CONGRUENCE-DETERMINING CHAIN IDEALS OF SEMIMODULAR LATTICES

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Abstract. Every semimodular lattice $L$ has a congruence-preserving extension to a semimodular lattice $K$ such that: $K$ contains as ideal a congruence-determining chain $C$ and every congruence is principal.

1. Introduction

If every congruence of $A$ is determined by its restriction to $B$, then $B$ is called a congruence-determining sublattice of $A$.

We prove:

Theorem. Every semimodular lattice $L$ has a congruence-preserving extension $K$ such that:

1. $L$ is the cover-preserving sublattice of a filter of $K$,
2. $K$ is a semimodular lattice,
3. $K$ contains as ideal a chain $C$,
4. $C$ is congruence-determining,
5. every congruence is principal.

2. The planar case

Proof. First, we consider the planar case. By Theorem 7. of G. Grätzer, E. Knapp, [1] $L$ has a congruence-preserving extension to a planar rectangular semimodular lattice $R = \mathcal{T}$, see on Figure 1. $[c, a]$ and $[c, b]$ are chain ideals. Let $n$ be the length of $[c, a]$, i.e. $n = l([c, a])$. We color the prime intervals of $[c, a]$ by $p_1, p_2, ..., p_n$. The colors correspond to the congruences of $L$. Let $\pi = \{p_{i_1}, p_{i_2}, ..., p_{i_k}\}$ be a permutation a subset of the color set $\{p_1, p_2, ..., p_n\}$.

The congruences of $R$ are determined by the restrictions to $V = [c, a] \cup [c, b]$. (Let us remark that the chain $[b, 1]$ can be shorter then $[c, a]$.)

$M$ is direct product of two chains of length $n$ and length $m$, $n \leq m$, in some covering squares we insert a double irreducible element to get $M_3$-s, see Figure 2. We color the prime intervals of $[e', c']$ in $M$ by given colors such that a subinterval $[u, v]$ of $[e', c']$ is colored by $\pi$. Insert a double-irreducible element into every monochromatic covering square to get $M_3$-s. To get $K$ we apply gluing for $R, M$, see Figure 3. Then $R$ is a filter of $K$.

$C$ is the ideal $[e, b]$ on the right side (black filled circles). It is easy to prove, that $K$ satisfies the properties, listed in the Theorem 1.

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2.1. Minimal set of principals.

3. Higher dimension

New, we consider the \( \text{Jw}(L) \geq 3 \) case. We construct \( K \) in several steps.

Step 1. Let \( L \) be a semimodular lattice of length \( n \), \( \text{Jw}(L) = 3 \). Then there is a chain \( D \) of length \( n \) such that \( L \) is the cover-preserving join-homomorphic image of \( D^3, \varphi(D^3) \to L \). \( L \) is the (0, 1)-sublattice of a rectangular lattice \( R \), the rectangular hull of \( L \), \([2]\) (see the next remark). This has three chain-deals \( C_1 \) (green), \( C_2 \) (blue) and \( C_3 \) (red) of length \( n \). \( C_1 \cup C_2 \cup C_3 \) is a congruence-determining meet-sublattice (order ideal).

Remark: the construction of \( R \). We denote by \( \Theta \) the cover-preserving join-congruence induced by \( \varphi \). Take the source \( S \) of \( \Theta \). A source element \( s \in S \) is called bastard if \( s \) itself or at least one of its lower covers \( t \) is join-irreducible. Let \( S' \subseteq S \) the set of all non bastard source elements and \( \Theta' \) denotes the corresponding cover-preserving join-congruence. Then \( R = D^3/\Theta' \) is a rectangular lattice (envelop) and \( L = D^3/\Theta \) is a cover-preserving sublattice of \( R \).
**Step 2.** Follow the construction on Figure 3. We glue together $B \cong D^3$ and $R$ identifying the zero element of $R$ with the unit element of $B$. Take $M$ (presented on Figure 1) in three examples $M_1, M_2$ and $M_3$ (flaps) and attach to $R \cup B$. We denote this lattice by $K_0$.

**Step 3.** $[v,e]$ and $[w,a]$ are isomorphic to $M \times C_2$. Apply again gluing twice: first we glue together $K_0$ and $[v,e]$. Finally we glue this lattice with $[w,a]$. This results $K$. The ideal $[w,u]$ is a chain, which contains the three colors as subchains. Let us remark that $[w,y] \cong [v,x] \cong M$.  

\[ \square \]
Let $\mathcal{K}$ denote the class of finite length semimodular lattices that have congruence-determining chain ideals. From [3] we get:

**Corollary.** Let $L$ be a semimodular lattice and let $D$ be a $(0,1)$-sublattice of $\text{Con} L$. Then there exists an $\mathcal{T} \in \mathcal{K}$ such that $K$ contains as ideal a chain $C$, $C$ is congruence-determining and the restriction mapping $\rho : \text{Con}\mathcal{T} \rightarrow \text{Con} L$, $\theta \mapsto [\theta]_L$, is actually a $(0,1)$-lattice isomorphism $\text{Con}\mathcal{T} \rightarrow D$; in particular, $\text{Con}\mathcal{T} \cong D$.

**References**


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**Figure 5.** 3D final, the queuing: $K$