REPRESENTING CONGRUENCE LATTICES OF LATTICES WITH PARTIAL UNARY OPERATIONS AS CONGRUENCE LATTICES OF LATTICES. I. INTERVAL EQUIVALENCE

G. GRÄTZER AND E. T. SCHMIDT

ABSTRACT. Let L be a bounded lattice, let $[a, b]$ and $[c, d]$ be intervals of L, and let $\varphi\colon [a, b] \to [c, d]$ be an isomorphism between these two intervals. Let us consider the algebra $L_{\overset{\leftrightarrow}{\varphi}} = \langle L; \wedge, \vee, \varphi, \varphi^{-1} \rangle$, which is a lattice with two partial unary operations. We construct a bounded lattice K (in fact, a convex extension of L) such that the congruence lattice of $L_{\overset{\leftrightarrow}{\varphi}}$ is isomorphic to the congruence lattice of K, and extend this result to (many) families of isomorphisms.

This result presents a lattice K whose congruence lattice is derived from the congruence lattice of L in a novel way.

1. Introduction

1.1. **The "magic wand" approach to constructing congruence lattices.** A typical way of constructing an algebra A with a given congruence lattice C is to construct an algebra B with a much larger congruence lattice and then "collapsing" principal congruences $\Theta(a, b)$ and $\Theta(c, d)$ in B in sufficient numbers so that the congruence lattice "shrinks" to C. To do this we need a "magic wand" that will make $a \equiv b$ equivalent to $c \equiv d$. Such a magic wand may be a pair of partial operations f and g such that $f(a) = c$, $f(b) = d$, and $g(c) = a$, $g(d) = b$. For instance, this is the start of the Congruence Lattice Characterization Theorem of Universal Algebras of the authors (see [9], and also [1], [3].)

If you want to construct a lattice K with a given congruence lattice C, how do you turn the action of the "magic wand" into lattice operations? To construct a simple modular lattice, E. T. Schmidt [17] started with the rational interval $L = [0, 1]$ and by a "magic wand" he required that all $[a, b]$ ($0 \leq a < b \leq 1$) satisfy that $a \equiv b$ be equivalent to $0 \equiv 1$. The action of the magic wand was realized with the $M_3[D]$ construction (which is the same as the Boolean triple construction in Section 2 except that it applies only to bounded distributive lattices D). This method was successfully used for the representation of finite distributive lattices as congruence lattices of modular lattices in E. T. Schmidt [17].

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In this paper, we prove that one can apply the magic wand to arbitrary lattices with zero.

1.2. **The "magic wand" for lattices.** If we are considering a "magic wand" that will realize that $a \equiv b$ be equivalent to $c \equiv d$ in the lattice L, we immediately notice that we have to say something about the intervals $[a, b]$ and $[c, d]$. For instance, if $a \equiv b$ $(\Theta \vee \Psi)$, for congruences Θ and Φ of L, then $c \equiv d$ $(\Theta \vee \Psi)$, therefore the sequence in $[a, b]$ that forces $a \equiv b$ $(\Theta \vee \Psi)$ (see, for instance, Theorem I.3.9 in [4]), must somehow be mapped to a sequence in $[c, d]$ to force $c \equiv d$ $(\Theta \vee \Psi)$. So for lattices, "magic wands" must act on intervals, not on pairs of elements.

The lattice K is an *extension* of the lattice L, if L is a sublattice of K. The lattice K is a *convex extension* of the lattice L, if L is a convex sublattice of K. A *convex embedding* is defined analogously.

To set up "magic wands"—as (convex) extensions—for lattices formally, let L be a bounded lattice, let $[a, b]$ and $[c, d]$ be intervals of L, and let $\varphi\colon [a, b] \to [c, d]$ be an isomorphism between these two intervals. We can consider φ and φ^{-1} as partial unary operations. Let us call a congruence Θ of L a $\overset{\leftrightarrow}{\varphi}$*-congruence* iff Θ satisfies the Substitution Property with respect to the partial unary operations φ and φ^{-1}. (The $\leftrightarrow$ signifies that the partial operations go both ways; in the paper G. Grätzer, M. Greenberg, and E. T. Schmidt [7], we take up partial operations going only one way.) Let $L_{\overset{\leftrightarrow}{\varphi}}$ denote the partial algebra obtained from L by adding the partial operations φ and φ^{-1}. Thus, a congruence relation of $L_{\overset{\leftrightarrow}{\varphi}}$ is the same as a $\overset{\leftrightarrow}{\varphi}$-congruence of L. We call an extension K of L a $\overset{\leftrightarrow}{\varphi}$*-congruence-preserving extension* of L, if a congruence of L extends to K iff it is a $\overset{\leftrightarrow}{\varphi}$-congruence and every $\overset{\leftrightarrow}{\varphi}$-congruence of L has *exactly one* extension to K. As a special case, we get the well-known concept of a congruence-preserving extension (in case, φ is trivial).

Let us call φ (resp., φ^{-1}) *algebraic* iff there is a unary algebraic function $\mathbf{p}(x)$ (that is, $\mathbf{p}(x)$ is obtained from a lattice polynomial by substituting all but one variables by elements of L) such that $x\varphi = \mathbf{p}(x)$, for all $x \in [a, b]$ (resp., $x\varphi^{-1} = \mathbf{p}(x)$, for all $x \in [c, d]$).

We prove the following result:

Theorem 1. *Let L be a bounded lattice, let $[a, b]$ and $[c, d]$ be intervals of L, and let $\varphi\colon [a, b] \to [c, d]$ be an isomorphism between these two intervals. Then L has a $\overset{\leftrightarrow}{\varphi}$-congruence-preserving convex extension into a bounded lattice K such that both φ and φ^{-1} are algebraic in K. In particular, the congruence lattice of the partial algebra $L_{\overset{\leftrightarrow}{\varphi}}$ is isomorphic to the congruence lattice of the bounded lattice K.*

So the lattice K constructed in this result is the magic wand for φ.

We base the realization of the magic wand on a construction of G. Grätzer and F. Wehrung [12] and an application of this construction in G. Grätzer and E. T. Schmidt [11].

1.3. **Outline.** Section 2 deals with the boolean triple construction. The relevant results from G. Grätzer and F. Wehrung [12] and G. Grätzer and E. T. Schmidt [11] are summarized and generalized.

Section 3 states two trivial lemmas on gluing and congruences from the folklore.

Section 4 constructs the lattice K of Theorem 1, while Section 5 verifies that K has the required properties.

In Section 6, we state and prove Theorem 2, which generalizes Theorem 1 to a family of pairwise isomorphic intervals; in Section 7, we further generalize Theorem 1 to any number of intervals (Theorem 3), allowing us to generalize Theorem 1 from isomorphic intervals to isomorphic convex sublattices.

In Section 8, we point out that Theorem 1 and its generalizations hold for lattices with zero.

In Section 9.1, we discuss how the results of this paper relate to the congruence lattice characterization problem of lattices. Section 9.2 comments on the congruence distributivity of the partial algebra $L_{\vec{\varphi}}$. Section 9.3 lists some open problems.

We use the standard notation, as in [4].

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2. The boolean triple construction

Following G. Grätzer and F. Wehrung [12], for a lattice L, let us call the triple $\langle x, y, z\rangle \in L^3$ *boolean* iff the following equations hold:

$$
\begin{aligned}
x &= (x \vee y) \wedge (x \vee z),\\
y &= (y \vee x) \wedge (y \vee z),\\
z &= (z \vee x) \wedge (z \vee y).
\end{aligned}
$$

We denote by $\mathbf{M}_3\langle L\rangle \subseteq L^3$ the poset of all boolean triples of L. We summarize the relevant results of G. Grätzer and F. Wehrung [12] in two lemmas.

Lemma 1. *Let L be a bounded lattice.*

(i) *For every triple $\langle x, y, z\rangle \in L^3$, there is a smallest boolean triple $\overline{\langle x, y, z\rangle} \in L^3$; in fact,*

$$\overline{\langle x, y, z\rangle} = \langle (x \vee y) \wedge (x \vee z), (y \vee x) \wedge (y \vee z), (z \vee x) \wedge (z \vee y)\rangle.$$

(ii) *$\mathbf{M}_3\langle L\rangle$ is a bounded lattice, with bounds $\langle 0,0,0\rangle$ and $\langle 1,1,1\rangle$, with the meet operation defined as*

$$\langle x, y, z\rangle \wedge \langle x', y', z'\rangle = \langle x \wedge x', y \wedge y', z \wedge z'\rangle$$

and the join operation defined by

$$\langle x, y, z\rangle \vee \langle x', y', z'\rangle = \overline{\langle x \vee x', y \vee y', z \vee z'\rangle}.$$

(iii) *The lattice $\mathbf{M}_3\langle L\rangle$ has a* spanning M_3, *that is, a $\{0,1\}$-sublattice isomorphic to M_3, namely,*

$$\{\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle, \langle 1,1,1\rangle\}.$$

(iv) *The intervals $[\langle 0,0,0\rangle, \langle 1,0,0\rangle]$, $[\langle 0,0,0\rangle, \langle 0,0,1\rangle]$ and $[\langle 1,0,0\rangle, \langle 1,1,1\rangle]$ of $\mathbf{M}_3\langle L\rangle$ are isomorphic to L. We identify the lattice L with the interval $[\langle 0,0,0\rangle, \langle 1,0,0\rangle]$ under the isomorphism $x \mapsto \langle x,0,0\rangle$, for $x \in L$. The natural isomorphisms of these intervals with one another are algebraic.*

For a congruence Θ of L, let Θ^3 denote the congruence of L^3 defined componentwise. Let $\mathbf{M}_3\langle\Theta\rangle$ be the restriction of Θ^3 to $\mathbf{M}_3\langle L\rangle$. In [12] the congruences of $\mathbf{M}_3\langle L\rangle$ were characterized as follows:

Lemma 2. *$\mathbf{M}_3\langle\Theta\rangle$ is a congruence relation of $\mathbf{M}_3\langle L\rangle$ and every congruence of $\mathbf{M}_3\langle L\rangle$ is of the form $\mathbf{M}_3\langle\Theta\rangle$, for a unique congruence Θ of L.*

Corollary 3. *Let L be a bounded lattice. The lattice $\mathbf{M}_3\langle L\rangle$ is a congruence-preserving convex extension of $L = [\langle 0,0,0\rangle, \langle 1,0,0\rangle]$ and also of $[\langle 1,0,0\rangle, \langle 1,1,1\rangle]$.*

In [11], the authors introduced, for arbitrary $a \in L$, the principal dual ideal $\mathbf{M}_3\langle L,a\rangle$ of $\mathbf{M}_3\langle L\rangle$:

$$\mathbf{M}_3\langle L,a\rangle = [\langle 0,a,0\rangle) \subseteq \mathbf{M}_3\langle L\rangle,$$

and observed (see Lemma 1 and Section 4 of [11]) that

$$\varphi_a\colon x \mapsto \langle x, a, x \wedge a\rangle$$

is a congruence-preserving convex embedding of L into $\mathbf{M}_3\langle L,a\rangle$.

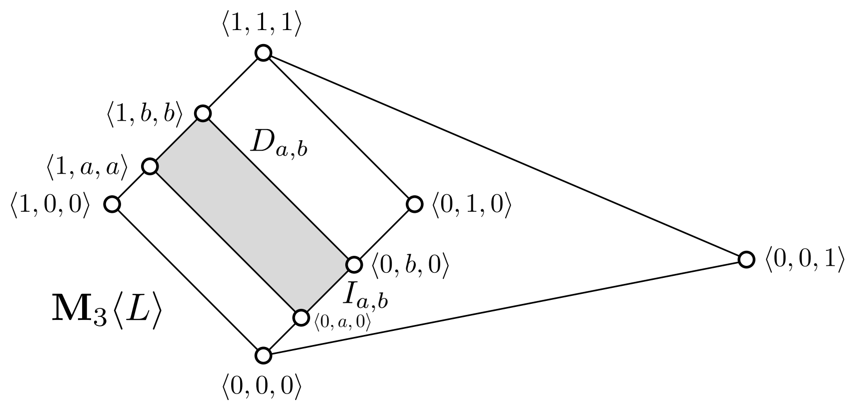

FIGURE 1. The shaded area: The lattice $\mathbf{M}_3\langle L,a,b\rangle$.

In this paper, we need a new variant of this construction; namely, for a, $b \in L$ with $a < b$, we introduce the interval $\mathbf{M}_3\langle L,a,b\rangle$ of $\mathbf{M}_3\langle L\rangle$:

$$\mathbf{M}_3\langle L,a,b\rangle = [\langle 0,a,0\rangle, \langle 1,b,b\rangle] \subseteq \mathbf{M}_3\langle L\rangle.$$

Again,

$$\varphi_a\colon x \mapsto \langle x, a, x \wedge a\rangle$$

is a (convex) embedding of L into $\mathbf{M}_3\langle L,a,b\rangle$. (Note that if L is bounded, then $\mathbf{M}_3\langle L,a\rangle = \mathbf{M}_3\langle L,a,1\rangle$.)

Using the notation (illustrated in Figure 2; the black filled elements form J):

$$\begin{aligned} B &= \{\, \langle x,a,x\wedge a\rangle \mid x \in L \,\} \ (= L\varphi_a),\\ I_{a,b} &= [\langle 0,a,0\rangle, \langle 0,b,0\rangle],\\ J &= \{\, \langle x\wedge a, a, x\rangle \mid x \le b \,\}, \end{aligned}$$

we can now generalize three lemmas (Lemmas 5–7) of [11]:

Lemma 4. *Let $\mathbf{v} = \langle x,y,z\rangle \in \mathbf{M}_3\langle L,a,b\rangle$. Then $\mathbf{v}$ has a decomposition in $\mathbf{M}_3\langle L,a,b\rangle$:*

$$\mathbf{v} = \mathbf{v}_B \vee \mathbf{v}_{I_{a,b}} \vee \mathbf{v}_J,$$

where

$$\begin{aligned} \mathbf{v}_B &= \langle x,y,z\rangle \wedge \langle 1,a,a\rangle = \langle x,a,x\wedge a\rangle \in B,\\ \mathbf{v}_{I_{a,b}} &= \langle x,y,z\rangle \wedge \langle 0,b,0\rangle = \langle 0,y,0\rangle \qquad \in I_{a,b},\\ \mathbf{v}_J &= \langle x,y,z\rangle \wedge \langle a,a,b\rangle = \langle z\wedge a,a,z\rangle \in J. \end{aligned}$$

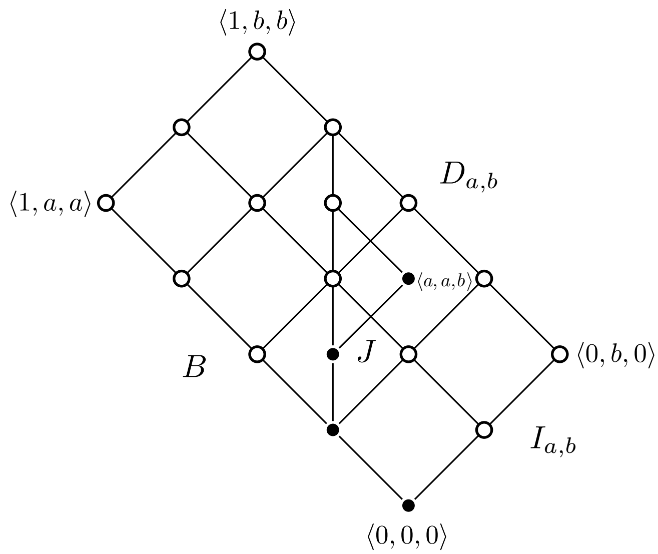

FIGURE 2. The lattice $\mathbf{M}_3\langle L, a, b\rangle$.

Lemma 5. *Let Φ be a congruence of $\mathbf{M}_3\langle L, a, b\rangle$ and let $\mathbf{v}$, $\mathbf{w} \in \mathbf{M}_3\langle L, a, b\rangle$. Then*

$$\mathbf{v} \equiv \mathbf{w} \quad (\Phi)$$

iff

$$\begin{aligned} \mathbf{v}_B &\equiv \mathbf{w}_B &&(\Phi),\\ \mathbf{v}_{I_{a,b}} &\equiv \mathbf{w}_{I_{a,b}} &&(\Phi),\\ \mathbf{v}_J &\equiv \mathbf{w}_J &&(\Phi). \end{aligned}$$

Lemma 6. *For a congruence Θ of L, let $\mathbf{M}_3\langle\Theta, a, b\rangle$ be the restriction of Θ^3 to $\mathbf{M}_3\langle L, a, b\rangle$. Then $\mathbf{M}_3\langle\Theta, a, b\rangle$ is a congruence of $\mathbf{M}_3\langle L, a, b\rangle$, and every congruence of $\mathbf{M}_3\langle L, a, b\rangle$ is of the form $\mathbf{M}_3\langle\Theta, a, b\rangle$, for a unique congruence Θ of L. It follows that φ_a is a congruence-preserving convex embedding of L into $\mathbf{M}_3\langle L, a, b\rangle$.*

We shall use the notation

$$D_{a,b} = [\langle 0, b, 0\rangle, \langle 1, b, b\rangle] = \{\, \langle x, b, x \wedge b\rangle \mid x \in L \,\}.$$

The following two observations are trivial.

Lemma 7.

(i) *$I_{a,b}$ is an ideal of $\mathbf{M}_3\langle L, a, b\rangle$ and $I_{a,b}$ is isomorphic to the interval $[a, b]$ of L.*

(ii) *$D_{a,b}$ is a dual ideal of $\mathbf{M}_3\langle L, a, b\rangle$ and $D_{a,b}$ is isomorphic to L.*

We can say a lot more about $D_{a,b}$. But first we need another lemma:

Lemma 8. *Let L be a lattice, let $[u, v]$ and $[u', v']$ be intervals of L, and let $\varphi\colon [u, v] \to [u', v']$ be an isomorphism between these two intervals. Let φ and φ^{-1} be algebraic in L. Then L is a congruence-preserving extension of $[u, v]$ iff it is a congruence-preserving extension of $[u', v']$.*

Proof. Let us assume that L is a congruence-preserving extension of $[u', v']$. Let Θ be a congruence relation of $[u, v]$ and let $\Theta\varphi$ be the image of Θ under φ. Since φ is an isomorphism, it follows that $\Theta\varphi$ is a congruence of $[u', v']$, and so $\Theta\varphi$ has a unique extension to a congruence Φ of L. We claim that Φ extends Θ to L and extends it uniquely.

1. Φ *extends* Θ. Let $x \equiv y \ (\Theta)$. Then $x\varphi \equiv y\varphi \ (\Theta\varphi)$, since φ is an isomorphism. By definition, Φ extends $\Theta\varphi$, so $x\varphi \equiv y\varphi \ (\Phi)$. Since φ^{-1} is algebraic, the last congruence implies that $x \equiv y \ (\Phi)$. Conversely, let $x \equiv y \ (\Phi)$ and x, $y \in [u, v]$. Then $x\varphi$, $y\varphi \in [u', v']$; since φ is algebraic, it follows that $x\varphi \equiv y\varphi \ (\Phi)$. Since Φ extends $\Theta\varphi$, we conclude that $x\varphi \equiv y\varphi \ (\Theta\varphi)$. Using that φ^{-1} is an isomorphism, we obtain that $x \equiv y \ (\Theta)$, verifying the claim.

2. Φ *extends* Θ *uniquely*. Let Ψ extend Θ to L. As in the previous paragraph—*mutatis mutandis*—we conclude that Ψ extends $\Theta\varphi$ to L, hence $\Psi = \Phi$, proving the uniqueness.

By symmetry, the lemma is proved. □

Note that the lemma is true for any two sublattices; however, we shall only use it for intervals, as stated.

We have already observed in Lemma 6 that $\mathbf{M}_3\langle L, a, b\rangle$ is a congruence-preserving convex extension of $[\langle 0, a, 0\rangle, \langle 1, a, a\rangle]$ $(= L\varphi_a)$. Since the isomorphism

$$x \mapsto x \vee \langle 0, b, 0\rangle$$

between $[\langle 0, a, 0\rangle, \langle 1, a, a\rangle]$ and $[\langle 0, b, 0\rangle, \langle 1, b, b\rangle] = D_{a,b}$ is algebraic, and so is the inverse

$$x \mapsto x \wedge \langle 1, a, a\rangle,$$

from Lemma 8 we conclude the following:

Corollary 9. $\mathbf{M}_3\langle L, a, b\rangle$ *is a congruence-preserving convex extension of* $D_{a,b}$.

We summarize our results:

Lemma 10. *Let L be a bounded lattice, and let a, $b \in L$ with $a < b$. Then there exists a bounded lattice $L_{a,b}$ (with bounds $0_{a,b}$ and $1_{a,b}$) and $u_{a,b}$, $v_{a,b} \in L_{a,b}$, such that the following conditions are satisfied:*

(i) *$v_{a,b}$ is a complement of $u_{a,b}$.*
(ii) *$D_{a,b} = [v_{a,b}, 1_{a,b}] \cong L$.*
(iii) *$I_{a,b} = [0_{a,b}, v_{a,b}] \cong [a, b]$.*
(iv) *$L_{a,b}$ is a congruence-preserving (convex) extension of $[0_{a,b}, u_{a,b}]$ and of $[v_{a,b}, 1_{a,b}]$.*
(v) *The congruences on $I_{a,b}$ and $D_{a,b}$ are* synchronized, *that is, if Θ is a congruence on L, $\overline{\Theta}$ is the extension of Θ to $L_{a,b}$ (we map Θ to $D_{a,b}$ under the isomorphism, and then by* (iv) *we uniquely extend it to $L_{a,b}$), and x, $y \in [a, b]$, then we can denote by $x_{D_{a,b}}$, $y_{D_{a,b}} \in D_{a,b}$ the images of x, y in $D_{a,b}$ and by $x_{I_{a,b}}$, $y_{I_{a,b}} \in I_{a,b}$ the images of x, y in $I_{a,b}$; synchronization means that $x_{D_{a,b}} \equiv y_{D_{a,b}} \ (\overline{\Theta})$ iff $x_{I_{a,b}} \equiv y_{I_{a,b}} \ (\overline{\Theta})$.*

Proof. Of course, $L_{a,b} = \mathbf{M}_3\langle L, a, b\rangle$. □

The lattice $L_{a,b}$ is illustrated by $L = C_5$ in Figure 3, the five-element chain, a is the atom and b is the dual atom of C_5. This figure is the same as Figure 2, only the notation is changed.

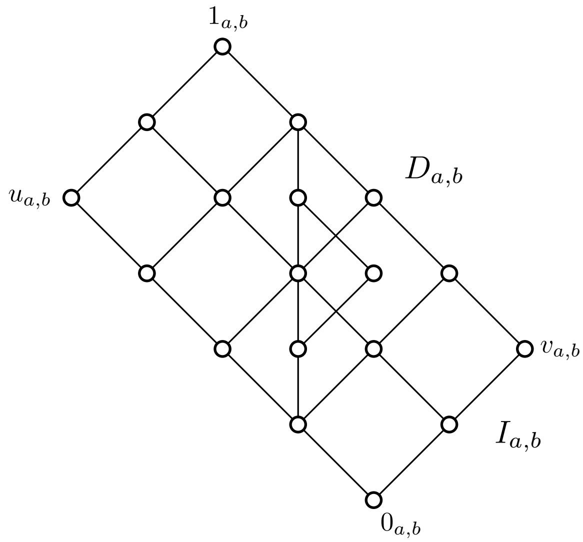

FIGURE 3. The lattice $L_{a,b}$.

We shall continue to use the notations: $I_{a,b} = [0_{a,b}, v_{a,b}]$ (an ideal of $L_{a,b}$) and $D_{a,b} = [v_{a,b}, 1_{a,b}]$ (a dual ideal of $L_{a,b}$).

3. Gluing and congruences

We briefly digress to state two trivial lemmas on gluing and congruences.

Let S and T be lattices, let F be a dual ideal of S, and let G be an ideal of T. If F is isomorphic to G (with ψ the isomorphism), then we can form the lattice Z, the *gluing* of S and T over F and G (with respect to ψ), where $Z = S \cup T$ and every $a \in F$ is identified with $a\psi \in G$; the partial order on Z is the natural one inherited from S and T with one step transitivity.

Now if Θ_S is a binary relation on S and Θ_T is a binary relation on T, we define the *reflexive binary product* $\Theta_S \overset{r}{\circ} \Theta_T$ as $\Theta_S \cup \Theta_T \cup (\Theta_S \circ \Theta_T)$.

The following statement is folklore:

Lemma 11. *A congruence Θ of Z can be uniquely written in the form*

$$\Theta = \Theta_S \overset{r}{\circ} \Theta_T,$$

where Θ_S is a congruence of S and Θ_T is a congruence of T satisfying the condition that Θ_S restricted to F equals Θ_T restricted to G (under the identification of elements by ψ).

Conversely, if Θ_S is a congruence of S and Θ_T is a congruence of T satisfying the condition that Θ_S restricted to F equals Θ_T restricted to G, then $\Theta = \Theta_S \overset{r}{\circ} \Theta_T$ is a congruence of Z.

Lemma 12. *Let us further assume that S is a congruence-preserving extension of F. Then Z is a congruence-preserving extension of T.*

Proof. Represent the congruence Θ of Z as in Lemma 11 in the form $\Theta = \Theta_S \overset{r}{\circ} \Theta_T$. Since S is a congruence-preserving extension of F, the congruence $\Theta_S = \Theta\rceil_S$

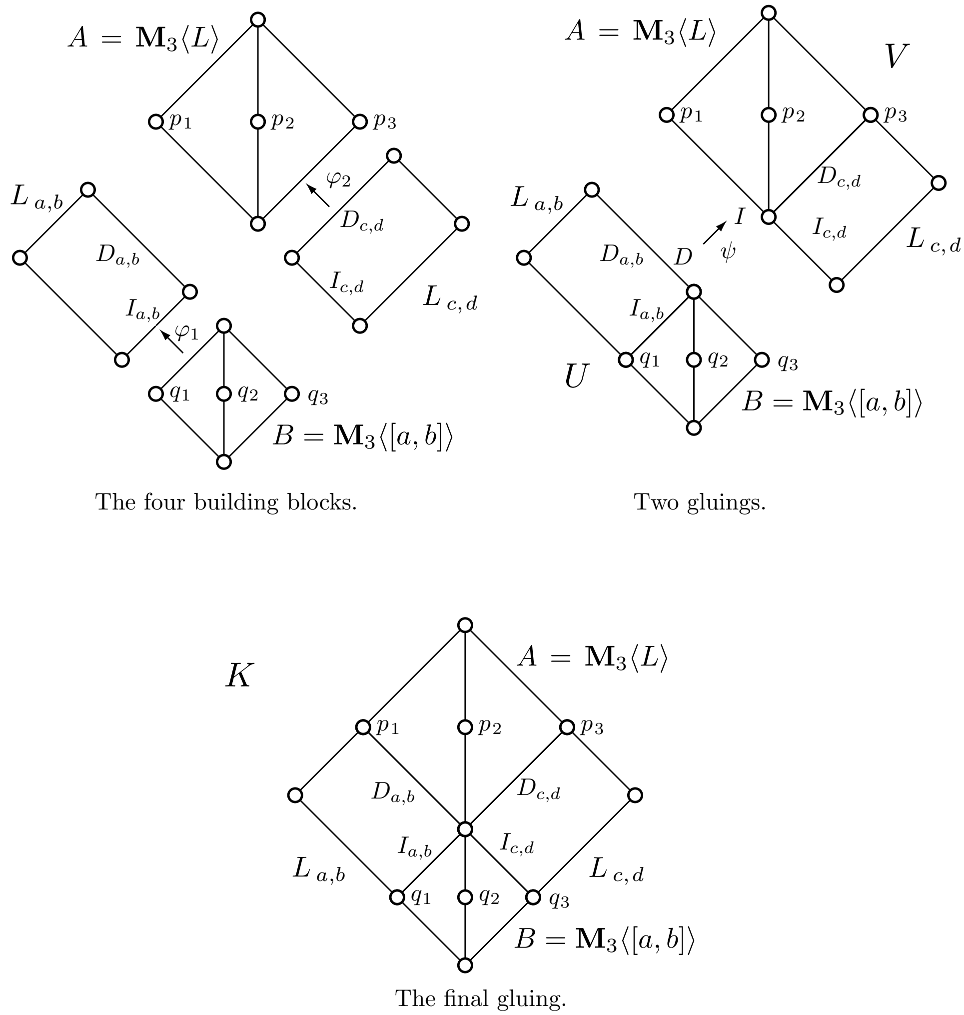

FIGURE 4. Constructing the lattice K.

(where $\rceil$ indicates restriction) is determined by its restriction to F, $\Theta\rceil_F$. But $\Theta\rceil_F = \Theta_T\rceil_G$, so Θ_T determines Θ. ☐

4. The construction of K

In this section, we build the lattice K of Theorem 1.

Let the bounded lattice L, the intervals $[a, b]$ and $[c, d]$, and the isomorphism $\varphi\colon [a, b] \to [c, d]$ be given as in Theorem 1. We start the construction with four lattices, which are assumed to be pairwise disjoint, see the top left of Figure 4: The four building blocks.

(i) $A = \mathbf{M}_3\langle L\rangle$. Let $\{0_A, p_1, p_2, p_3, 1_A\}$ be the spanning M_3 in A—as in Lemma 1.(iii).
(ii) $B = \mathbf{M}_3\langle [a, b]\rangle$, with the spanning M_3: $\{0_B, q_1, q_2, q_3, 1_B\}$.
(iii) The lattice $L_{a,b}$—as in Lemma 10.

(iv) The lattice $L_{c,d}$—as in Lemma 10.

Some notation: An element of one of the four building blocks is described as a triple $\langle x, y, z\rangle \in L^3$ belonging to the particular building block. Note that a triple $\langle x, y, z\rangle$ may belong to two or more building blocks. If it is not clear the element of which building block a triple is representing, we shall make it clear with subscripts:

$$\begin{aligned} &\langle x, y, z\rangle_A, && \text{for } \langle x, y, z\rangle \text{ as an element of } A,\\ &\langle x, y, z\rangle_{a,b}, && \text{for } \langle x, y, z\rangle \text{ as an element of } L_{a,b},\\ &\langle x, y, z\rangle_{c,d}, && \text{for } \langle x, y, z\rangle \text{ as an element of } L_{c,d},\\ &\langle x, y, z\rangle_B, && \text{for } \langle x, y, z\rangle \text{ as an element of } B. \end{aligned}$$

We start the construction by gluing together B and $L_{a,b}$ to obtain the lattice U, see the top right of Figure 4: Two gluings.

In B, we use the dual ideal

$$[q_1) = \{\, \langle 1, x, x\rangle \mid a \leq x \leq b \,\},$$

while in $L_{a,b}$ we utilize the ideal

$$I_{a,b} = \{\, \langle 0, x, 0\rangle \mid a \leq x \leq b \,\},$$

and we consider the natural isomorphism

$$\varphi_1 \colon \langle 1, x, x\rangle_B \mapsto \langle 0, x, 0\rangle_{a,b}, \quad x \in [a, b],$$

between the dual ideal $[q_1)$ of B and the ideal $I_{a,b}$ of $L_{a,b}$ to glue B and $L_{a,b}$ together to obtain the lattice U.

Similarly, we glue $L_{c,d}$ and A over the dual ideal

$$D_{c,d} = \{\, \langle x, d, x \wedge d\rangle \mid x \in L \,\}$$

of $L_{c,d}$ and the ideal

$$(p_3] = \{\, \langle 0, 0, x\rangle \mid x \in L \,\}$$

of A, with respect to the natural isomorphism

$$\varphi_2 \colon \langle x, d, x \wedge d\rangle_{c,d} \mapsto \langle 0, 0, x\rangle_A, \quad x \in L,$$

to obtain the lattice V.

In U, we define the dual ideal

$$D = [q_3, 1_B] \cup D_{a,b},$$

which is the union of $[q_3, 1_B]$ and $D_{a,b}$, with the unit of $[q_3, 1_B]$ identified with the zero of $D_{a,b}$.

In V, we define the ideal

$$I = I_{c,d} \cup [0_A, p_1],$$

which is the union of $I_{c,d}$ and $[0_A, p_1]$, with the unit of $I_{c,d}$ identified with the zero of $[0_A, p_1]$.

Next we set up an isomorphism $\psi \colon D \to I$. Since

$$[q_3, 1_B] = \{\, \langle x, x, b\rangle \mid a \leq x \leq b \,\}$$

and

$$I_{c,d} = \{\, \langle 0, x, 0\rangle \mid c \leq x \leq d \,\},$$

we define ψ on $[q_3, 1_B]$ by

$$\psi \colon \langle x, x, b\rangle_B \mapsto \langle 0, x\varphi, 0\rangle_{c,d}, \quad x \in [a, b],$$

where $\varphi\colon [a,b] \to [c,d]$ is the isomorphism given in Theorem 1. We define ψ on $D_{a,b}$ by

$$\psi\colon \langle x, b, x \wedge b\rangle_{a,b} \mapsto \langle x, 0, 0\rangle_A, \quad x \in L.$$

It is clear that $\psi\colon D \to I$ is well-defined and it is an isomorphism.

Finally, we construct the lattice K of Theorem 1 by gluing U over I with V over D with respect to the isomorphism $\psi\colon D \to I$, see the bottom half of Figure 4: The final gluing.

By Lemma 1.(iv), the map $x \mapsto \langle x, 0, 0\rangle_A$ is a natural isomorphism between L and the principal ideal $(p_1]$ of A; this gives us a convex embedding of L into A. We identify L with its image, and regard L as a convex sublattice of A and therefore of K. So K is a convex extension of L. We have completed the construction of the bounded lattice K of Theorem 1.

5. The verification

We now verify that the lattice extension K of L satisfies the conditions stated in Theorem 1.

First, we shall describe the congruences of K. We need some notation: For a congruence Θ of L,

Θ_A denotes the congruence Θ^3 restricted to A;

$\Theta_{a,b}$ denotes the congruence Θ^3 restricted to $L_{a,b}$;

$\Theta_{c,d}$ denotes the congruence Θ^3 restricted to $L_{c,d}$;

Θ_B denotes the congruence Θ^3 restricted to B.

Let us start with U. The lattices B and $L_{a,b}$ are glued together over $[q_1)$ and $I_{a,b}$ with the isomorphism $\varphi_1\colon \langle 1, x, x\rangle_B \mapsto \langle 0, x, 0\rangle_{a,b}$, and obviously $\langle 1, x, x\rangle \equiv \langle 1, y, y\rangle$ (Θ^3) iff $\langle 0, x, 0\rangle \equiv \langle 0, y, 0\rangle$ (Θ^3). Hence by Lemma 11, the congruences of U are of the form $\Theta_B \overset{r}{\circ} \Theta_{a,b}$.

The lattice B is a congruence-preserving extension of $[q_1)$ (formed in B) by Lemma 1.(iv); therefore, by Lemma 12 and Corollary 9, the lattice U is a congruence-preserving extension of $L_{a,b}$, which, in turn, is a congruence-preserving extension of $D_{a,b}$ ($\cong L$). So U is a congruence-preserving extension of $D_{a,b}$ ($\cong L$). Similarly, V is a congruence-preserving extension of A, which, in turn, is a congruence-preserving extension of $[0_A, p_1] = L$, so V is a congruence-preserving extension of $[0_A, p_1] = L$.

We glue U and V together over D and I over ψ; equivalently, we identify the dual ideal

$$D_{a,b} = \{\, \langle x, b, x \wedge b\rangle \mid x \in L \,\} \subseteq U$$

of U with

$$[0_A, p_1] = \{\, \langle x, 0, 0\rangle \mid x \in L \,\} \subseteq A,$$

and note that for any congruence Θ of L,

$$\langle x, b, x \wedge b\rangle \equiv \langle y, b, y \wedge b\rangle \quad (\Theta^3)$$

iff

$$\langle x, 0, 0\rangle \equiv \langle y, 0, 0\rangle \quad (\Theta^3),$$

so Θ^3 restricted to $D_{a,b}$ is mapped by ψ to Θ^3 restricted to $[0_A, p_1]$—and we identify the dual ideal

$$[q_3, 1_B] = \{\, \langle x, x, b\rangle \mid a \leq x \leq b \,\} \subseteq B$$

of B with the ideal

$$I_{c,d} = \{\, \langle 0, x, 0\rangle \mid c \leq x \leq d \,\} \subseteq L_{c,d}$$

of $L_{c,d}$ by identifying $\langle x, x, b\rangle$ with $\langle 0, x\varphi, 0\rangle$, for $x \in [a, b]$. So in K, $\langle x, x, 1\rangle \equiv \langle y, y, 1\rangle$ iff $\langle 0, x\varphi, 0\rangle \equiv \langle 0, y\varphi, 0\rangle$; translating this back to L, we obtain that $x \equiv y$ (Θ) iff $x\varphi \equiv y\varphi$ (Θ). This condition is equivalent to the statement that Θ has the Substitution Property with respect to the partial unary operations φ and φ^{-1}.

This proves, on the one hand, that if Θ extends to K, then Θ is a $\overleftrightarrow{\varphi}$-congruence, and, on the other hand, that a $\overleftrightarrow{\varphi}$-congruence Θ extends uniquely to K, that is, K is a $\overleftrightarrow{\varphi}$-congruence-preserving extension of L.

Second, we have to show that φ and φ^{-1} are algebraic in K. We define

$$\mathbf{p}(x) = ((((((x \wedge \langle a, a, b\rangle_{a,b}) \vee \langle b, a, a\rangle_{a,b}) \wedge q_2) \vee \langle d, c, c\rangle_{c,d}) \wedge \langle c, c, d\rangle_{c,d}) \vee p_2) \wedge p_1.$$

For $x \in [a, b]$, we want to compute $\mathbf{p}(x)$. There are seven steps in the computation of $\mathbf{p}(x)$ (see Figure 5):

$$\begin{aligned}
x &= \langle x, 0, 0\rangle_A = \langle x, b, x\rangle_{a,b} && \text{(in } A \text{ and in } L_{a,b}\text{)},\\
x_1 &= x \wedge \langle a, a, b\rangle && \text{(computed in } L_{a,b}\text{)},\\
x_2 &= x_1 \vee \langle b, a, a\rangle && \text{(computed in } L_{a,b}\text{)},\\
x_3 &= x_2 \wedge q_2 && \text{(computed in } U\text{)},\\
x_4 &= x_3 \vee \langle d, c, c\rangle && \text{(computed in } K\text{)},\\
x_5 &= x_4 \wedge \langle c, c, d\rangle && \text{(computed in } L_{c,d}\text{)},\\
x_6 &= x_5 \vee p_2 && \text{(computed in } V\text{)},\\
x_7 &= x_6 \wedge p_1 && \text{(computed in } A\text{)}.
\end{aligned}$$

Our goal is to prove that $x_7 = x\varphi$.

By the definition of ψ, when gluing U and V together, we identify $x = \langle x, 0, 0\rangle \in A$ with $\langle x, b, x \wedge b\rangle = \langle x, b, x\rangle \in L_{a,b}$, so $x = \langle x, b, x\rangle \in L_{a,b}$. Therefore, $x_1 = \langle x, b, x\rangle \wedge \langle a, a, b\rangle = \langle a, a, x\rangle$, computed in $L_{a,b}$.

We compute x_2 completely within $L_{a,b}$, utilizing Lemma 1:

$$x_2 = x_1 \vee \langle b, a, a\rangle = \langle a, a, x\rangle \vee \langle b, a, a\rangle = \overline{\langle b, a, x\rangle} = \langle b, x, x\rangle.$$

$x_3 = x_2 \wedge q_2$ is computed in U, which we obtained by gluing $L_{a,b}$ and B together with respect to the isomorphism

$$\varphi_1 \colon \langle 1_B, x, x\rangle_B \mapsto \langle 0, x, 0\rangle_{a,b}, \quad x \in [a, b],$$

between the dual ideal $[q_1)$ of B and the ideal $I_{a,b}$ of $L_{a,b}$. So x_3 is computed in two steps. First, in $L_{a,b}$:

$$x_2 \wedge v_{a,b} = \langle b, x, x\rangle \wedge \langle 0, b, 0\rangle = \langle 0, x, 0\rangle.$$

The image of $\langle 0, x, 0\rangle$ under φ_1^{-1} is $\langle b, x, x\rangle = \langle b, x, x\rangle$, so in B:

$$x_3 = \langle b, x, x\rangle \wedge q_2 = \langle b, x, x\rangle \wedge \langle a, b, a\rangle = \langle a, x, a\rangle.$$

Now comes the crucial step. To compute $x_4 = x_3 \vee \langle d, c, c\rangle$, we first compute in B

$$x_3 \vee q_3 = \langle a, x, a\rangle \vee \langle a, a, b\rangle = \overline{\langle a, x, b\rangle} = \langle x, x, b\rangle.$$

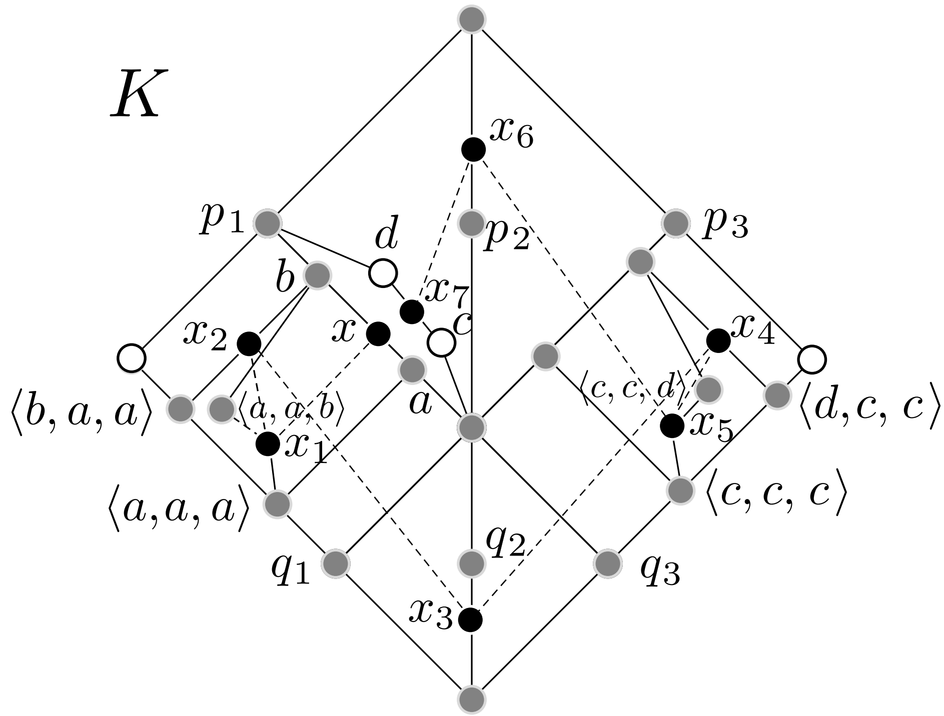

FIGURE 5. The seven steps.

Take the image of $\langle x, x, b\rangle$ under ψ and join it with $\langle d, c, c\rangle$ in $L_{c,d}$:

$$x_4 = \langle x, x, b\rangle\psi \vee \langle d, c, c\rangle = \langle 0, x\varphi, 0\rangle \vee \langle d, c, c\rangle = \overline{\langle d, x\varphi, c\rangle} = \langle d, x\varphi, x\varphi\rangle.$$

So

$$x_5 = x_4 \wedge \langle c, c, d\rangle = \langle d, x\varphi, x\varphi\rangle \wedge \langle c, c, d\rangle = \langle c, c, x\varphi\rangle,$$

computed in $L_{c,d}$.

To compute $x_6 = x_5 \vee p_2$, we note that

$$x_5 \vee v_{c,d} = \langle c, c, x\varphi\rangle \vee \langle 0, d, 0\rangle = \overline{\langle c, d, x\varphi\rangle} = \langle x\varphi, d, x\varphi\rangle.$$

Then we take the image of $\langle x\varphi, d, x\varphi\rangle_{c,d}$ under φ_2 and join it with p_2 in A:

$$x_6 = \langle x\varphi, d, x\varphi\rangle\varphi_2 \vee p_2 = \langle 0, 0, x\varphi\rangle \vee \langle 0, 1, 0\rangle = \overline{\langle 0, 1, x\varphi\rangle} = \langle x\varphi, 1, x\varphi\rangle.$$

Finally, in A,

$$x_7 = x_6 \wedge p_1 = \langle x\varphi, 1, x\varphi\rangle \wedge \langle 1, 0, 0\rangle = \langle x\varphi, 0, 0\rangle,$$

and $x\varphi$ is identified with $\langle x\varphi, 0, 0\rangle$, so $x_7 = x\varphi$, as claimed.

The proof for φ^{-1} is similar, using the algebraic function

$$\mathbf{q}(y) = ((((((y \vee p_2) \wedge \langle c, c, d\rangle_{c,d}) \vee \langle d, c, c\rangle_{c,d}) \wedge q_2) \vee \langle b, a, a\rangle_{a,b}) \wedge \langle a, a, b\rangle_{a,b}) \vee a.$$

This completes the proof of Theorem 1.

As you can see, the four sublattices of K isomorphic to M_3 play a crucial role in the proof; the elements forming these M_3-s are gray-filled in Figure 5.

6. A family of intervals

Let L be a bounded lattice, let $[a_i, b_i]$, $i < \alpha$, be intervals of L (α is an initial ordinal ≥ 2), and let

$$\varphi_{i,j} \colon [a_i, b_i] \to [a_j, b_j], \quad \text{for } i,\ j < \alpha,$$

be an isomorphism between the intervals $[a_i, b_i]$ and $[a_j, b_j]$. For notational convenience, we write $[a, b]$ for $[a_0, b_0]$. Let

$$\Phi = \{\, \varphi_{i,j} \mid i,\ j < \alpha \,\}$$

be subject to the following conditions, for i, $j < \alpha$:

Φ.(1) $\varphi_{i,i}$ is the identity map on $[a_i, b_i]$.
Φ.(2) $\varphi_{i,j}^{-1} = \varphi_{j,i}$.
Φ.(3) $\varphi_{i,j} \circ \varphi_{j,k} = \varphi_{i,k}$.

Let L_Φ denote the partial algebra obtained from L by adding the partial operations $\varphi_{i,j}$, i, $j < \alpha$. (Note that within the framework of Theorem 1, the partial algebra $L_{\overleftrightarrow{\varphi}}$ is the same as L_Φ with $\Phi = \{\varphi, \varphi^{-1}, \mathrm{id}_{[a,b]}, \mathrm{id}_{[c,d]}\}$.) Let us call a congruence Θ of L a Φ-*congruence* iff Θ satisfies the Substitution Property with respect to the partial unary operations $\varphi \in \Phi$. Thus, a congruence relation of L_Φ is the same as a Φ-congruence of L. We call K a Φ-*congruence-preserving extension* of L, if a congruence of L extends to K iff it is a Φ-congruence of L_Φ and every Φ-congruence of L has *exactly one* extension to K.

Theorem 2. *Let L be a bounded lattice, and let $\alpha \geq 2$ be an ordinal. Let $[a_i, b_i]$, $i < \alpha$, be intervals of L, and let $\varphi_{i,j} \colon [a_i, b_i] \to [a_j, b_j]$ be an isomorphism between the intervals $[a_i, b_i]$ and $[a_j, b_j]$, for i, $j < \alpha$, subject to the conditions Φ.(1)–Φ.(3), where $\Phi = \{\, \varphi_{i,j} \mid i,\ j < \alpha \,\}$. Then the partial algebra L_Φ has a Φ-congruence-preserving convex extension into a bounded lattice K such that all $\varphi_{i,j}$, i, $j < \alpha$, are algebraic in K. In particular, the congruence lattice of the partial algebra L_Φ is isomorphic to the congruence lattice of the lattice K.*

To prove this result, we have to go somewhat beyond what we did in Section 4. We have to replace A and B with lattices of similar structure but with spanning M_α-s (as opposed to M_3-s), and we must have α lattices L_{a_i,b_i}-s to insert (not two). As a result, the construction cannot be done by a series of gluing as illustrated in Figure 4.

We assume that $\alpha > 2$; the $\alpha = 2$ case was done in Theorem 1. For notational convenience, we write φ_i for $\varphi_{0,i}$, $i < \alpha$.

Let L_{Φ_0} denote the partial algebra obtained from L by adding the partial operations φ_i, $i < \alpha$. In view of Φ.(3), it is sufficient to prove the theorem for L_{Φ_0}.

Our first task is to find a substitute for the $\mathbf{M}_3\langle L\rangle$ construction.

In G. Grätzer and F. Wehrung [13], the *lattice tensor product* of two lattices is introduced (for an alternative approach, see G. Grätzer and M. Greenberg [6]). We state some of the basic properties of this construction for M_α and L.

Lemma 13. *Let L be a bounded lattice. Then the lattice tensor product of M_α and L, denoted by $M_\alpha \boxtimes L$, is a bounded lattice with the following properties:*

(i) *The lattice $M_\alpha \boxtimes L$ has a* spanning M_α, *that is, a $\{0,1\}$-sublattice isomorphic to M_α; we denote by $\{\, p_i \mid i < \alpha \,\}$ the atoms of this spanning M_α.*
(ii) *The intervals $[0, p_i]$ and $[p_i, 1]$ are isomorphic to L, for all $i < \alpha$. The natural isomorphisms of these intervals with one another are algebraic.*

(iii) *Let $J \subseteq \alpha$ be a three-element set. Then the sublattice of $M_\alpha \boxtimes L$ generated by $\bigcup([0,p_i] \cup [p_i,1] \mid i \in J)$ is naturally isomorphic to $\mathbf{M}_3\langle L\rangle$.*

(iv) *The lattice $M_\alpha \boxtimes L$ is a congruence-preserving convex extension of the interval $[0,p_i]$, for any $i < \alpha$.*

All these statements are explicit or implicit in G. Grätzer and F. Wehrung [13], with the exception of (iv); a direct proof can easily be obtained from Theorem 9.3 of [13] utilizing the embedding of L into $M_\alpha \boxtimes L$ given by $x \mapsto p_i \boxtimes x$.

To start proving Theorem 2, let $A = M_\alpha \boxtimes L$, with spanning M_α with atoms p_i, $i < \alpha$, and let $B = M_\alpha \boxtimes [a,b]$, with spanning M_α with atoms q_i, $i < \alpha$.

For $i < \alpha$, we define the lattice $K(i)$.

First, let $i = 0$. We obtain $K(0)$ by gluing together, as in Section 4, the lattice B with the dual ideal $[q_0, 1_B]$ and the lattice $L_{a,b}$ with the ideal $I_{a,b}$ with respect to the appropriate isomorphism (as in Section 4); the resulting lattice has a dual ideal $D_{a,b}$ with an isomorphism to $[0_A, p_0]$ (an ideal of A), and we perform this gluing to A, as illustrated in Figure 4—*mutatis mutandis*. As before, $K(0)$ is a congruence-preserving extension of $[0_A, p_0] = L$, and the congruences are of the form $\Theta_0 = (\Theta_B \overset{r}{\circ} \Theta_{a,b}) \overset{r}{\circ} \Theta_A$ (the definition of these congruences is also borrowed from Section 5, here A and B denote the lattices defined in this section not the lattices of Section 5).

Second, let $i > 0$. Glue the lattice B with the dual ideal $[q_i, 1_B]$ and the lattice L_{a_i,b_i} with the ideal I_{a_i,b_i} with respect to the isomorphism induced by φ_i; the resulting lattice has a dual ideal D_{a_i,b_i} with an isomorphism to $[0_A, p_i]$ (an ideal of A), and we perform this gluing, as illustrated with $L_{c,d}$ in Figure 4—*mutatis mutandis*. Again, $K(i)$ is a congruence-preserving extension of $[0_A, p_i] \cong L$, and the congruences are of the form $\Theta_i = (\Theta_B \overset{r}{\circ} \Theta_{a_i,b_i}) \overset{r}{\circ} \Theta_A$.

We define

$$K = \bigcup(K(i) \mid i < \alpha),$$

partially ordered by $\bigcup(\leq_i \mid i < \alpha)$, where $\leq_i$ is the partial ordering of $K(i)$, $i < \alpha$.

Observe that K is a lattice, and each $K(i)$, $i < \alpha$, is a sublattice. If x, $y \in K$, and x, $y \in K(i)$, for some $i < \alpha$, then $x \wedge y$ and $x \vee y$ are formed in $K(i)$. The result does not depend on i, because if also x, $y \in K(j)$, for $i \neq j < \alpha$, then x, $y \in K(i) \cap K(j) = A \cup B$, independent of i and j, and the operations are performed in the lattice $A \cup B$. If x, $y \in K$, but x, $y \notin K(i)$, for any $i < \alpha$, then $x \in K(i) - K(j)$ and $y \in K(j) - K(i)$, for some $i \neq j$ with i, $j < \alpha$, and then

$$x \wedge y = (x \wedge_{K(i)} 1_B) \wedge_B (y \wedge_{K(j)} 1_B),$$

and dually,

$$x \vee y = (x \vee_{K(i)} 0_A) \vee_A (y \vee_{K(j)} 0_A).$$

For a congruence Θ of L_Φ, we define the binary relation $\overline{\Theta}$ on K:

$$\overline{\Theta} = \bigcup(\Theta_i \mid i < \alpha).$$

It is obvious that $\overline{\Theta}$ restricted to $K(i)$ is Θ_i. To prove that $\overline{\Theta}$ is a congruence of K, we shall utilize the following lemma from [4] (see also [2], G. Grätzer and E. T. Schmidt [8], and F. Maeda [14]):

Lemma I.3.8. *A reflexive binary relation Θ on a lattice L is a congruence relation iff the following three properties are satisfied, for x, y, z, $t \in L$:*

(i) $x \equiv y \ (\Theta)$ *iff* $x \wedge y \equiv x \vee y \ (\Theta)$.

(ii) *$x \leq y \leq z$, $x \equiv y \pmod{\Theta}$, and $y \equiv z \pmod{\Theta}$ imply that $x \equiv z \pmod{\Theta}$.*
(iii) *$x \leq y$ and $x \equiv y \pmod{\Theta}$ imply that $x \wedge t \equiv y \wedge t \pmod{\Theta}$ and $x \vee t \equiv y \vee t \pmod{\Theta}$.*

Now observe that any two comparable elements and any three element chain of K is contained in some $K(i)$, $i < \alpha$; moreover, given two comparable elements in some $K(i)$, $i < \alpha$ and an element not in the same $K(i)$, then the meet and join formulas exhibited above take care of the computation. So it follows from Theorem 1 and Lemma I.3.8 that $\overline{\Theta}$ is a congruence on K.

The remainder of the proof of Theorem 2 is similar to the proof in Section 5.

7. Any number of intervals

Let L be a bounded lattice, let α be an ordinal, and for $i < \alpha$, let φ_i be an isomorphism between the interval $[a_i, b_i]$ and the interval $[c_i, d_i]$:

$$\varphi_i \colon [a_i, b_i] \to [c_i, d_i].$$

Let

$$\Phi = \{\, \varphi_i \mid i < \alpha \,\},$$

and let L_Φ denote the partial algebra obtained from L by adding the partial operations φ_i, for $i < \alpha$. Let us call a congruence Θ of L a Φ-*congruence* iff Θ satisfies the Substitution Property with respect to the partial unary operations φ_i, $i < \alpha$, that is, $x \equiv y \pmod{\Theta}$ implies that $x\varphi_i \equiv y\varphi_i \pmod{\Theta}$, for all x, $y \in [a_i, b_i]$ and $i < \alpha$. Thus, a congruence relation of L_Φ is the same as a Φ-congruence of L. We call K a Φ-*congruence-preserving extension* of L, if a congruence of L extends to K iff it is a Φ-congruence of L and every Φ-congruence of L has *exactly one* extension to K.

Theorem 3. *Let L be a bounded lattice, let Φ be given as above. Then the partial algebra L_Φ has a Φ-congruence-preserving convex extension into a lattice K such that all φ_i, $i \in I$, are algebraic in K. In particular, the congruence lattice of the partial algebra L_Φ is isomorphic to the congruence lattice of the lattice K.*

Note that we do not claim that the lattice K is bounded; in the construction, the lattice A is bounded but the lattice B (as the dual discrete direct product of—possibly—infinitely many bounded lattices) is not, so K has a unit but no zero, in general.

Observe that this is a very general setup. It allows us to consider a single interval $[a, b]$ and some (or all) automorphisms of it. Or we may take two intervals $[a, b]$ and $[c, d]$, and two isomorphisms $\varphi \colon [a, b] \to [c, d]$ and $\varphi \colon [c, d] \to [a, b]$ without requiring that $\varphi^{-1} = \psi$.

It would be nice to be able to claim that this theorem can be proved by applying Theorem 2 to the isomorphisms φ_i one at a time, and then forming a direct limit. Unfortunately, the direct limit at ω produces a lattice with no zero or unit, so we cannot continue with the construction.

Nevertheless, the proof of Theorem 3 can be obtained by one modification of the proof of Theorem 2.

To start proving Theorem 3, let $A = M_\alpha \boxtimes L$, with spanning M_α and with atoms p_i, $i < \alpha$. Let $B_i = M_\alpha \boxtimes [a_i, b_i]$, with spanning M_α and with atoms $q_{i,j}$, $j < \alpha$. Now we define the lattice B as the dual discrete direct product of the B_i, $i < \alpha$. So B has a unit element; it contains (after the obvious identifications) each B_i, $i < \alpha$, as a dual ideal, and every element of B is a finite meet of elements from these dual ideals.

For $i < \alpha$, let us define the lattice $K(i)$ in a manner very similar to the construction in Section 4, in the proof of Theorem 1, starting with the lattice A and gluing to it L_{a_i,b_i} over the ideal $[0_A, p_i]$ of A and the dual ideal D_{a_i,b_i} of L_{a_i,b_i}; and then gluing the resulting lattice to B over the ideal I_{a_i,b_i} and the dual ideal $[q_i, 1_B]$ of B. Observe that for i, $i' < \alpha$

$$K(i) \cap K(i') = A \cup B, \text{ if } i \neq i',$$

and the convex sublattice of K generated by $A \cup B$ is equal to K. Now for a congruence Θ of L_Φ, let $\Theta(i)$ be the unique extension of Θ to $K(i)$, as defined in Section 5.

We define the lattice

$$K = \bigcup (\, K(i) \mid i < \alpha\,),$$

and for a congruence Θ of L_Φ, the binary relation $\overline{\Theta}$ on K:

$$\overline{\Theta} = \bigcup (\, \Theta(i) \mid i < \alpha\,).$$

The rest of the proof is analogous to the proofs of Theorems 1 and 2. The difference is, of course, that while in any B_i, any two intervals $[p_{i,j}, q_{i,j}]$ and $[p_{i,j'}, q_{i,j'}]$ are congruence equivalent (i, j, $j' < \alpha$), the intervals $[p_{i,j}, q_{i,j}]$ and $[p_{i',j'}, q_{i',j'}]$, for $i \neq i'$, j, $j' < \alpha$, are congruence independent, since they appear in distinct direct factors of B.

Theorem 3 allows us to generalize Theorem 1 from isomorphic intervals to isomorphic convex sublattices. Let L be a bounded lattice, let U and V be convex sublattices of L, and let $\varphi\colon U \to V$ be an isomorphism between these two convex sublattices. We introduce a $\overset{\leftrightarrow}{\varphi}$*-congruence*, the partial algebra $L_{\overset{\leftrightarrow}{\varphi}}$, and a $\overset{\leftrightarrow}{\varphi}$*-congruence-preserving extension* as in Section 1.2, *mutatis mutandis*.

Here is the generalization of Theorem 1:

Theorem 1′. *Let L be a bounded lattice, let U and V be convex sublattices of L, and let $\varphi\colon U \to V$ be an isomorphism between these two convex sublattices. Then L has a $\overset{\leftrightarrow}{\varphi}$-congruence-preserving convex extension into a lattice K such that both φ and φ^{-1} are algebraic in K. In particular, the congruence lattice of the partial algebra $L_{\overset{\leftrightarrow}{\varphi}}$ is isomorphic to the congruence lattice of the lattice K.*

Note that K is no longer claimed to be bounded because we obtain it from Theorem 3.

Proof. Let $[a_i, b_i]$, $i < \alpha$, be the family of all subintervals of U. Let $[c_i, d_i] = [a_i, b_i]\varphi$, for $i < \alpha$, be the family of all corresponding subintervals of V. Let

$$\varphi_i \colon [a_i, b_i] \to [c_i, d_i], \quad \text{for } i < \alpha,$$

be the restriction of φ to $[a_i, b_i]$. Now we get Theorem 1′ by a straightforward application of Theorem 3. □

Of course, we can similarly generalize Theorems 2 and 3.

8. Lattices with zero but no unit

Theorems 1–3 remain valid if we only assume that the lattice L have a zero (of course, then we conclude the existence of a lattice only with zero). The existence of the unit element was required only for the clarity of the exposition. The lattice A in the proof of Theorem 1 (and similarly, in the proofs of Theorems 2 and 3), then, will not necessarily have a spanning M_3, which we utilized in Section 5. However, we can easily find p_1, p_2, and p_3 in A that generate a sublattice M_3 and will have all the properties required in Section 5.

Theorem $1'$ requires a trivial modification.

Let L be a lattice, let A, $B \subseteq L$, and let $\varphi\colon A \to B$ be a map. We call the map φ *locally algebraic* iff for every $[u, v] \subseteq A$, there is a unary algebraic function $\mathbf{p}(x)$ such that $x\varphi = \mathbf{p}(x)$, for all $x \in [u, v]$.

Theorem $1''$. *Let L be a lattice with zero, let U and V be convex sublattices of L, and let $\varphi\colon U \to V$ be an isomorphism between these two convex sublattices. Then L has a $\overleftrightarrow{\varphi}$-congruence-preserving convex extension into a lattice K such that both φ and φ^{-1} are locally algebraic in K. In particular, the congruence lattice of the partial algebra $L_{\overleftrightarrow{\varphi}}$ is isomorphic to the congruence lattice of the lattice K.*

9. Discussion

9.1. **Congruence lattices of lattices.** This field is dominated by the following problem (see Problem II.7 of [2]): *Can every distributive algebraic lattice be represented as the congruence lattice of a lattice?* Despite the fact that this question had already been considered by R. P. Dilworth in the early forties, and that so many of the best in the field spent so much time trying to answer it, an answer still eludes us. For a detailed review of this field as of 1998, see Appendix C of [4], and for a recent survey, see J. Tůma and F. Wehrung [19].

Based on Theorems 2 and 3, this problem can be restated as follows:

Problem 1. Which distributive algebraic lattices can be represented as congruence lattices of partial algebras of the type L_Φ introduced in this paper.

This is not a trivial reduction; as mentioned in the Introduction, this method was successfully applied in E. T. Schmidt [17]. Of course, the techniques existing then allowed Schmidt to apply this method only to isomorphic *distributive* intervals. Also, the theorem: *Every lattice has a regular congruence-preserving extension,* was first proved using the construction developed in this paper; the proof published in [11] uses a different construction.

To illustrate this problem, consider that in $F(3)$ (the free lattice on 3 generators) there are infinitely many prime intervals. Any partition Π of these yields a Φ for Theorem 3 (since any two prime intervals are isomorphic). Hence we get a lattice $K(\Phi)$ whose congruence lattice is isomorphic to $F(3)_\Phi$.

E. T. Schmidt [16] states the best result in the field of congruence lattice representations:

Theorem 4. *Let D be a distributive algebraic lattice in which the meet of two compact elements is compact again. Then D can be represented as the congruence lattice of a lattice.*

Recent developments include the following result, see F. Wehrung [20] and M. Ploščica, J. Tůma, and F. Wehrung [15]:

Theorem 5.

(i) *Let D be a distributive algebraic lattice in which the meet of two compact elements is compact again. Then D can be represented as the congruence lattice of a relatively complemented lattice with zero.*

(ii) *Congruence lattices of free lattices with $\geq \aleph_2$ generators cannot be represented as congruence lattices of relatively complemented lattices.*

See also F. Wehrung [21] and J. Tůma and F. Wehrung [18].

These results are basically negative. They state that the classical approaches (E. T. Schmidt and P. Pudlák) have limited reach: they do not even get to the congruence lattice of a free lattice with $\geq \aleph_2$ generators.

Our approach is completely different from the classical approaches. We hope that for many new classes of distributive algebraic lattices it will provide representation theorems as congruence lattices of lattices.

9.2. **Congruence distributivity.** A trivial corollary of Theorems 2 and 3 is the following:

Corollary 14. *The congruence lattice of the partial algebra* L_Φ *is distributive.*

Let us prove this directly. To simplify the presentation, we prove this under the assumptions of Theorem 1 for $L_{\overleftrightarrow{\varphi}}$. Corollary 14 trivially follows from the following statement:

Lemma 15. $\operatorname{Con} L_{\overleftrightarrow{\varphi}}$ *is a sublattice of* $\operatorname{Con} L$.

Proof. Let Θ, $\Phi \in \operatorname{Con} L_{\overleftrightarrow{\varphi}}$. We only have to prove that $\Theta \vee \Phi$ (the join formed in $\operatorname{Con} L$) has the Substitution Property with respect to φ and φ^{-1}. Let $x \equiv y \pmod{\Theta \vee \Phi}$, where $a \leq x \leq y \leq b$. Then there exists a sequence $x = z_0 \leq z_1 \leq \cdots \leq z_n = y$ of elements of L such that for every $i < n$, either $z_i \equiv z_{i+1}$ (Θ) or $z_i \equiv z_{i+1}$ (Φ). But then either $z_i\varphi \equiv z_{i+1}\varphi$ (Θ) or $z_i\varphi \equiv z_{i+1}\varphi$ (Φ), because Θ and Φ are $\overleftrightarrow{\varphi}$-congruences. So the sequence $x\varphi = z_0\varphi, z_1\varphi, \ldots, z_n\varphi = y\varphi$ establishes that $x\varphi \equiv y\varphi$ $(\Theta \vee \Phi)$. The same argument applied to φ^{-1} establishes that $x\varphi^{-1} \equiv y\varphi^{-1}$ $(\Theta \vee \Phi)$. ◻

9.3. **Problems.** Theorem 1 is proved in this paper only for lattices with zero. So the following is natural to raise:

Problem 2. Let L be a lattice, let $[a, b]$ and $[c, d]$ be intervals of L, and let $\varphi\colon [a, b] \to [c, d]$ be an isomorphism between these two intervals. Does L have a $\overleftrightarrow{\varphi}$-congruence-preserving (convex) extension into a lattice K such that both φ and φ^{-1} are algebraic in K?

In Theorem 1, we start with a bounded lattice L, and obtain a bounded lattice K. However, the construction does not preserve the bounds.

Problem 3. Let L be a bounded lattice, let $[a, b]$ and $[c, d]$ be intervals of L, and let $\varphi\colon [a, b] \to [c, d]$ be an isomorphism between these two intervals. Does L have a $\overleftrightarrow{\varphi}$-congruence-preserving, $\{0, 1\}$-preserving, (convex) extension into a lattice K such that both φ and φ^{-1} are algebraic in K?

In Section 8 we discuss that Theorems 1, 2, and 3 are valid for lattices with zero, and in Theorems 1 and 2 we obtain lattices with zero. So we ask:

Problem 4. Let L be a lattice with zero, let $[a, b]$ and $[c, d]$ be intervals of L, and let $\varphi\colon [a, b] \to [c, d]$ be an isomorphism between these two intervals. Does L have a $\overleftrightarrow{\varphi}$-congruence-preserving, $\{0\}$-preserving, (convex) extension into a lattice K such that both φ and φ^{-1} are algebraic in K?

Of course, we can raise Problems 2–4 also for Theorem 2. For Theorem 3, we do not even know the existence of a K with zero.

Magic wands extend to unary algebraic functions. But algebraic functions, though isotone, are not necessarily isomorphisms. An affirmative answer to the following problem would be a two-fold generalization of Theorem 1.

Problem 5. Let L be a lattice and let φ be an isotone map of L into L. Does there exist an extension K of L and a unary polynomial p of K such that the following conditions hold:

(i) The restriction of p to L is φ.
(ii) A congruence relation Θ of L can be extended to K if and only if $x \equiv y$ (Θ) implies that $x\varphi \equiv y\varphi$ (Θ), for $x, y \in L$.
(iii) Every congruence of L has at most one extension to K.

Note that the argument in the proof of Lemma 15 applies to this situation, so the algebra L with the unary operation φ has a distributive congruence lattice.

There are, of course, variants of this problem for more than one isotone map and for $\{0\}$-preserving and $\{0, 1\}$-preserving extensions.

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Department of Mathematics, University of Manitoba, Winnipeg, MB R3T 2N2, Canada
E-mail address, George Grätzer: gratzer@ms.umanitoba.ca
URL, George Grätzer: http://server.math.umanitoba.ca/homepages/gratzer/

Mathematical Institute of the Budapest University of Technology and Economics, Műegyetem rkp. 3, H-1521 Budapest, Hungary
E-mail address, E. T. Schmidt: schmidt@math.bme.hu
URL, E. T. Schmidt: http://www.math.bme.hu/~schmidt/