DIAMOND-FREE 3-DIMENSIONAL SEMIMODULAR LATTICES

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Abstract. A semimodular lattice L is called *diamond-free* if has no sublattice isomorphic to M_3. In [1] there are several characterizations of these lattices. In the present paper we give a new characterization of these lattices which is a **visual structure theorem**.

1. Introduction

We begin with the following representation theorem [2]:

Theorem 1. *Each finite semimodular lattice L is a cover-preserving join-homomorphic image of a distributive lattice G - the grid - which is the direct product of $n = \mathrm{w}(\mathbf{J}(L))$ finite chains.*

$n = \mathrm{w}(\mathbf{J}(L))$ is the dimension of L and $\mathrm{w}(P)$ is the width of the order P. $G = C^3$, where C is a chain. Let Θ be the join-congruence of $G = C^3$ induced by the cover-preserving join-homorphism. Θ_i denotes the projction of Θ to the components.

Theorem 2. *The 3-dimensional semimodular lattice $L = (G, \Theta)$ is diamond-free if and only if Θ satisfies the following property:*

(DF) let $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3), a \prec b$, $a \equiv b$ (Θ) implies that for one of $i \in \{1, 2, 3\}$ $a_i \equiv b_i$ (Θ_i).

Θ is determined by a subset S of G.

Definition 1. *An element $s \in G$ is called a* source element *of Θ if there is a $t, t \prec s$ such that $s \equiv t$ (Θ) and for every prime quotient u/v if $s/t \searrow u/v, s \neq u$ imply $u \not\equiv v$ (Θ). The set $\mathcal{S}_\Theta$ of all source elements of Θ is the* source *of Θ.*

If we have a class $\mathcal{K}$ of diamond-free semimodular lattices then we can get further lattices of this kind by applying the following constructions:

(1) Direct product of elements of $\mathcal{K}$,
(2) Sublattice of a lattice $L \in \mathcal{K}$,
(3) The Hall-Dilworth gluing of members of $\mathcal{K}$.

A semimodular lattice L has dimension n if n is the smallest natural number suc that L is the cover-preserving join-homomorphic image of the direct product of n chains, (see more in the dimension in the next section). In this paper we start with a subclass $\mathcal{D}_2$ of diamond-free 2-dimensional semimodular lattices, we call these 2-dimensional patch lattices, these are the building stones of the 2-dimensional semimodular lattices. We define the generalized Edelman-Jaison lattices, which are diamond-free 3- dimensional lattices. Then we apply (1)-(3) and we get all 3-dimensional diamond-free semimodular lattices.

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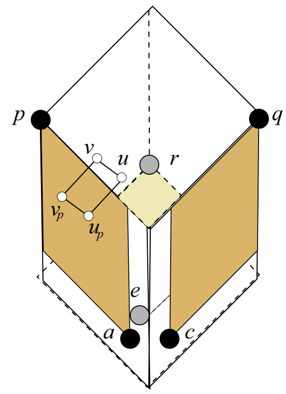

FIGURE 1. Diamond-free area of the grid

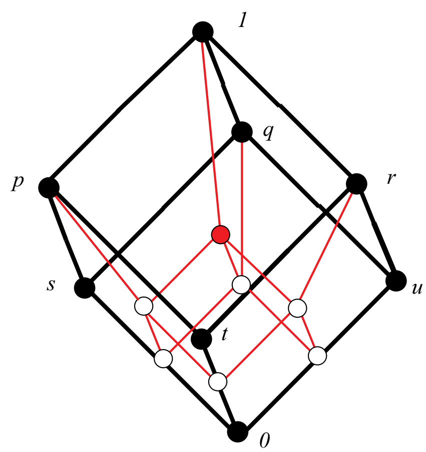

FIGURE 2. The Edelman-Jaison lattice

Let D be a finite planar distributive lattice. This looks like the lattice presented in Figure 3. You can see that this is a special glued sum of covering squares (mosaics), in this case the patch lattice is the four-element boolean lattice. Two neighboring mosaics are glued over an edge. We say that D is the pathwork of the given mosaics. In [3] we generalized this construction for the class of slim semimodular lattices and we characterized the 2-dimensional patch lattices. We define further 3-dimensional diamond-free lattices.

Definition 2. *A 2-dimensional patch lattice P, is a planar semimodular lattice which has two dual atoms p and q such that $p \wedge q = 0$. The sublattice $\{0, p, q, 1\}$ is the skeleton $\mathbf{Sk}(P)$ of P.*

In the distributive case the ony one 2-dimensional patch lattice is the four element boolean lattice. The smallest non-distributive patch lattice is N_7 (see in Figure 4). A 3-dimensional diamond-free semimodular lattice is the *Edelman-Jaison lattice* [4], see Figure 2.

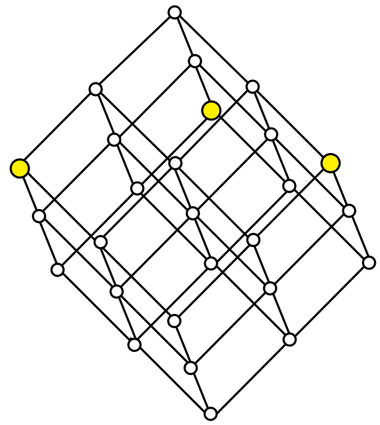

FIGURE 3. The source of the Edelman-Jaison lattice

Definition 3. *A 3-dimensional semimodular lattice L, is a generalized Edelman-Jaison lattice (*$\mathbf{EJ}$ *lattice) if it has skeleton* $\mathbf{Sk}(L) = \{1, p, q, r, s, t, u, 0\}$ *such that* p, q *and* r *are dual atoms and* $s \prec p, s \prec q, t \prec p, t \prec r, u \prec q, u \prec r$.

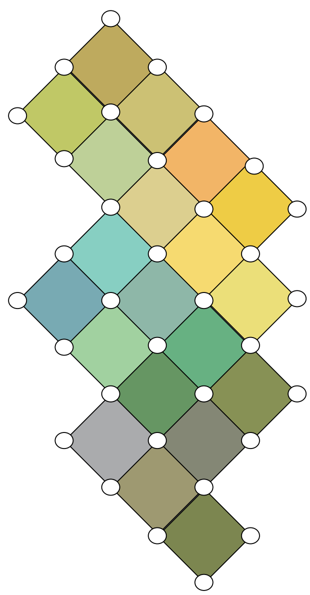

FIGURE 4. A planar distributive lattice

Patchwork can be defined for higher dimesional semimodular lattices. We glue together over faces lattices of this type we obtain a lattice presented in Figure 3.

Conjecture. *A 3-dimensional semimodular lattice L is diamand-free if and only if L is the patchwork of generalized Edelman-Jaison lattices and lattices of type $P \times C_2$, where P is a patch lattice.*

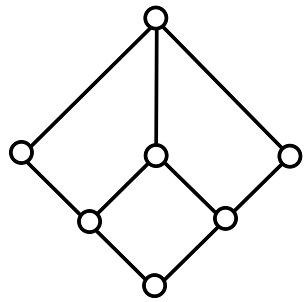

FIGURE 5. Semimodular but non modular lattice S_7

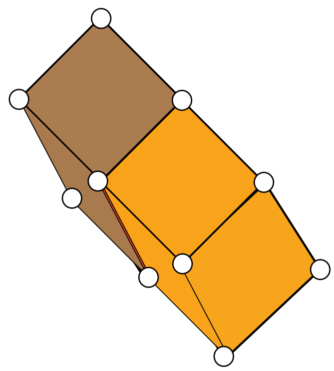

FIGURE 6. A 3-dimensional patchwork: gluing ot two cubes over faces

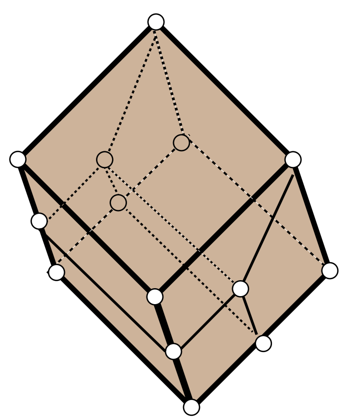

FIGURE 7. A monochromatic building stone $S_7 \times C_2$

The patch lattices - in the two-dimensional case - doesn't contain M_3 and therefore the slim semimodular lattices are diamond-free. dimensional lattice.

Theorem . asserts that in the 3-dimensional case we have the following lattices as building stones of the diamond-free semimodular lattices: $P, P \times C_2, C_2, {C_2}^2, {C_2}^3$ and lattices of type presented in Figure 2 ($S_7 \times C_2$). In Figure 5 the green cube

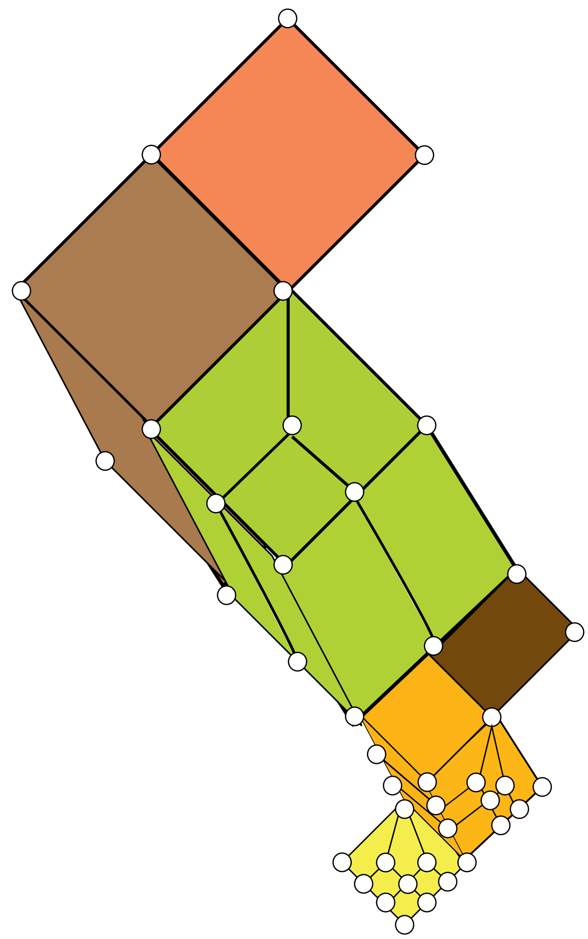

FIGURE 8. A 3-dimensional diamond-free semimodular lattice

is $S_7 \times C_2$. Color these cubes with different colors and glue together to get a diamand-free 3-dimensional patchwork.

2. Source lattices

In Figure 8. we have a 3-dimensional patchwork of 8 monochromatic cubes.

Definition 4. *The* beret $\Delta = \{1, q_i \prec 1\}$ *(swiss cup) of a distributive lattice L is the unit element and the dual atoms of L.*

The beret is a join-sublattice (order dual-ideal) of L. Take the cover-preserving join-congruence of L the where the beret is the only one non-trivial congruence class. We denote this cover-preserving join-congruence by the same letter Δ

We introduce the following notations:

$$\mathbb{S}_k = {C_3}^k/\Delta \text{ and } \mathbb{B}_k = {C_2}^k/\Delta.$$

Definition 5. $\mathbb{S}_k$ *is called the k-dimensional* source lattice *and* $\mathbb{B}_k$ *is the k-dimensional* source kernel.

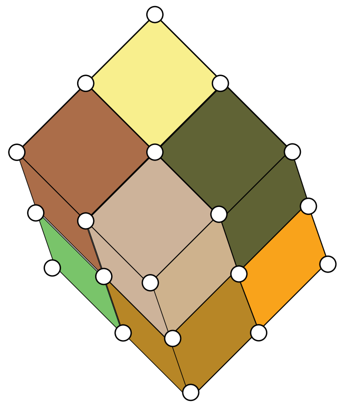

FIGURE 9. A 3-dimensional rectangular patchwork

$\mathbb{B}_k$ is a filter of $\mathbb{S}_k$. The source lattices are rectangular lattices. The ${C_2}^k$ boolean lattice is a (0,1)-sublattice of $\mathbb{S}_k$. This is the *skeleton* of $\mathbb{S}_k$ and will be denoted by $\mathbf{Sk}(\mathbb{S}_k)$. The dimension of ${C_2}^k$ is k and therefore: $\mathbf{Dim}(\mathbb{S}_k) = \mathbf{Dim}(\mathbb{B}_k) = \mathbf{dim}(\mathbb{S}_k) = k$. $\mathbb{B}_3$ is M_3 which means that these lattice has dimension 3.

2.1. **Diamond-free area.** The *rank*, $\mathrm{r}(g)$ of a grid element $g \in G$ is the number of the lower covers of g. Obviously, $\mathrm{r}(g) = 1$ means that g is a join-irreducible element.

Let s be an element of the grid $G = {C_n}^3$. We denote by Θ_s the smallest cover-preserving join-congruence, where $s \equiv x$ (Θ_s) for some $x \prec s$. We ask: for which s is G/Θ_s diamond-free ? In Figure 9 the yellow area $F = [a, b] \cup [c, d] \cup [e, f]$. $a = (1, 1, 0), b = (n-1, n-1, 0), c = (0, 1, 1), d = (0, n-1, n-1), e = (1, 0, 1), f = (n-1, 0, n-1)$. The three intervals in Figure 5. is the set of all elements of rank two.

Lemma 1. *G/Θ_s is diamond-free if and only if $\mathrm{r}(s) = 2$ (i.e. $s \in F$).*

By Theorem 1 L is a glued sum of patchwork irreducible lattices. If the dimension of each component is already defined then the dimension of L is the maximum of the dimensions of the components.

3. PATCH LATTICES

3.1. **The 2-dimensional patch lattices.** . In [3] we geved several characterizations of 2-dimensional patch lattices. The original definition is the following.

Firstly, we take $P_1 = {C_2}^2$ and we insert into this lattice $S_7 = \mathbb{S}_2$, we get obviously $P_2 = S_7$ and $\{a \wedge b, a, b, a \vee b\}$ form the skeleton of P_2. With other words, we replace ${C_2}^2$ by a copy of S_7. This lattice has three covering squares (cells). By the next step we insert into some of this covering squares (cells) a copy of S_7 getting P_2.

Secondly, as long as there is a chain $f \prec c \prec a$ such that v is a new element and $T = \{x = f \wedge z,\ z,\ a,\ w = z \vee f\}$ is a 4-cell in the original lattice L but $x \prec z$ at the present stage, see Figure10, we insert a new element y such that $x \prec y \prec z$ and $y \prec c$. (This way we get two 4-cells to replace the cell T.) When this "downward-going" procedure terminates, we obtain L'. (This construction was introduced in [3]. The collection of all new elements, which is a poset, will be called a *fork*. We

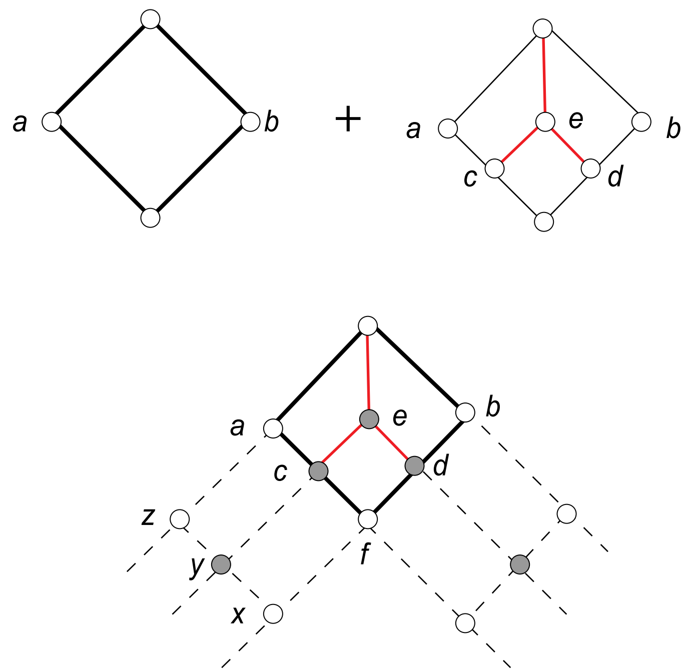

FIGURE 10. How can we get patch lattices ?

say that L' is obtained from L by *adding a fork to* L (*at the* 4*-cell* S), see Figure 9 for an illustration. Adding forks to L means to add several forks to L one by one). The 2-dimensional patch lattices are $P_1 = {C_2}^2$, $P_2 = S_7$, P_3, ...

The patch lattices are rectangular lattices for which the Kuroš-Ore dimension exist. The skeleton of these lattices is the four element Boolean lattice. In [3] we have given several characterizations of 2-dimensional patch lattices.) In [3], Theorem 3.4. we have given several characterizations of the patch lattices among them (Theorem 3.4. is formulated for planar semimodular lattices, reformulate for slim lattices):

Lemma 2. *Let P be a slim semimodular lattice. Then the following conditions are equivalent:*

(1) P is a patch lattice, i.e. we obtain P from the four-element boolean lattice by adding finitely many forks one by one;
(2) P is a patchwork irreducible lattice,
(3) P is indecomposable with respect to the Hall-Dilworth gluing;
(4) P is indecomposable with respect to the Hall-Dilworth gluing over chains;
(5) P has two dual atoms p and q such that $p \wedge q = 0$.

If P is a patch lattice then $\mathbf{Dim}(P) = \mathbf{dim}(P) = 2$.

3.2. **The 3-dimensional patch lattices.** 3-dimensional patch lattices are:

These are 3-dimensional lattices such that the matrix is an invertible 3-matrix. In the modular case we have only one, M_3. In the non-modular case case these can be caracterised with the property that these have a skeleton which isthe ${C_2}^3$

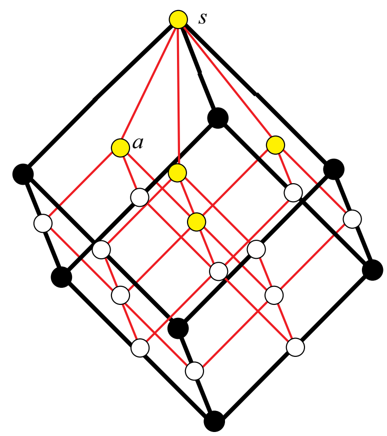

FIGURE 11. The source lattice $\mathbb{S}_3$ with a diamond and skeleton

boolean lattice and the dual atoms of this boolean lattice are dual atoms of of the given lattice.

3-dimensional patch lattices (see in [5]) are for example:

(1) the boolean lattice ${C_2}^3$, (distributiv, diamond-free),
(2) 3-dimensional projective geometries, (modular)
(3) $Q \times C_2$, where Q is a 2-dimensional patch lattice, (diamand-free),
(4) 3-dimensional source lattice $\mathbb{S}_3$,
(5) the Edelman-Jaison lattice.

We begin with the eight element Boolean lattice $P_1 = {C_2}^3$, this is a patch lattice and this will be the skeleton of the patch lattices. We consider this as a 8- cell. We make word by word the same as in the 2-dimensional case, we insert into P_1 the source lattice $\mathbb{S}_3$, getting P_2. Don't forget we must go downward and add some further new elements (Figure 8). We can consider in this lattice a 8-cell and insert into this cell $\mathbb{S}_3$.

4. PATCHWORK

Let K and L be semimodular lattices, let F be a filter of K, and I be an ideal of L, then we can form the the lattice G, the well-known Hall-Dilworth gluing of K and L over F and I. Assume that $\mathbf{Dim}(K)$, $\mathbf{Dim}(L)$, $\mathbf{Dim}(F)$, $\mathbf{Dim}(I)$ are defined. We call the gluing G the *patching* of K and L if:

$$\text{(Dim)} \qquad \mathbf{Dim}(I) < \min(\mathbf{Dim}(K), \mathbf{Dim}(L)).$$

Let $\{M_i\}$ be a system of intervalls - called blocks if $\{M_i\} = L$ and if $M_i \cap M_j \neq \emptyset, i \neq j$, then the union, $M_i \cup M_j$ is the Hsall-Dilworth gluing satisfying (DID)(i.e. the gluing is via an edge or face. If K_1, K_2 and K_3 are face (2-dimesional) of blocks such that (see in Figure 1) $K_1 \cap K_2$ is an ideal of K_2 and similarly $K_1 \cap K_3$ is an ideal of K_3 and

$$1_2 \wedge 1_3 = 1_1$$

then this is a *patching system*, see Figure 2.

The blocks of the patchwork system are special rectangular lattices the patch latt. To make the figures more spectacular we can use colors. If we color the blocks, that means we have a set $\mathcal{C}$ – the elements are called *colors* – and a mapping $\psi : \{M_i\} \to \mathcal{C}$ such that if $M_i \cap M_j \neq \emptyset, i \neq j$ M_i and M_j have different colors. The patchwork is considered always in respect a dimension concept.

5. The proof of Theorem 1

Lemma 3. *A rectangular semimodular lattice L is diamond-free if and only if it every source element is of rank two.*

Let L be a semimodular lattice. By [6] every semimodular lattice is the cover-preserving sublattice of a rectangular lattice and therefore we may assume that L is rectangular. It is clear that the given conditions are sufficient.

To prove that the conditions are necessary let L be a semimodular lattice of dimension 3, $\mathbf{dim}(L) = 3$ (the general case is similar). Then the grid G is the direct product of three chains, $G = C^3$. By Theorem 2 there is a congruence-preserving join-congruence Θ on G such that $L \cong G/\Theta$. Let $\mathcal{S}$ be the source of Θ, then the rank, $r(s)$ of a source element is $r(s) \leq 3$. $r(s)$ can't be 3, the source lattice $\mathbb{S}_3$ contains a diamond i.e. the source lattice of $\mathcal{S}$ is $\mathbb{S}_2 = {C_3}^2 \cong S_7$, which is diamond-free. Then $r(s) = 2$, i.e. s is in the flap of the grid G (which is a cube) see in Figure 8.

Then s is the unit element of this source lattice (it is a cover preserving sublattice). The dimension is three which implies the existence of an upper cover c of s. The elements a, b, c span a covering cube. Take the sublattice of L generated by the interval $[a \wedge b, s] \cong S_7$, this is obviously $S_7 \times C_2$. Let a and b the two lower cover of s.

If $r(s) = 2$ (see Figure 8) then there is another source element t then we "extened" the interval $[a \wedge b, s] \cong S_7$ to a patch lattice Q and on this way we get a covering cube isomorphic to $Q \times C_2$. This process is described in [7].

The sublattices of form $Q \times C_2$ belongs to source elements. the remaining blocks of the patchwork are unit-cubes of G.

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