AN EXTENSION THEOREM FOR FINITE SEMIMODULAR LATTICES (SKETCH)

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Abstract. Every semimodular lattice L has a congruence-preserving extension to a semimodular lattice K such that K is congruence-preserving, every congruence is principal and has many other useful properties. Thies generalize the result proved in [2].

1. Introduction

1.1. **Notions.** As usual, $\mathrm{J}(L)$ stands for the poset of all nonzero join-irreducible elements of L. The *width* $\mathrm{w}(P)$ of a (finite) poset P is defined to be $\max\{n\colon P$ has an n-element antichain$\}$. We define the dimension of a finite semimodular lattice: $\mathbf{Dim}(L) = \mathrm{w}(\mathrm{J}(L))$.

An embedding $\varphi : L \to K$ is said to be *cover-preserving* iff it preserves the relation $\preceq$.

A *rectangular* lattice L is a finite semimodular lattice in which $\mathrm{J}(L)$ is the disjoint sum of chains C_i.

Let K be a lattice. A lattice L is a *congruence-preserving,* extension of K if L is an extension and every congruence Θ of K has exactly one extension $\overline{\Theta}$ to L satisfying $\overline{\Theta}|K = \Theta$.

If every congruence of A is determined by its restriction to B, then B is called a *congruence-determining* sublattice of A.

1.2. **The Theorem.** The following result was proved in [2] for planar (especially for 2-dimensional) semimodular lattices. We prove that similar therem holds for all finite semimodular lattices.

Theorem. *Every semimodular lattice L has an extension K satisfying:*

(1) K is a congruence-preserving extension of L,
(2) L is the cover-preserving sublattice of a filter of K,
(3) K is a semimodular lattice,
(4) K is a rectangular lattice,
(5) every congruoence of K is principal,
(6) K contains as ideal a chain C,
(7) C is congruence-determining.

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2. The proof of the Theorem

First, we consider the planar case. By Theorem 7. of G. Grätzer, E. Knapp, [1] L has a congruence-preserving extension to a planar rectangular semimodular lattice, the rectangular hull, $R = \overline{L}$, see on Figure 1. $[c, a]$ and $[c, b]$ are chain ideals. Let n be the length of $[c, a]$, i.e. $n = \mathfrak{l}([c, a])$.

The congruences of R are determined by the restrictions to $V = [c, a] \cup [c, b]$. (Let us remark that the chain $[b, 1]$ can be shorter then $[c, a]$.)

M is direct product of two chains of length n, in the vertical diagonal covering squares we insert a double irreducible element to get M_3-s, see Figure 2. To get K we apply the Hall-Dilworth gluing for R, M, see Figure 3. Then R is a filter of K.

C is the ideal $[e, b]$ on the right side (black filled circles). It is easy to proof, that K satisfies the properties, listed in the Theorem 1.

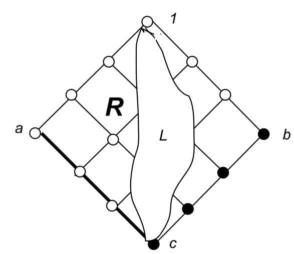

Figure 1. The rectangular extension (hull) $R = \overline{L}$

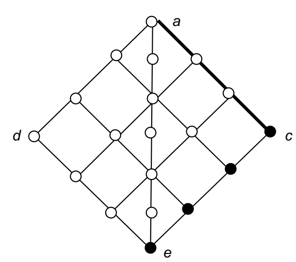

Figure 2. The lattice M

We consider the $\mathrm{Jw}(L) \geq 3$ **case.** and we construct K in several steps.

Step 1. Let L be a semimodular lattice of length n, $\mathrm{Jw}(L) = 3$. Then there is a chain D of length n such that L is the cover-preserving join-homomorphic image of D^3, $\varphi(D^3) \to L$. L is the $(0, 1)$-sublattice of a rectangular lattice R, the rectangular hull of L, [3] (see the next remark). This has three chain-deals C_1 (green), C_2 (blue) and C_3 (red) of length n. $C_1 \cup C_2 \cup C_3$ is a congruence-determining meet-sublattice (order ideal). It easy t prove that R is a congruence-preserving extension.

Step 2. Follow the construction on Figure 3. We glue together $B \cong D^3$ and R identifying the zero element of R with the unit element of B. Take M (presented on Figure 1) in three examples $\boldsymbol{M}_1, \boldsymbol{M}_2$ and $\boldsymbol{M}_3$ (flaps) and attach to $R \cup B$. We denote this lattice by K_0.

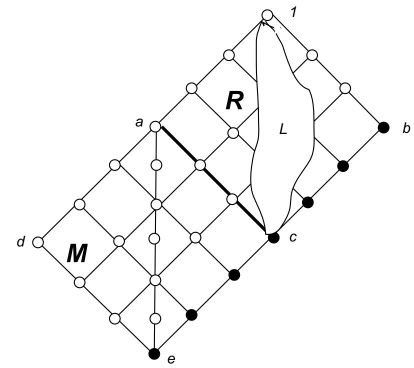

FIGURE 3. The planar lattice K

Step 3. $[v, e]$ and $[w, a]$ are ismorphic to $M \times C_2$. Applay again the Hall-Dilworth gluing twice: first we glue together K_0 and $[v, e]$. Finally we glue this lattice with $[w, a]$. This results K. The ideal $[w, u]$ is a chain, which contains the three colors as subchains. Let us remark that $[w, y] \cong [v, x] \cong M$.

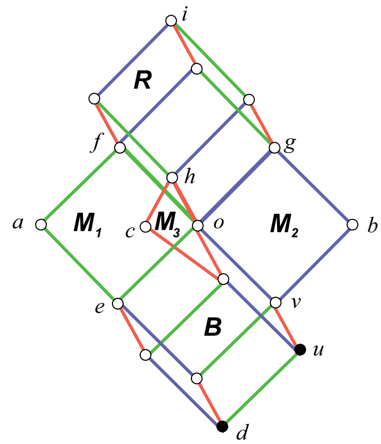

FIGURE 4. 3-flaps extension: K_0

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Let $\mathcal{K}$ denote the class of finite length semimodular lattices that have congruence-determining chain ideals. From [4] we get:

Corollary. *Let L be a semimodular lattice and let D be a $(0,1)$-sublattice of* $\operatorname{Con} L$*. Then there exists an $\overline{L} \in \mathcal{K}$ such that K contains as ideal a chain C, C is congruence-determining and the the restriction mapping $\rho : \operatorname{Con} \overline{L} \to \operatorname{Con} L$, $\theta \mapsto \theta\rceil_L$, is actually a $(0,1)$-lattice isomorphism $\operatorname{Con} \overline{L} \to D$; in particular, $\operatorname{Con} \overline{L} \cong D$.*

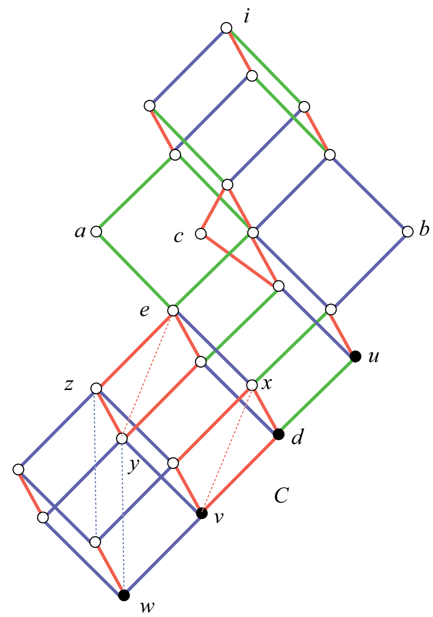

Figure 5. 3D final, the queuing: K

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