COMPLETE CONGRUENCE REPRESENTATIONS WITH 2-DISTRIBUTIVE MODULAR LATTICES

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Abstract. In 1990, we published the following result:

Let $\mathfrak{m}$ *be a regular cardinal* $> \aleph_0$*. Every* $\mathfrak{m}$-algebraic lattice L *can be represented as the lattice of* $\mathfrak{m}$*-complete congruence relations of an* $\mathfrak{m}$-complete modular lattice K*.*

In this note, we present a short proof of this theorem. In fact, we present a significant improvement: The lattice K we construct is 2-distributive.

1. Introduction

We prove the following result:

Theorem. *Let* $\mathfrak{m}$ *be a regular cardinal* $> \aleph_0$*. For every* $\mathfrak{m}$-algebraic lattice L*, there exists a lattice* K *with the following properties:*

(i) K *is an* $\mathfrak{m}$*-complete modular lattice.*
(ii) *The lattice* $\operatorname{Con}_{\mathfrak{m}} K$ *of* $\mathfrak{m}$*-complete congruences of* K *is isomorphic to* L*.*
(iii) *Every* $\mathfrak{m}$*-compact congruence of* K *is principal, generated by a prime interval.*
(iv) K *is* 2*-distributive.*

This theorem was first proved in [4] (an alternative proof was published in [3]); only (iv) is new. This paper presents a very simple proof of this stronger result.

We refer the reader to [4] or [2] for background information. As in [4], we outline the construction and leave the easy computations to the reader.

The construction of K uses two basic constructions: collar sums (see Section 2) and towers (see Section 3). The tower construction is a simplified version of the tower construction in [5].

In R. Freese, G. Grätzer, and E. T. Schmidt [1] we proved that every complete lattice L can be represented as the lattice of complete congruence relations of a complete modular lattice K. This is a special case of our Theorem: $|L| \leq \mathfrak{m}$. The lattice K we construct in [1] also has the property that every complete congruence is generated by a prime interval. The lattice K in [1] is 3-distributive but not 2-distributive.

The strongest result of this type is in [5]: every $\mathfrak{m}$-algebraic lattice K can be represented as the lattice of $\mathfrak{m}$-complete congruences of an $\mathfrak{m}$-complete *distributive* lattice L. This lattice L, however, does not have property (iii); not every $\mathfrak{m}$-compact

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congruence of K is principal, generated by a prime interval. In fact, the lattice L constructed in [5] has no prime interval. It is easy to prove that this in necessarily so. Let $\mathfrak{p}$ be a prime interval of an $\mathfrak{m}$-complete complete distributive lattice L. Then $\Theta(\mathfrak{p})$, the congruence generated by $\mathfrak{p}$, is, in fact, a $\mathfrak{m}$-complete complete congruence, that is, $\Theta(\mathfrak{p}) = \Theta_{\mathfrak{m}}(\mathfrak{p})$ by Lemma 2 of [1]. Therefore, $\Theta_{\mathfrak{m}}(\mathfrak{p})$ is an atom of $\operatorname{Con}_{\mathfrak{m}} K$. So if L has no atoms, then K cannot have a prime interval.

Notation. For basic concepts and notation, see [2]. In this paper, let $\mathfrak{m}$ be a regular cardinal. A set of cardinality less than $\mathfrak{m}$ is said to be *small*. A concept modified with "$\mathfrak{m}$-" has the obvious meaning: $\mathfrak{m}$-product is $\mathfrak{m}$-discrete direct product; $\mathfrak{m}$-congruence is $\mathfrak{m}$-complete congruence (the Substitution Property holds for small sets of congruences), $\mathfrak{m}$-compact and $\mathfrak{m}$-algebraic, and so on. In notation, the subscripted $\mathfrak{m}$ plays the same role: $\operatorname{Con}_{\mathfrak{m}} K$, $\Theta_{\mathfrak{m}}(\mathfrak{p})$, $\operatorname{Id}_{\mathfrak{m}} C$, and so on. $\Theta_c(a, b)$ is the complete congruence collapsing a and b.

L will denote the $\mathfrak{m}$-algebraic lattice in the Theorem; K will be the lattice we construct to prove the Theorem. If $|L| \leq 2$, then we can take $K = L$. We shall henceforth assume that $|L| > 2$.

Let C denote the set of *nonzero* $\mathfrak{m}$-compact elements of L. Since $|L| > 2$, it follows that $|C| \geq 2$. The set C is closed under small nonempty joins in L. We denote by $\operatorname{Id}_{\mathfrak{m}} C$ the complete lattice of $\mathfrak{m}$-ideals of C together with the "empty ideal" $\varnothing$. Clearly, $x \mapsto \{c \in C \mid c \leq x\}$ is an isomorphism between L and $\operatorname{Id}_{\mathfrak{m}} C$.

The concept of n-distributive lattices was introduced in A. P. Huhn [6].

2. Collar sum

Let E and F be $\mathfrak{m}$-complete lattices; let E have the unit element 1_E and F the zero 0_F. We denote by $E \oplus F$ the ordinal sum of E and F, where 1_E and 0_F are identified. Let us further assume that F contains a set P of atoms, E contains a set Q of dual atoms, and there is a bijection $\varphi \colon P \to Q$.

Consider the lattice $E \oplus F$ and, for every $p \in P$, we adjoin two new elements r_p, s_p, such that $M_p = \{p\varphi, r_p, s_p, 0_F, p\}$ form a sublattice isomorphic to the five-element, nondistributive, modular lattice $\mathfrak{M}_3$. The set

$$G = E \oplus F \cup \bigcup(\, \{r_p, s_p\} \mid p \in P\,) = E \oplus F \cup \bigcup(\, M_p \mid p \in P\,)$$

is naturally partially ordered by the requirement that $E \oplus F$, and all the M_p, $p \in P$, be sublattices; we call D the *collar sum* of E and F with respect to φ—see Figure 1.

Lemma 1. *The collar sum G of E and F has the following properties:*

(i) *G is an $\mathfrak{m}$-complete lattice.*
(ii) *If E and F are modular, then so is G.*
(iii) *If E and F are modular and 2-distributive, then so is G.*
(iv) *An $\mathfrak{m}$-congruence Ψ of G can be described by an $\mathfrak{m}$-congruence Θ of E and an $\mathfrak{m}$-congruence Φ of F satisfying*

$$p\varphi \equiv 1_E \quad (\Theta) \qquad \textit{iff} \qquad 0_F \equiv p \quad (\Phi),$$

for all $p \in P$.

Proof.

(i) The first statement is trivial.

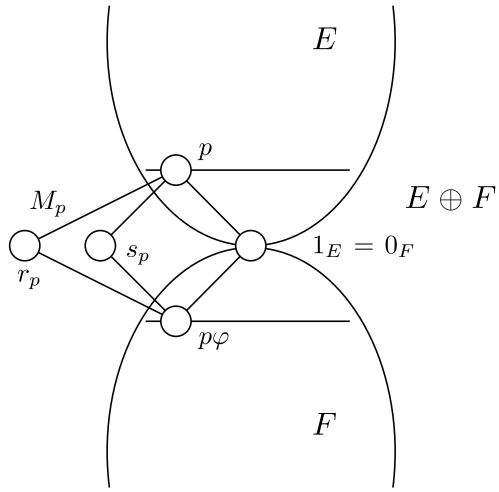

FIGURE 1. Collar sum of E and F with respect to φ

(ii) Assume that E and F are both modular. Then $E \oplus F$ is obviously modular. If G is not modular, then it contains $\mathfrak{N}_5$ as a sublattice. Since, all r_p, s_p ($p \in P$) are doubly irreducible elements in G, it is easy to see that $\mathfrak{N}_5$ must be a sublattice of E or F, a contradiction.

(iii) By the previous statement, G is modular. If it is not 2-distributive, then by A. P. Huhn [6], it contains a nine element subset $R = B \cup \{c\}$, where B is an eight element Boolean sublattice with zero 0_B and unit 1_B, such that c is a complement of all three atoms of B in $[0_B, 1_B]$. As in (ii), it is easy to see that R must be in E or F, a contradiction.

(iv) If Θ and Φ satisfy the condition in (iv), then we can define an $\mathfrak{m}$-congruence Ψ on G as follows:
   (a) $\Psi = \Theta$ on E.
   (b) $\Psi = \Phi$ on F.
   (c) For $p \in P$, $\{r_p\}$ and $\{s_p\}$ are both singleton congruence classes under Ψ unless $p\varphi \equiv 1_E \ (\Theta)$ *and* $0_F \equiv p \ (\Phi)$, in which case the congruence class under Ψ containing r_p is

$$[1_E]\Theta \cup [0_F]\Phi \cup \{r_p, s_p\}.$$

The converse is obvious.

□

3. The tower construction

In this section, let A be a small set with at least two elements. We build a 2-distributive modular lattice $T(A)$, the *tower over A.*

Let $D = \mathbf{2}^A$ with zero $\mathbf{o}_D$ and unit $\mathbf{i}_D$, where $\mathbf{2}$ denotes the two-element lattice with zero $0_\mathbf{2}$ and unit $1_\mathbf{2}$. We regard the elements of D as functions $\mathbf{v}$; so $\mathbf{v}(a) \in \mathbf{2}$, for $a \in A$. Since A is small, D is the complete direct product as well as the $\mathfrak{m}$–discrete direct product of $|A|$ copies of $\mathbf{2}$—see Figure 2.

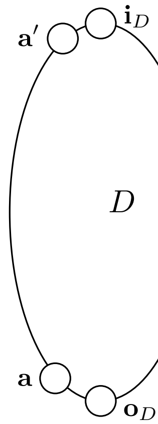

FIGURE 2. The lattice D

For $a \in A$, let $\mathbf{a} \in D$ be defined as follows:

$$\mathbf{a}(x) = \begin{cases} 1_{\mathbf{2}}, & \text{for } x = a; \\ 0_{\mathbf{2}}, & \text{for } x \neq a. \end{cases}$$

Let $\mathbf{a}'$ be the complement of $\mathbf{a}$ in D, that is,

$$\mathbf{a}'(x) = \begin{cases} 0_{\mathbf{2}}, & \text{for } x = a; \\ 1_{\mathbf{2}}, & \text{for } x \neq a. \end{cases}$$

Observe that in D, the intervals $[\mathbf{o}_D, \mathbf{a}]$ and $[\mathbf{a}', \mathbf{i}_D]$ are prime.

The elements of D^ω are vectors of type ω. For every natural number n, we define the element $\mathbf{o}_n$ of D^ω as follows:

$$\mathbf{o}_n = \langle \underbrace{\mathbf{i}_D, \dots, \mathbf{i}_D}_{n\text{-times}}, \mathbf{o}_D, \dots \rangle.$$

Then

$$\mathbf{o}_0 = \langle \mathbf{o}_D, \mathbf{o}_D, \mathbf{o}_D, \dots \rangle < \mathbf{o}_1 = \langle \mathbf{i}_D, \mathbf{o}_D, \mathbf{o}_D, \dots \rangle < \mathbf{o}_2 = \langle \mathbf{i}_D, \mathbf{i}_D, \mathbf{o}_D, \dots \rangle < \dots$$

Let $D_n = [\mathbf{o}_n, \mathbf{o}_{n+1}] \subseteq D^\omega$, for $n < \omega$.

Obviously, there is a natural isomorphism between D and D_n. Let $\mathbf{o}_n$, $\mathbf{i}_n$, $\mathbf{a}_n$, and $\mathbf{a}'_n$ be the image of $\mathbf{o}$, $\mathbf{i}$, $\mathbf{a}$, and $\mathbf{a}'$, respectively, under this isomorphism; of course, $\mathbf{i}_n = \mathbf{o}_{n+1}$ and $[\mathbf{o}_n, \mathbf{a}_n]$, $[\mathbf{a}'_n, \mathbf{i}_n]$ are prime intervals.

The poset $\bigcup(D_n \mid n < \omega)$ is obviously the sum $\bigoplus_{i=1}^{\infty} D_i$.

For every atom $\mathbf{a} \in D$ and for every natural number n, the interval $[\mathbf{a}'_n, \mathbf{a}_{n+1}]$ is the three element chain: $\{\mathbf{a}'_n, \mathbf{o}_{n+1}, \mathbf{a}_{n+1}\}$; define two new elements, r^a_n, s^a_n such that

$$M_{n,a} = \{\mathbf{a}'_n, r^a_n, s^a_n, \mathbf{o}_{n+1}, \mathbf{a}_{n+1}\}$$

forms an $\mathfrak{M}_3$.

Now, we define the tower $T(A)$ as the poset

$$\begin{aligned} T(A) &= \bigcup(D_n \mid n < \omega) \cup \bigcup(\{r^a_n, s^a_n\} \mid n < \omega \text{ and } a \in A) \\ &= \bigcup(D_n \mid n < \omega) \cup \bigcup(M_{n,a} \mid n < \omega \text{ and } a \in A). \end{aligned}$$

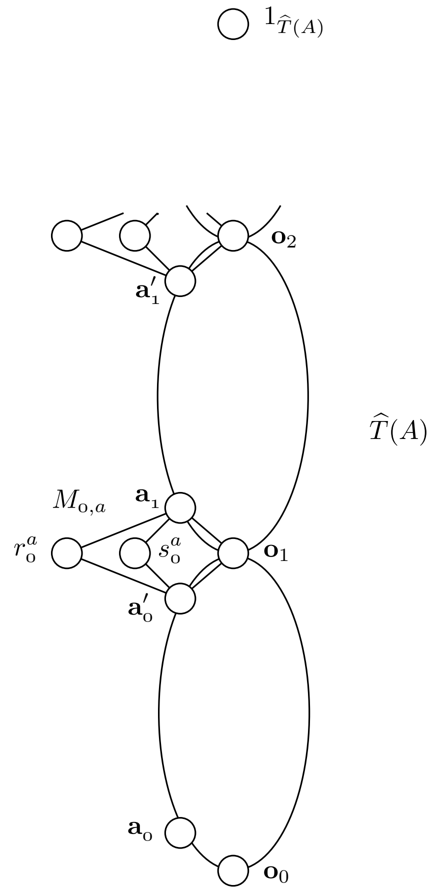

FIGURE 3. The lattice $\widehat{T}(A)$

The zero of $T(A)$ will be denoted by $0_{T(A)}$. Let $\widehat{T}(A)$ be $T(A)$ with a unit $1_{\widehat{T}(A)}$ adjoined—see Figure 3.

Lemma 2. *The towers $T(A)$ and $\widehat{T}(A)$ have the following properties:*

(i) *$T(A)$ is a 2-distributive modular lattice.*

(ii) *$\widehat{T}(A)$ is 2-distributive modular and also complete.*

(iii) *Let $a \in A$. We define an equivalence relation Θ^a_A on $\widehat{T}(A)$. The nontrivial classes of Θ^a_A are of the following three types:*

(a) *the congruence class of $\Theta(\mathbf{o}_0, \mathbf{a}_0)$ on D_0 containing $\mathbf{o}_0$;*

(b) *the congruence classes of $\Theta(\mathbf{o}_n, \mathbf{a}_n)$ on D_n not containing $\mathbf{o}_n$ or $\mathbf{i}_n$;*

(c) *the intervals $[\mathbf{a}'_n, \mathbf{a}_{n+1}]$, for all $n < \omega$.*

Then Θ^a_A is a complete congruence on $\widehat{T}(A)$, for all $a \in A$. In fact, $\Theta^a_A = \Theta_c(\mathbf{o}_0, \mathbf{a}_0) = \Theta_{\mathfrak{m}}(\mathbf{o}_0, \mathbf{a}_0)$, and the interval $[\mathbf{o}_0, \mathbf{a}_0]$ of $\widehat{T}(A)$ is prime.

(iv) *There is a one-to-one correspondence between the subsets B of A and the ($\mathfrak{m}$-)congruences of $\widehat{T}(A)$, given by*

$$B \to \bigvee\nolimits_{\mathfrak{m}} (\Theta^a_A \mid a \in B).$$

Proof. This lemma follows from Lemma 1, since we obtain $T(A)$ as a direct limit of collar sums. □

For every small $A \subseteq C$ (recall that C denotes the set of nonzero $\mathfrak{m}$-compact elements of L), $|A| > 1$, we build a tower $T(A)$. In addition, for every $a \in C$, we "double" a to obtain the two-element set $\{\dot{a}, \ddot{a}\}$, and carry out the tower construction for this set; the resulting lattices will be denoted by $T(a)$ and $\widehat{T}(a)$, respectively. Note that in this case $D = \mathbf{2}^2$, and the lattice of ($\mathfrak{m}$-)congruence relations of $\widehat{T}(A)$ is isomorphic to $\mathbf{2}^2$.

4. Merging the towers

The next two constructions bring all the towers together into one lattice.

Let $A \subseteq C$, $|A| > 1$ be a small set; first we merge the towers $\widehat{T}(A)$ and $\widehat{T}(\bigvee A)$ by forming the direct product $T(A) \times T(\bigvee A)$ and adjoining a new unit element; let $M(A)$ denote this lattice with zero $0_{M(A)}$ and unit $1_{M(A)}$—see Figure 4. . Observe that $\widehat{T}(A)$ and $\widehat{T}(\bigvee A)$ are 2-distributive modular lattices, hence so is $M(A)$.

Lemma 3.

(i) *The $\mathfrak{m}$-congruences of $M(A)$ different from $\iota_{M(A)}$ are in one-to-one correspondence with pairs $\langle \Theta, \Psi \rangle$, where $\Theta < \iota_{T(A)}$ is an $\mathfrak{m}$-congruence of $T(A)$ and $\Phi < \iota_{T(\bigvee A)}$ is an $\mathfrak{m}$-congruence of $T(\bigvee A)$.*

(ii) *The $\mathfrak{m}$-congruence lattice* $\mathrm{Con}_{\mathfrak{m}}\, M(A)$ *of $M(A)$ is* atomistic (*that is, every element is a complete join of atoms*). *The atoms of* $\mathrm{Con}_{\mathfrak{m}}\, M(A)$ *are of the form*

$$\Theta_{\mathfrak{m}}(0_{M(A)}, \langle \mathbf{a}_0, 0_{T(\bigvee A)} \rangle),$$

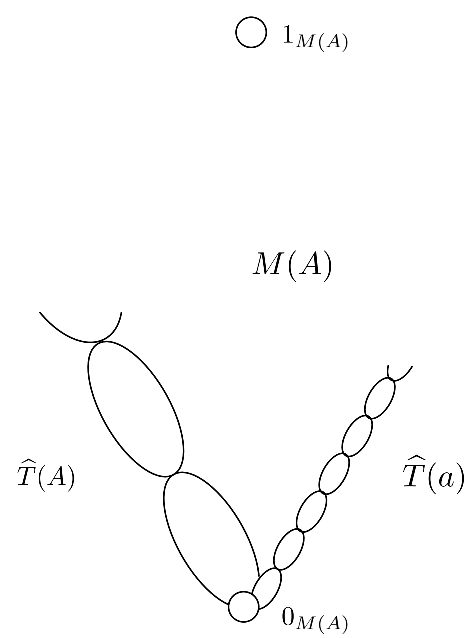

Figure 4. The lattice $M(A)$

where $a \in A$, or of the form

$$\Theta_{\mathfrak{m}}(0_{M(A)}, \langle 0_{T(A)}, \mathbf{b}_0 \rangle),$$

where $b = \dot{u}$ or $b = \ddot{u}$ and $u = \bigvee A$.

Proof. This follows from Lemma 2. $\square$

We merge all the towers, by forming the $\mathfrak{m}$-direct product:

$$M(L) = \Pi_{\mathfrak{m}}\, (M(X) \mid X \subseteq C \text{ and } 1 < |X| < \mathfrak{m}),$$

with zero $0_{M(L)}$—see Figure 5. Observe that, in general, $M(L)$ does not have a unit element.

The following lemma trivially follows from Lemma 3.

Lemma 4. *The $\mathfrak{m}$-congruence lattice $\operatorname{Con}_{\mathfrak{m}} M(L)$ is atomistic. The atoms Θ of $\operatorname{Con}_{\mathfrak{m}} M(L)$ are in one-to-one correspondence with atoms $\mathbf{a}_\Theta$ of $M(L)$ such that Θ is generated as an $\mathfrak{m}$-congruence by the prime interval $[0_M(L), \mathbf{a}_\Theta]$.*

This motivates the following notation. Let q be an atom of $M(L)$. Then q is associated with an atom Θ of $\operatorname{Con}_{\mathfrak{m}} M(L)$ and an element $a = a_\Theta \in C$. We shall call a the color of q, and write $\operatorname{col} q = a$.

5. The construction of K

We construct K as the collar sum of $F = M(L)$ with P the set of all atoms of F and the lattice E defined as follows: For every $a \in C$, we define I_a as the set of all atoms of $M(L)$ of color a. Let N_a be the modular lattice of length 2 with $|I_a|$ atoms if $|I_a| > 2$; if $|I_a| = 2$, then we choose $N_a = \mathfrak{M}_3$. Then we form the $\mathfrak{m}$-product $N = \Pi_{\mathfrak{m}}\, (N_a \mid a \in C)$. Let E be the dual of N; let $x \to x^*$ denote this dual isomorphism. Under this dual isomorphism, the image of N_a will be denoted by H_a. Then H_a is a dual ideal of E. So for every $a \in C$, there is a bijection φ_a between I_a and the a subset of all (dual) atoms of H_a. The union φ of the maps φ_a defines a bijection between P and a subset Q of all dual atoms of E. We apply the collar construction to obtain K—see Figure 6.

Every N_a is a 2-distributive modular lattice, hence so is N and E. By Lemma 1, K is a 2-distributive modular lattice. Furthermore, K is $\mathfrak{m}$-complete.

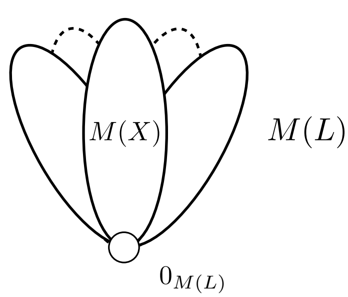

FIGURE 5. The lattice $M(L)$

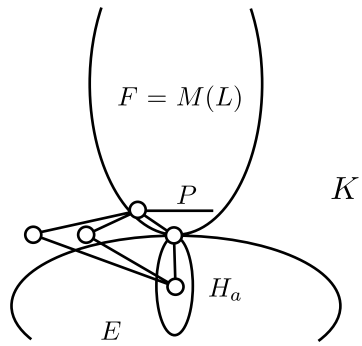

FIGURE 6. The lattice K

6. The proof of the Theorem

For $a \in C$, we define an equivalence relation Θ^a on K in several steps.

Firstly, let A be a small subset of C with $|A| > 1$, and we define $\Theta^a_{M(A)}$ on $M(A)$. If $A \subseteq (a]$ (or equivalently, if $\bigvee A \leq a$), then we define $\Theta^a_{M(A)}$ on $M(A)$ as $\iota_{M(A)}$. Otherwise, using Lemma 3, we take the $\mathfrak{m}$-congruence Φ of $\widehat{T}(A)$ that corresponds to the set $A \cap (a]$, and let $\Theta^a_{M(A)}$ be the $\mathfrak{m}$-congruence on $M(A)$ that is Φ on $\widehat{T}(A)$ and $\omega_{\widehat{T}(\bigvee A)}$ on $\widehat{T}(\bigvee A)$; by Lemma 3, this describes an $\mathfrak{m}$-congruence $\Theta^a_{M(A)}$.

Secondly, we form the $\mathfrak{m}$-direct product of these $\Theta^a_{M(A)}$, and obtain $\Theta^a_{M(L)}$ on $M(L)$.

Thirdly, observe that, for every $p \in P$, the elements $0_{M(L)}$ and p are collapsed by $\Theta^a_{M(L)}$ iff $\operatorname{col} p \leq a$. Since E is the dual of the $\mathfrak{m}$-direct product $N = \Pi_{\mathfrak{m}} (N_b \mid b \in C)$, we can define Θ^a_E on E as the unique $\mathfrak{m}$-congruence with the property that the dual of Θ^a_E restricted to N_b is ω_{N_b} iff $b \nleq a$, otherwise, it is ι_{N_b}.

Fourthly, for every $p \in P$, we have the sublattice M_p in the collar sum. On M_p, we define $\Theta^a_{M_p}$ as ω_{M_p} iff $\operatorname{col} p \leq a$, otherwise, it is ι_{M_p}.

Finally, we define Θ^a on K as the transitive closure of the relation

$$\Theta^a_{M(L)} \cup \Theta^a_F \cup \bigcup(\, \Theta^a_{M_p} \mid p \in P\,).$$

Here are some easy consequences of this definition.

Lemma 5. *Let $a \in C$.*

(i) Θ^a *is an $\mathfrak{m}$-congruence of K.*

(ii) *For p, $q \in I_a$,*

$$\Theta_{\mathfrak{m}}(0_F, \mathbf{p}) = \Theta_{\mathfrak{m}}(0_F, \mathbf{q})$$

in K.

(iii) *Choose a small set $A \subseteq C$ such that $a \in A$, $|A| > 1$. Let $\mathbf{a}$ denote the element of K that corresponds to the element $\mathbf{a}_0 \in T(A)$. Then $\Theta^a = \Theta_{\mathfrak{m}}(0_F, \mathbf{a})$ and $[0_F, \mathbf{a}]$ is prime.*

Proof. (i) It is easy to compute that Θ^a is an $\mathfrak{m}$-congruence of K.

(ii) The second statement follows from the fact that N_a is a simple lattice.

(iii) This is obvious from the definition of Θ^a and from the previous lemmas. $\square$

Obviously, the map $a \mapsto \Theta^a$ from C to the $\mathfrak{m}$-join semilattice of $\mathfrak{m}$-compact congruences of K is one-to-one and isotone. We shall need the following statement:

Lemma 6. *For every small nonempty X in C,*

$$\bigvee\nolimits_{\mathfrak{m}}(\Theta^x \mid x \in X) = \Theta^{\bigvee X}$$

holds in K.

Proof. Since $x \to \Theta^x$ is isotone, it follows that

$$\bigvee\nolimits_{\mathfrak{m}}(\Theta^x \mid x \in X) \leq \Theta^{\bigvee X}.$$

To prove the reverse inequality, we can assume that $|X| > 1$, and consider the lattice $M(X)$. Let $y = \bigvee X$ in C. By definition, in $M(X)$ all $\mathbf{x}_0$ ($x \in X$) are collapsed with $0_{M(X)}$ by $\bigvee_{\mathfrak{m}}(\Theta^x \mid x \in X)$. It follows then that in $M(X)$:

$$\bigvee\nolimits_{\mathfrak{m}}(\Theta^x \mid x \in X) = \iota_{M(X)}.$$

Therefore, $0_M \equiv \mathbf{b}_0 \ (\bigvee_{\mathfrak{m}}(\Theta^x \mid x \in X))$, where $b = \dot{y}$ and $\mathbf{b}_0$ is formed in $\widehat{T}(\bigvee X)$. Now take any small $Y \subseteq C$, such that $y \in Y$, $|Y| > 1$. By Lemma 5(ii), $\Theta_{\mathfrak{m}}(0_M, \mathbf{b}) = \Theta_{\mathfrak{m}}(0_M, \mathbf{y})$, and $\Theta_{\mathfrak{m}}(0_M, \mathbf{y}) = \Theta^{\bigvee X}$, completing the proof of the lemma. $\square$

Hence to prove the Theorem, it is sufficient to observe the following statement.

Lemma 7. *Every $\mathfrak{m}$-compact $\mathfrak{m}$-congruence $\Theta \neq \omega$ of K is of the form $\Theta = \Theta^a$, for some $a \in C$.*

Proof. This is trivial from Lemma 6. $\square$

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