}(K_2)$ and a congruence $\boldsymbol{\alpha}$ of L. We can view $\boldsymbol{\alpha}$ as a congruence of K_2 and let $\boldsymbol{\alpha} = \boldsymbol{\gamma}_1 \vee \cdots \vee \boldsymbol{\gamma}_n$

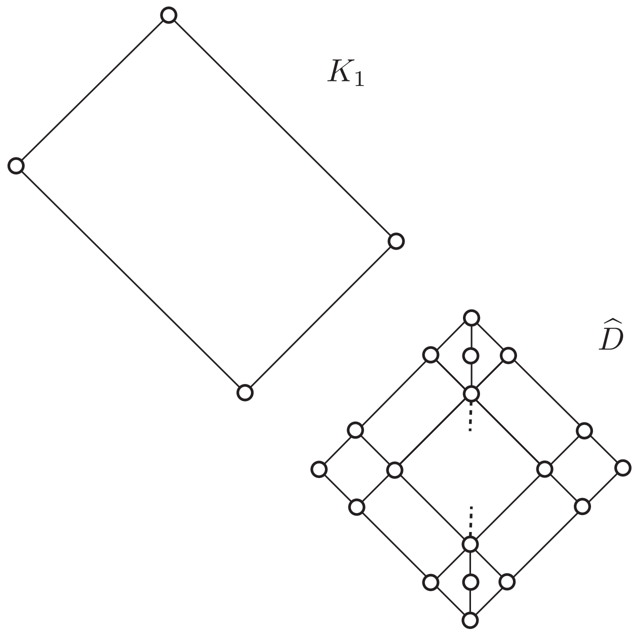

FIGURE 4. Step 2 of construction

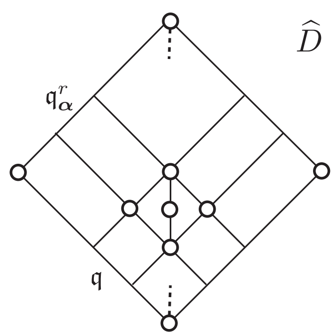

FIGURE 5. Step 2 of construction: a detail of the lattice $\widehat{D}$

be a join-decomposition of $\boldsymbol{\alpha}$ into join-irreducible congruences. By Theorem 9 and (P), we can associate with each $\boldsymbol{\gamma}_i$, for $i = 1, \dots, n$, a prime interval $\mathfrak{p}_i$ of C so that $\operatorname{con}(\mathfrak{p}_i) = \boldsymbol{\gamma}_i$.

We construct a rectangular lattice $C[\boldsymbol{\alpha}]$ (a cousin of $\widehat{D}$) as follows:

Let $\mathsf{C}_{n+1} = \{0 < 1 < \cdots < n\}$. Take the direct product $C \times \mathsf{C}_{n+1}$. We think of this direct product as consisting of n columns, column 1 (the bottom one), ..., column n (the top one).

In column i, for $1 \leq i \leq n$, we take the covering square whose upper right edge is perspective to $\mathfrak{p}_i$ and insert an eye. In the covering M_3 sublattice we obtain, every prime interval $\mathfrak{p}$ satisfies $\operatorname{con}(\mathfrak{p}) = \boldsymbol{\gamma}_i$. See Figure 6 for an illustration with $n = 3$; a prime interval $\mathfrak{p}$ is labelled with $\boldsymbol{\gamma}_i$ if $\operatorname{con}(\mathfrak{p}) = \boldsymbol{\gamma}_i$.

Let b denote the top element of the M_3 we constructed for $\mathfrak{p}_n$, clearly, we have $b \in C_{\mathrm{ur}}(C[\boldsymbol{\alpha}])$. Take the element $a \in C_{\mathrm{ll}}(C[\boldsymbol{\alpha}])$ so that the interval $[a,b]$ is a chain of length n. Then the n prime intervals $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ of $[a,b]$ satisfy

$$\mathrm{con}(\mathfrak{q}_1) = \boldsymbol{\gamma}_1, \ldots, \mathrm{con}(\mathfrak{q}_n) = \boldsymbol{\gamma}_n,$$

so $\mathrm{con}([a,b]) = \boldsymbol{\alpha}$, finding that in the lattice $C[\boldsymbol{\alpha}]$, the congruence $\boldsymbol{\alpha}$ is principal.

We identify C with $C_{\mathrm{ur}}(C[\boldsymbol{\alpha}])$; note that this is a "congruence preserving" isomorphism: for a prime interval $\mathfrak{p}$ of C, the image $\mathfrak{p}'$ of $\mathfrak{p}$ in $C_{\mathrm{ur}}(C[\boldsymbol{\alpha}])$ satisfies $\mathrm{con}(\mathfrak{p}) = \mathrm{con}(\mathfrak{p}')$.

Now we form the rectangular gluing of $C[\boldsymbol{\alpha}]$ with filter C and K_2 with the ideal C to obtain the lattice $K_2[\boldsymbol{\alpha}]$. Obviously, $K_2[\boldsymbol{\alpha}]$ is a rectangular lattice, it is a cover-preserving congruence-preserving extension of K_2 and, therefore, of L.

It is easy to see that $C_{\mathrm{ll}}(K_2[\boldsymbol{\alpha}])$ is still (congruence) isomorphic to C; for a rigorous treatment see the Corner Lemma and the Eye Lemma in G. Czédli [1] as they are used in the proof of [1, Lemma 7.2]. We can continue this expansion with all the congruences of L. In the last step, we get the lattice $K_3 = K$, satisfying all the conditions of Theorem 2.

4.1. **Discussion.** Let L be a rectangular lattice and let $\boldsymbol{\alpha}$ be a join-irreducible congruence of L. We call $\boldsymbol{\alpha}$ *left-sided*, if there a prime interval $\mathfrak{p} \subseteq C_{\mathrm{ll}}(L)$ such that $\mathrm{con}(\mathfrak{p}) = \boldsymbol{\alpha}$ but there is no such $\mathfrak{p} \subseteq C_{\mathrm{lr}}(L)$. In the symmetric case, we call $\boldsymbol{\alpha}$ *right-sided*. The congruence $\boldsymbol{\alpha}$ is *one-sided* if it is left-sided or right-sided. The congruence $\boldsymbol{\alpha}$ is *two-sided* if it is not one-sided.

Using these concepts, we can further analyze Theorem 9 and condition (P). By Theorems 6 and 7, we build a rectangular lattice from a grid (the direct product of two chains) by inserting first forks and then eyes. At the start, all join-irreducible congruences are one-sided. When we insert a fork, we introduce a two-sided congruence. When we insert an eye, we identify two congruences, resulting in a two-sided congruence.

What congruence pairs occur in Theorem 9? Let $\boldsymbol{\beta}_l$ be a congruence of $C_{\mathrm{ll}}(L)$ and let $\boldsymbol{\beta}_r$ be a congruence of $C_{\mathrm{lr}}(L)$. Under what conditions is there a congruence $\boldsymbol{\alpha}$ of L such that $\boldsymbol{\alpha}^l = \boldsymbol{\beta}_l$ and $\boldsymbol{\alpha}^r = \boldsymbol{\beta}_r$? Here is the condition: If $\mathfrak{p}$ is a prime

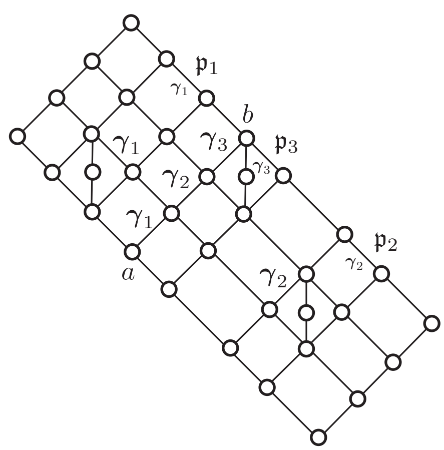

FIGURE 6. Step 3 of construction: the lattice $C[\boldsymbol{\alpha}]$

interval of $C_{\mathrm{ll}}(L)$ collapsed by $\boldsymbol{\beta}_l$ and there is a prime interval $\mathfrak{q}$ of $C_{\mathrm{lr}}(L)$ with $\mathrm{con}(\mathfrak{p}) = \mathrm{con}(\mathfrak{q})$, then $\mathfrak{q}$ is collapsed by $\boldsymbol{\beta}_r$; and symmetrically.

In Step 3 of the construction, we use the chain C_{n+1}. Clearly, C_n would have sufficed. Can we use, in general, shorter chains?

In a finite sectionally complemented lattice, the congruences are determined around the zero element. So it is clear that for finite sectionally complemented lattices, all congruences are principal.

For a finite semimodular lattice, the congruences are scattered all over. So it is somewhat surprising that Theorem 1 holds.

For modular lattices, the situation is similar to the semimodular case. E. T. Schmidt [21] proved that every finite distributive lattice D can be represented as the congruence lattice of a countable modular lattice K. (See also G. Grätzer and E. T. Schmidt [17] and [18].) It is an interesting question whether Theorem 1 holds for countable modular lattices.

The congruence structure of planar semimodular lattices is further investigated in three recent papers: G. Czédli [2], [3] and G. Grätzer [12].

References

[1] G. Czédli, *Representing homomorphisms of distributive lattices as restrictions of congruences of rectangular lattices.* Algebra Universalis **67** (2012) 313–345.

[2] G. Czédli, *Patch extensions and trajectory colourings of slim rectangular lattices.* Algebra Universalis.

[3] G. Czédli, *A note on congruence lattices of slim semimodular lattices.* Algebra Universalis.

[4] G. Czédli and G. Grätzer, *Planar Semimodular Lattices: Structure and Diagrams.* Chapter in [20].

[5] G. Czédli and E. T. Schmidt, *Slim semimodular lattices. I. A visual approach,* Order **29** (2012), 481-497.

[6] ______, *Slim semimodular lattices. II. A description by patchwork systems,* Order **30** (2013), 689–721.

[7] G. Grätzer, The Congruences of a Finite Lattice, A *Proof-by-Picture* Approach. Birkhäuser Boston, 2006. xxiii+281 pp. ISBN: 0-8176-3224-7.

[8] G. Grätzer, Lattice Theory: Foundation. Birkhäuser Verlag, Basel, 2011. xxix+613 pp. ISBN: 978-3-0348-0017-4.

[9] G. Grätzer, *Planar Semimodular Lattices: Congruences.* Chapter in [20].

[10] ______, *Notes on planar semimodular lattices. VI. On the structure theorem of planar semimodular lattices.* Algebra Universalis.

[11] ______, *The order of principal congruences of a lattice.* Algebra Universalis **70** (2013), 95–105. arXiv: 1302.4163

[12] ______, *Congruences of fork extensions of lattices.* Acta Sci. Math. (Szeged). arXiv: 1307.8404

[13] G. Grätzer and E. Knapp, *Notes on planar semimodular lattices. III. Rectangular lattices.* Acta Sci. Math. (Szeged) **75** (2009), 29–48.

[14] ______, *Notes on planar semimodular lattices. IV. The size of a minimal congruence lattice representation with rectangular lattices.* Acta Sci. Math. (Szeged) **76** (2010), 3–26.

[15] G. Grätzer, H. Lakser, and E. T. Schmidt, *Congruence lattices of finite semimodular lattices.* Canad. Math. Bull. **41** (1998), 290–297.

[16] G. Grätzer and E. T. Schmidt, *On congruence lattices of lattices,* Acta Math. Acad. Sci. Hungar. **13** (1962), 179–185.

[17] ______, *On finite automorphism groups of simple arguesian lattices,* Studia Sci. Math. Hungar. **35** (1999), 247–258.

[18] ______, *On the Independence Theorem of related structures for modular (arguesian) lattices.* Studia Sci. Math. Hungar. **40** (2003), 1–12.

[19] ______, *A short proof of the congruence representation theorem of rectangular lattices.* arXiv: 1303.4464. Algebra Universalis (2013).

[20] G. Grätzer and F. Wehrung eds., Lattice Theory: Empire. Special Topics and Applications. Birkhäuser Verlag, Basel.

[21] E. T. Schmidt, *Every finite distributive lattice is the congruence lattice of some modular lattice.* Algebra Universalis **4** (1974), 49-57.

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