REGULAR CONGRUENCE-PRESERVING EXTENSIONS OF LATTICES

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Dedicated to the memory of Viktor Gorbunov

Abstract. In this paper, we prove that every lattice L has a congruence-preserving extension into a regular lattice $\tilde{L}$, moreover, every compact congruence of $\tilde{L}$ is principal. We construct $\tilde{L}$ by iterating a construction of the first author and F. Wehrung and taking direct limits.

We also discuss the case of a finite lattice L, in which case $\tilde{L}$ can be chosen to be finite, and of a lattice L with zero, in which case $\tilde{L}$ can be chosen to have zero and the extension can be chosen to preserve zero.

1. Introduction

We use the standard terminology, as in [1]: Let L and K be lattices. If L is a sublattice of K, we call K an *extension* of L; if, in addition, L has a zero and the zero of L is the zero of K, we call K a $\{0\}$-*extension* of L. If K is an extension of L, Θ is a congruence of L, and Φ is a congruence of K, then Φ is an *extension* of Θ to K iff the restriction of Φ to L equals Θ. We call K a *congruence-preserving extension* of L iff every congruence of L has *exactly one* extension to K.

Let φ be an embedding of L into K. If K is a congruence-preserving extension of $L\varphi$, then we call φ a *congruence-preserving embedding* of L into K. If L has a zero, 0, and φ preserves the zero (that is, 0φ is the zero of K), then we call φ a $\{0\}$-*embedding*; we define, similarly, a $\{0,1\}$-*embedding*. We combine these in the obvious way, e.g., *congruence-preserving* $\{0\}$-*embedding*.

We call the lattice L *regular*, if whenever Θ and Φ are congruences of L and Θ and Φ share a congruence class, then $\Theta = \Phi$.

We prove the following three results:

Theorem 1. *Every lattice L has a congruence-preserving embedding into a regular lattice $\tilde{L}$.*

Theorem 2. *Every lattice L with zero has a congruence-preserving $\{0\}$-embedding into a regular lattice $\tilde{L}$ with zero.*

Theorem 3. *Every finite lattice L has a congruence-preserving $\{0\}$-embedding into a finite regular lattice $\tilde{L}$.*

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Theorem 1 for bounded lattices was first announced by the second author at the Szeged meeting on universal algebra and lattice theory in the summer of 1998.

2. Preliminaries

In this section, we introduce a construction of the first author and F. Wehrung [5], which generalizes a lattice construction introduced for *bounded distributive lattices* by the second author [7].

Let L be a lattice. As in [5], we call $\langle x, y, z\rangle \in L^3$ *boolean*, if

$$\begin{aligned} x &= (x \vee y) \wedge (x \vee z),\\ y &= (y \vee x) \wedge (y \vee z),\\ z &= (z \vee x) \wedge (z \vee y). \end{aligned}$$

$\mathbf{M}_3\langle L\rangle \subseteq L^3$ denotes the poset of all boolean triples of L. Boolean triples are *balanced*, that is, they satisfy

$$x \wedge y = y \wedge z = z \wedge x.$$

We summarize some of the results of G. Grätzer and F. Wehrung [5]:

Theorem 4. *Let L be a lattice.*

(i) *For every triple $\langle x, y, z\rangle \in L^3$, there is a* smallest *boolean triple $\overline{\langle x, y, z\rangle} \in L^3$ with $\langle x, y, z\rangle \leq \overline{\langle x, y, z\rangle}$; in fact,*

$$\overline{\langle x, y, z\rangle} = \langle (x \vee y) \wedge (x \vee z), (y \vee x) \wedge (y \vee z), (z \vee x) \wedge (z \vee y)\rangle.$$

(ii) $\mathbf{M}_3\langle L\rangle$ *is a lattice, where the meet is componentwise:*

$$\langle x, y, z\rangle \wedge \langle x', y', z'\rangle = \langle x \wedge x', y \wedge y', z \wedge z'\rangle$$

and the join is defined by

$$\langle x, y, z\rangle \vee \langle x', y', z'\rangle = \overline{\langle x \vee x', y \vee y', z \vee z'\rangle}.$$

(iii) *For a congruence Θ of L, let Θ^3 denote the congruence of L^3 defined componentwise. Let $\mathbf{M}_3\langle\Theta\rangle$ be the restriction of Θ^3 to $\mathbf{M}_3\langle L\rangle$. Then $\mathbf{M}_3\langle\Theta\rangle$ is a congruence of $\mathbf{M}_3\langle L\rangle$, and every congruence of $\mathbf{M}_3\langle L\rangle$ is of the form $\mathbf{M}_3\langle\Theta\rangle$, for a unique congruence Θ of L.*

Choose an arbitrary $a \in L$. Consider the map

$$\varphi_a \colon x \mapsto \langle x, a, x \wedge a\rangle.$$

Lemma 1. *φ_a is a congruence-preserving embedding of L into $\mathbf{M}_3\langle L\rangle$.*

Proof. φ_a is obviously one-to-one and meet-preserving. It also preserves the join because

$$\overline{\langle x \vee y, a, (x \wedge a) \vee (y \wedge a)\rangle} = \langle x \vee y, a, (x \vee y) \wedge a\rangle,$$

for x, $y \in L$. Finally, φ_a is congruence-preserving in view of Theorem 4.(iii). ☐

We need the following form of regularity:

Lemma 2. *The regularity of the lattice L is equivalent to the condition:*

(R) *For all a, b, $c \in L$ with $a < b$, there exist d, $e \in L$ with $d < e$ such that $c \in [d, e]$ and $\Theta(a, b) = \Theta(d, e)$.*

Proof. Let L be regular and let a, b, $c \in L$ with $a < b$. Let $\Theta = \Theta(a, b)$, and let Φ be the smallest congruence collapsing the congruence class $[c]\Theta$. Obviously, $[c]\Theta = [c]\Phi$, so by the definition of regularity, $\Theta = \Phi$. Therefore, $a \equiv b\ (\Phi)$ and, by the definition of Φ, there are $d < e$ in $[c]\Theta$ such that $a \equiv b\ (\Theta(d, e))$. Since $c \in [c]\Theta$, we can choose $d < e$ so that $c \in [d, e]$.

Conversely, assume that (R) holds for L. Let Θ and Φ be congruences of L and let Θ and Φ share a congruence class, that is, let $[c]\Theta = [c]\Phi$, for some $c \in L$. Let Ψ be the smallest congruence under which $[c]\Theta = [c]\Phi$ is a congruence class. Then $[c]\Theta = [c]\Psi$ and, obviously, $\Psi \subseteq \Theta$. Now let $a \equiv b\ (\Theta)$ with $a < b$. By (R), there exist d, $e \in L$ with $d < e$ such that $c \in [d, e]$ and $\Theta(a, b) = \Theta(d, e)$. Since $a \equiv b$ (Θ), it follows that $d \equiv e\ (\Theta)$. Using that $c \in [d, e]$, we conclude that d, $e \in [c]\Theta$. By the definition of Ψ, we get that $d \equiv e\ (\Psi)$, so $\Theta(a, b) = \Theta(d, e)$ implies that $a \equiv b\ (\Psi)$, proving that $\Theta \subseteq \Psi$. Thus $\Theta = \Psi$. Similarly, $\Phi = \Psi$. Therefore, $\Theta = \Phi$, concluding the proof of regularity. ☐

3. General lattices

The crucial observation for general lattices is the following:

Lemma 3. *Let L be a lattice, let a, b, $c \in L$. The element $e = \langle c, b, c \wedge b \rangle \in \mathbf{M}_3\langle L \rangle$ satisfies*

$$\Theta(a\varphi_a, b\varphi_a) = \Theta(c\varphi_a, e).$$

Proof. Obviously, both $\Theta(a\varphi_a, b\varphi_a)$ and $\Theta(c\varphi_a, e)$ equal $\mathbf{M}_3\langle \Theta(a, b) \rangle$, so we get this statement from Theorem 4(iii). ☐

So let a, b, $c \in L$ with $a < b$. Then for $a\varphi_a$, $b\varphi_a$, $c\varphi_a \in \mathbf{M}_3\langle L \rangle$, the inequality $a\varphi_a < b\varphi_a$ holds and (R) is satisfied in $\mathbf{M}_3\langle L \rangle$ for these elements with $d = c\varphi_a$ and e.

To prove Theorem 1, form a transfinite sequence $\langle \langle a_\gamma, b_\gamma, c_\gamma \rangle \mid \gamma < \alpha \rangle$ with the following properties:

(i) α is a limit ordinal.
(ii) $\langle a_\gamma, b_\gamma, c_\gamma \rangle \in L^3$ and $a_\gamma < b_\gamma$, for $\gamma < \alpha$.
(iii) Every $\langle a, b, c \rangle \in L^3$ with $a < b$ occurs as $\langle a_\gamma, b_\gamma, c_\gamma \rangle$, for some $\gamma < \alpha$.

We construct a direct union of lattices as follows:

Let L_0 be $\mathbf{M}_3\langle L \rangle$, with L identified with $L\varphi_{a_0}$.

If $\gamma = \delta + 1$ and L_δ is defined, then let $L_\gamma = \mathbf{M}_3\langle L_\delta \rangle$, where L is identified in $\mathbf{M}_3\langle L_\delta \rangle$ with $L_\delta\varphi_{a_\gamma}$.

If γ is a limit ordinal and L_δ is defined, for all $\delta < \gamma$, then let $L_\gamma = \bigcup(L_\delta \mid \delta < \gamma)$, and L is identified with a sublattice of L_γ in the obvious fashion.

Obviously, L_α has the following properties:

(i) L_α is an extension of L.
(ii) Condition (R) holds in L_α for a, b, $c \in L$ with $a < b$.

Repeat this construction ω times. Obviously, the resulting lattice $\tilde{L}$ satisfies condition (R) and it is therefore regular. By Lemma 1, $L_{\delta+1}$ is a congruence-preserving extension of $\mathbf{M}_3\langle L_\delta \rangle$, for $\delta < \alpha$. It is trivial that a union of congruence-preserving extensions is a congruence-preserving extension again, hence $\tilde{L}$ is a congruence-preserving extension of L. This completes the proof of Theorem 1.

Note that we proved for $\tilde{L}$ a slightly stronger form of (R), which we call (RC) in Section 7.

4. Lattices with zero

Let L be a lattice with zero. To prove Theorem 2, we have to proceed as in Section 3, but we have to construct $\{0\}$-embeddings.

For arbitrary $a \in L$, consider the following principal dual ideal $\mathbf{M}_3\langle L, a\rangle$ of $\mathbf{M}_3\langle L\rangle$:

$$\mathbf{M}_3\langle L, a\rangle = [\langle 0, a, 0\rangle) \subseteq \mathbf{M}_3\langle L\rangle.$$

Theorem 4 (i) and (ii) obviously hold for $\mathbf{M}_3\langle L, a\rangle$ and φ_a (defined in Section 2) maps L into $\mathbf{M}_3\langle L, a\rangle$; in this section, we regard φ_a as a map of L into $\mathbf{M}_3\langle L, a\rangle$.

Lemma 4. *Let* $\langle x, y, z\rangle \in \mathbf{M}_3\langle L, a\rangle$. *Then* $a \leq y$ *and*

(1) $$x \wedge a = z \wedge a.$$

Proof. Since $\langle x, y, z\rangle$ is boolean, it is balanced, so $x \wedge y = z \wedge y$. Therefore, $x \wedge a = (x \wedge y) \wedge a = (z \wedge y) \wedge a = z \wedge a$, as claimed. □

We need an easy decomposition statement for the elements of $\mathbf{M}_3\langle L, a\rangle$. Let us use the notation

$$\begin{aligned} B &= \{\, \langle x, a, x \wedge a\rangle \mid x \in L \,\} \ (= L\varphi_a),\\ K &= \{\, \langle 0, x, 0\rangle \mid x \in L,\ x \geq a \,\},\\ J &= \{\, \langle x \wedge a, a, x\rangle \mid x \in L \,\}. \end{aligned}$$

Lemma 5. *Let* $\mathbf{v} = \langle x, y, z\rangle \in \mathbf{M}_3\langle L, a\rangle$. *Choose an arbitrary upper bound* i *of* $\{x, y, z\}$ *in* L. *Then* $\mathbf{v}$ *has a decomposition in* $\mathbf{M}_3\langle L, a\rangle$*:*

(2) $$\mathbf{v} = \mathbf{v}_B \vee \mathbf{v}_K \vee \mathbf{v}_J,$$

where

(3) $$\mathbf{v}_B = \langle x, y, z\rangle \wedge \langle i, a, a\rangle = \langle x, a, x \wedge a\rangle \in B,$$

(4) $$\mathbf{v}_K = \langle x, y, z\rangle \wedge \langle 0, i, 0\rangle = \langle 0, y, 0\rangle \qquad \in K,$$

(5) $$\mathbf{v}_J = \langle x, y, z\rangle \wedge \langle a, a, i\rangle = \langle z \wedge a, a, z\rangle \in J.$$

Proof. (3) follows from (1). By symmetry, (5) follows, and (4) is trivial. Finally, the right side of (2) componentwise joins into the left side in view of (1). □

Note that $\mathbf{v}_B$, $\mathbf{v}_K$, and $\mathbf{v}_J$ do not depend on i.

In terms of this decomposition, we can describe the congruences of $\mathbf{M}_3\langle L, a\rangle$.

Lemma 6. *Let* Φ *be a congruence of* $\mathbf{M}_3\langle \Theta, a\rangle$ *and let* $\mathbf{v}$, $\mathbf{w} \in \mathbf{M}_3\langle L, a\rangle$. *Then*

(6) $$\mathbf{v} \equiv \mathbf{w} \quad (\Phi),$$

iff

(7) $$\mathbf{v}_B \equiv \mathbf{w}_B \quad (\Phi),$$

(8) $$\mathbf{v}_K \equiv \mathbf{w}_K \quad (\Phi),$$

(9) $$\mathbf{v}_J \equiv \mathbf{w}_J \quad (\Phi).$$

Proof. (6) implies (7) by (3). Similarly, for (8) and (9).

Conversely, (7)–(9) imply (6) by (2). □

Now we can prove that Theorem 4 (iii) holds for $\mathbf{M}_3\langle L, a\rangle$.

Lemma 7. *For a congruence Θ of L, let $\mathbf{M}_3\langle \Theta, a\rangle$ be the restriction of Θ^3 to $\mathbf{M}_3\langle L, a\rangle$. Then $\mathbf{M}_3\langle \Theta, a\rangle$ is a congruence of $\mathbf{M}_3\langle L, a\rangle$, and every congruence of $\mathbf{M}_3\langle L, a\rangle$ is of the form $\mathbf{M}_3\langle \Theta, a\rangle$, for a unique congruence Θ of L.*

Proof. It follows from Theorem 4 that $\mathbf{M}_3\langle \Theta, a\rangle$ is a congruence of $\mathbf{M}_3\langle L, a\rangle$. Let $\mathbf{v} = \langle x, y, z\rangle$, $\mathbf{v}' = \langle x', y', z'\rangle \in \mathbf{M}_3\langle L, a\rangle$ and choose an $i \in L$ that is an upper bound for the set $\{x, y, z, x', y', z'\}$.

Let Φ be a congruence of $\mathbf{M}_3\langle L, a\rangle$, and let Θ be the restriction of Φ to L with respect to the embedding φ_a. By Lemma 6,

$$\mathbf{v} \equiv \mathbf{v}' \quad (\Phi),$$

iff (7)–(9) hold. Note that $\mathbf{v}_B$, $\mathbf{v}'_B \in L\varphi_a$, so (7) is equivalent to $\mathbf{v}_B \equiv \mathbf{v}'_B$ (Θ).

Now consider

$$p(\mathbf{x}) = (\mathbf{x} \vee \langle 0, i, 0\rangle) \wedge \langle i, a, a\rangle.$$

Then $p(\langle x \wedge a, a, x\rangle) = \langle x, i, x\rangle \wedge \langle i, a, a\rangle = \langle x, a, x \wedge a\rangle$. So (9) implies that $p(\mathbf{v}_J) \equiv p(\mathbf{v}'_J)$ (Φ), and symmetrically. Thus, (9) is equivalent to

$$p(\mathbf{v}_J) \equiv p(\mathbf{v}'_J) \quad (\Phi)$$

that is, to

$$p(\mathbf{v}_J) \equiv p(\mathbf{v}'_J) \quad (\Theta),$$

since $p(\mathbf{v}_J)$, $p(\mathbf{v}'_J) \in L\varphi_a$.

Now consider

$$q(\mathbf{x}) = (\mathbf{x} \vee \langle a, a, i\rangle) \wedge \langle i, a, a\rangle.$$

Then, for $x \geq a$, $q(\langle 0, x, 0\rangle) = \langle x, a, x \wedge a\rangle$. So $q(\mathbf{v}_K) \equiv q(\mathbf{v}'_K)$ (Φ), that is, $q(\mathbf{v}_K) \equiv q(\mathbf{v}'_K)$ (Θ).

Finally, define

$$r(\mathbf{x}) = (\mathbf{x} \vee \langle a, a, i\rangle) \wedge \langle 0, i, 0\rangle.$$

Then $q(\langle x, x \wedge a, a\rangle) = \langle 0, x, 0\rangle$. So $q(\mathbf{v}_B) \equiv q(\mathbf{v}'_B)$ (Φ). From these it follows that $\mathbf{v}_K \equiv \mathbf{v}'_K$ (Φ) is equivalent to $q(\mathbf{v}_K) \equiv q(\mathbf{v}'_K)$ (Φ) and $q(\mathbf{v}_K)$, $q(\mathbf{v}'_K) \in L\varphi_a$, so the latter is equivalent to $q(\mathbf{v}_K) \equiv q(\mathbf{v}'_K)$ (Θ).

We conclude that the congruence $\mathbf{v} \equiv \mathbf{v}'$ (Φ) in $\mathbf{M}_3\langle L, a\rangle$ is equivalent to the following three congruences in L (which we consider identified with $L\varphi_a$ by φ_a):

$$\begin{aligned} \mathbf{v}_B &\equiv \mathbf{v}'_B \quad (\Theta), \\ p(\mathbf{v}_J) &\equiv p(\mathbf{v}'_J) \quad (\Theta), \\ q(\mathbf{v}_K) &\equiv q(\mathbf{v}'_K) \quad (\Theta). \end{aligned}$$

□

Now to prove Theorem 2, we proceed as in the proof of Theorem 1, except that in the γ-th step we use $\mathbf{M}_3\langle L, a_\gamma\rangle$ instead of $\mathbf{M}_3\langle L\rangle$ and observe that φ_{a_γ} is a $\{0\}$-embedding and that the element e used in Lemma 3, in fact, belongs to $\mathbf{M}_3\langle L, a_\gamma\rangle$.

5. Finite lattices

To prove Theorem 3, we observe first

Lemma 8. *Every sectionally complemented lattice is regular.*

Proof. Let L be a sectionally complemented lattice, a, b, $c \in L$, $a < b$. We have to prove that there exist $e < f$ such that $c \in [e, f]$ and $\Theta(a, b) = \Theta(e, f)$.

Let u be a sectional complement of a in b. Let v be a sectional complement of $c \wedge u$ in c. Then $u \wedge v = 0$, so $\Theta(v, u \vee v) = \Theta(0, u) = \Theta(a, b)$ and $c \in [v, u \vee v]$, so $e = v$, $f = u \vee v$ satisfy condition (R). □

The authors proved in [3] that every finite lattice has a congruence-preserving embedding into a finite sectionally complemented lattice; in fact, the embedding constructed preserves the zero. So this result combined with Lemma 8 proves Theorem 3.

Note that not all finite atomistic lattices are regular; for instance, the seven-element lattice in Figure 1 is atomistic but not regular.

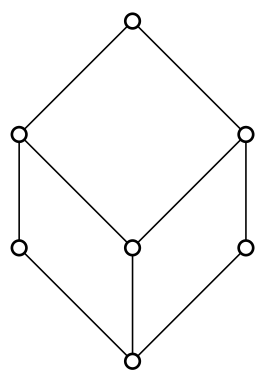

Figure 1. A seven-element lattice.

6. Compact congruences

The compact congruence relations of a regular lattice have a nice property:

Lemma 9. *Every compact congruence of a regular lattice L is principal.*

Proof. It is sufficient to prove that in a regular lattice the join of two principal congruences is again principal. So let L be a regular lattice and let a, b, c, $d \in L$, $a < b$, $c < d$. By condition (R), there are e, $f \in L$ with $e < f$ such that $c \in [e, f]$ and $\Theta(a, b) = \Theta(e, f)$. Then $\Theta(a, b) \vee \Theta(c, d) = \Theta(e, f \vee d)$.

Theorem 1 and Lemma 9 yield: □

Corollary. *Every lattice L has a congruence-preserving embedding into a lattice $\widetilde{L}$ in which every compact congruence is principal.*

This raises the question: what can we say about the compact congruences of $\mathbf{M}_3\langle L\rangle$? To be more precise, let κ be an arbitrary element of L and let $\mathbf{M}_{3,\varphi_\kappa}\langle L\rangle$ denote $\mathbf{M}_3\langle L\rangle$ in which L is identified with $L\varphi_\kappa$. So $\mathbf{M}_{3,\varphi_\kappa}\langle L\rangle$ is a congruence-preserving extension of L.

We prove that if Θ is a congruence of L and Θ is the join of two principal congruences, then the extension of Θ to $\mathbf{M}_{3,\varphi_\kappa}\langle L\rangle$ is principal:

Lemma 10. *In the lattice* $\mathbf{M}_{3,\varphi_\kappa}\langle L\rangle$,

$$\begin{aligned}(10)\qquad &\Theta(\langle a_1,\kappa,a_1\wedge\kappa\rangle,\langle b_1,\kappa,b_1\wedge\kappa\rangle)\vee\Theta(\langle a_2,\kappa,a_2\wedge\kappa\rangle,\langle b_2,\kappa,b_2\wedge\kappa\rangle)\\ &\qquad=\Theta(\langle a_1,a_2,a_1\wedge a_2\rangle,\langle b_1,b_2,b_1\wedge b_2\rangle).\end{aligned}$$

for all a_1, a_2, b_1, $b_2\in L$.

Proof. It follows immediately from Theorem 4 that, for any $\langle a_1,a_2,a_3\rangle$, $\langle b_1,b_2,b_3\rangle\in\mathbf{M}_{3,\varphi_\kappa}\langle L\rangle$,

$$\Theta_{\mathbf{M}_{3,\varphi_\kappa}\langle L\rangle}(\langle a_1,a_2,a_3\rangle,\langle b_1,b_2,b_3\rangle)=\mathbf{M}_3\langle\Theta_L(a_1,b_1)\vee\Theta_L(a_2,b_2)\vee\Theta_L(a_3,b_3)\rangle.$$

Therefore,

$$\begin{aligned}\Theta(\langle a_1,\kappa,a_1\wedge\kappa\rangle,&\langle b_1,\kappa,b_1\wedge\kappa\rangle)\vee\Theta(\langle a_2,\kappa,a_2\wedge\kappa\rangle,\langle b_2,\kappa,b_2\wedge\kappa\rangle)\\ &=\mathbf{M}_3\langle\Theta_L(a_1,b_1)\rangle\vee\mathbf{M}_3\langle\Theta_L(a_2,b_2)\rangle\\ &=\mathbf{M}_3\langle\Theta_L(a_1,b_1)\vee\Theta_L(a_2,b_2)\rangle\\ &=\Theta(\langle a_1,a_2,a_1\wedge a_2\rangle,\langle b_1,b_2,b_1\wedge b_2\rangle).\end{aligned}$$ ☐

It is now clear that Lemma 10 allows us to prove that every lattice has a congruence-preserving extension into a lattice in which every compact congruence is principal. Indeed, let $L_0=L$, and define L_i, $i<\omega$ inductively as follows: $L_{i+1}=\mathbf{M}_{3,\varphi_{\kappa_i}}\langle L_i\rangle$, where κ_i is an arbitrary element of L_i. Then, for every i, the lattice L_{i+1} is a congruence-preserving extension of L_i, so the direct limit $\bar{L}$ of these lattices is a congruence-preserving extension of L. It follows Lemma 10 that in $\bar{L}$ every compact congruence is principal.

This proof is marginally simpler than deriving this fact from Theorem 1 and Lemma 9. It is not clear whether $\bar{L}$ is regular.

7. Discussion

The most important class of regular lattices is the class of relatively complemented lattices. It is natural to ask: which lattices have a congruence-preserving embedding into a relatively complemented lattice? M. Ploščica, J. Tůma, and F. Wehrung [6] proved that not every lattice admits such an embedding. G. Grätzer, H. Lakser, and F. Wehrung [2] proved that if the congruence lattice is finite, then there is such an embedding. A very recent result of F. Wehrung [8] states that if the semilattice of all compact congruences is a lattice, then, again, there is such an embedding.

Let us call the lattice L *strongly regular*, if the following condition holds:

(SR) For all a, b, $c\in L$ with $a<b$, there exist d, $e\in L$ with $d<e$ such that $c\in[d,e]$ and $[a,b]$, $[d,e]$ are projective intervals of K.

It is easy to see that every relatively complemented lattice is strongly regular. Indeed, let L be a relatively complemented lattice. Let a, b, $c\in L$ with $a<b$. Let u be a relative complement of a in $[a\wedge c,b]$, let e be a relative complement of $c\wedge u$ in $[a\wedge c,c]$, and let $f=c\vee u$. Then it is easy to see that $[a,b]$ and $[e,f]$ are projective intervals and $c\in[e,f]$.

As we noted it in Section 3, (SR) is satisfied in the lattice $\tilde{L}$ constructed in this paper.

The lattice $\tilde{L}$ has another interesting property which is related to relative complementedness.

(RC) For $a,\ b,\ c \in L,\ a \leq b \leq c$, there exist $a_1,\ b_1,\ c_1 \in L$ with $a_1 \leq b_1 \leq c_1$ such that $[a, b]$, $[a_1, b_1]$ and $[b, c]$, $[b_1, c_1]$ are projective intervals and b_1 has a relative complement d_1 in the interval $[a_1, c_1]$.

Consider the following elements of $\mathbf{M}_3\langle L\rangle$: $a_1 = \langle a, b, a\rangle$, $b_1 = \langle b, b, b\rangle$, $c_1 = \langle b, c, b\rangle$, $d_1 = \langle a, c, a\rangle$. Then $[a\varphi_\kappa, b\varphi_\kappa]$, $[a_1, b_1]$ and $[b\varphi_\kappa, c\varphi_\kappa]$, $[b_1, c_1]$ are projective pairs of intervals, for any $\kappa \in L$.

8. Problems

Unquestionably, the most interesting open problem is the following:

Problem 1. Does every bounded lattice L have a congruence-preserving $\{0, 1\}$-embedding into a bounded regular lattice $\tilde{L}$?

Even the finite case is open:

Problem 2. Does every finite lattice L have a congruence-preserving $\{0, 1\}$-embedding into a finite regular lattice $\tilde{L}$?

In [3], the authors prove that every finite lattice has a congruence-preserving embedding into a finite sectionally complemented lattice. The embedding, in fact, preserves the zero.

Problem 3. Does every finite lattice L have a congruence-preserving $\{0, 1\}$-embedding into a finite sectionally complemented lattice?

In [4], the authors prove that every finite lattice K has a congruence-preserving embedding into a finite semimodular lattice L.

Problem 4. Does every finite lattice L have a congruence-preserving $\{0, 1\}$-embedding into a finite semimodular lattice?

Let us repeat the problem from Section 6:

Problem 5. Is the lattice $\bar{L}$ regular?

We can also consider stronger versions of regularity. Let us call a lattice L *homogeneous-regular*, if L is regular and for every congruence relation Θ of L, any two Θ classes are isomorphic.

Problem 6. Can one prove Theorems 1–3 for congruence-preserving embeddings into homogeneous-regular lattices?

Problem 7. Can one solve Problems 1 and 2 for congruence-preserving embeddings into homogeneous-regular lattices?

Let L be a homogeneous-regular lattice, let Θ be a congruence of L, and let $a \in L$. Then the assignment $\Theta \mapsto [a]\Theta$ does not depend on a, so, up to isomorphism, we assign to a congruence Θ a lattice $[a]\Theta$. If the lattice $[a]\Theta$ determines the congruence Θ, we call L *very regular*. In other words, in such a lattice, a congruence is determined by any one of its congruence classes, and the congruence classes of different congruences are nor isomorphic.

Problem 8. Can one prove Theorems 1–3 for congruence-preserving embeddings into very regular lattices?

Problem 9. Can one solve Problems 1 and 2 for congruence-preserving embeddings into very regular lattices?

We now state a more modest pair of problems for stronger versions of regularity:

Problem 10. Can every finite distributive lattice be represented as the congruence lattice of a homogeneous-regular (resp., very regular) finite lattice?

Finally, a somewhat technical problem. We can rewrite Lemma 3 as follows:

Lemma 11. *Let L be a lattice, and let $a \in L$. Then there is a congruence-preserving embedding $\varphi_a \colon L \to \mathbf{M}_3\langle L\rangle$ such that for all b, $c \in L$ with $a < b$ there are d, $e \in \mathbf{M}_3\langle L\rangle$ with*

$$d \leq c\varphi_a \leq e$$

and with

$$\Theta(d, e) = \Theta(a\varphi_a, b\varphi_a).$$

(We actually show that one can always take $d = c\varphi_a$.)

Is there a "better" embedding, that is, an embedding of L into $\mathbf{M}_3\langle L\rangle$ that accomplishes what φ_a does but for all triples of elements?

Problem 11. Let L be a lattice. Is it true that there is a congruence-preserving embedding $\varphi \colon L \to \mathbf{M}_3\langle L\rangle$ such that, for all a, b, $c \in L$ with $a < b$, there are d, $e \in \mathbf{M}_3\langle L\rangle$ with $d \leq c\varphi \leq e$ and $\Theta(d, e) = \Theta(a\varphi, b\varphi)$? [1]

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[1] This problem was solved by H. Lakser in the negative.