REPRESENTATIONS OF JOIN-HOMOMORPHISMS OF DISTRIBUTIVE LATTICES WITH DOUBLY 2-DISTRIBUTIVE LATTICES

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To the memory of András Huhn, 1947–1985.
We remember you.

Abstract. In the early eighties, A. Huhn proved that if D, E are finite distributive lattices and $\psi\colon D \to E$ is a $\{0\}$-preserving join-embedding, then there are finite lattices K, L and there is a lattice homomorphism $\varphi\colon K \to L$ such that $\operatorname{Con} K$ (the congruence lattice of K) is isomorphic to D, $\operatorname{Con} L$ (the congruence lattice of L) is isomorphic to E, and the natural induced mapping $\operatorname{ext}\varphi\colon \operatorname{Con} K \to \operatorname{Con} L$ represents ψ. The present authors with H. Lakser generalized this result to an arbitrary $\{0\}$-preserving join-homomorphism ψ.

It was also A. Huhn who introduced the 2-*distributive identity*:

$$x \wedge (y_1 \vee y_2 \vee y_3) = (x \wedge (y_1 \vee y_2)) \vee (x \wedge (y_1 \vee y_3)) \vee (x \wedge (y_2 \vee y_3)).$$

We shall call a lattice *doubly* 2-*distributive*, if it satisfies the 2-distributive identity and its dual.

In this note, we prove that *the lattices K and L in the above result can be constructed as doubly 2-distributive lattices.*

1. Introduction

The congruence lattice, $\operatorname{Con} L$, of a finite lattice L is a finite distributive lattice (N. Funayama and T. Nakayama [1]). The converse is a result of R. P. Dilworth, first published in G. Grätzer and E. T. Schmidt [6].

Many paper have been published making L planar, "small", modular (for countable L), and so on. See Appendix C in [2] for a review.

Recent publications consider *simultaneous representations*. Let K, L be lattices and let φ be a homomorphism of K into L. Then φ induces a map $\operatorname{ext}\varphi$ of $\operatorname{Con} K$ into $\operatorname{Con} L$: for a congruence relation Θ of K, let the image Θ under $\operatorname{ext}\varphi$ be the congruence relation of L generated by the set $\Theta\varphi = \{\, \langle a\varphi, b\varphi\rangle \mid a \equiv b \ (\Theta)\,\}$; obviously, φ is a $\{0, \vee\}$-homomorphism of $\operatorname{Con} K$ into $\operatorname{Con} L$. *The simultaneous representation problem* asks when a $\{0, \vee\}$-homomorphism between finite distributive lattices can be so represented.

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The following result was proved by A. P. Huhn in [9] in the special case when ψ is an embedding and was proved for arbitrary ψ in G. Grätzer, H. Lakser, and E. T. Schmidt [5]:

Theorem 1. *Let D and E be finite distributive lattices, and let*

$$\psi\colon D \to E$$

be a $\{0, \vee\}$-homomorphism. Then there are finite lattices K and L, a lattice homomorphism $\varphi\colon K \to L$, and isomorphisms

$$\alpha\colon D \to \operatorname{Con} K,$$
$$\beta\colon E \to \operatorname{Con} L$$

with

$$\psi\beta = \alpha(\operatorname{ext}\varphi).$$

Furthermore, φ is an embedding iff ψ separates 0.

Theorem 1 concludes that the following diagram is commutative:

$$\begin{array}{ccc} D & \xrightarrow{\psi} & E \\ \cong\big\downarrow\alpha & & \cong\big\downarrow\beta \\ \operatorname{Con} K & \xrightarrow{\operatorname{ext}\varphi} & \operatorname{Con} L \end{array}$$

See G. Grätzer, H. Lakser, and E. T. Schmidt [4] for related results.

A. Huhn introduced n-distributivity in [8]. Let $n \geq 1$ be an integer. A lattice L is *n-distributive*, if for all x, $y_1, \ldots, y_{n+1} \in L$,

$$x \wedge (\bigvee_{i=1}^{n+1} y_i) = \bigvee_{i=1}^{n+1} (x \wedge (\bigvee_{\substack{j=1\\ j\neq i}}^{n+1} y_j)).$$

In particular, a lattice L is 1-*distributive* iff it is distributive and 2-*distributive* iff it satisfies the identity:

(2D) $$x \wedge (y_1 \vee y_2 \vee y_3) = (x \wedge (y_1 \vee y_2)) \vee (x \wedge (y_1 \vee y_3)) \vee (x \wedge (y_2 \vee y_3)).$$

We shall call a lattice L *doubly* 2-*distributive*, if it satisfies the 2-distributive identity and its dual. For instance, N_5 and M_3 are doubly 2-distributive lattices.

Now we can state our main result:

Theorem 2. *In Theorem 1, K and L can be constructed as finite, doubly 2-distributive lattices.*

The background for this paper is briefly presented in Sections 2 and 3. The construction of the lattice L is presented in Section 4 in six easy steps. We verify in Section 5 that L is 2-distributive and in Section 6 that L is dually 2-distributive. The crucial step is Lemma 9, proving that under very special circumstances we can glue 2-distributive lattices and obtain a 2-distributive lattice. The concluding Section 7 provides some discussion.

2. Multi-coloring

The construction that leads to the proof of Theorem 2 is based on an extension lemma first proved in [4]. We state this lemma in this section. But first some concepts.

Let M be a finite lattice and let $\mathfrak{C}$ be a finite set; the elements of $\mathfrak{C}$ will be called *colors.* A *coloring* μ of M over $\mathfrak{C}$ is a map

$$\mu\colon \mathfrak{P}(M) \to \mathfrak{C}$$

of the set of prime intervals $\mathfrak{P}(M)$ of M into $\mathfrak{C}$ satisfying the condition: if two prime intervals generate the same congruence relation of M, then they have the same color; that is,

$$\mathfrak{p},\ \mathfrak{q} \in \mathfrak{P}(M) \text{ and } \Theta(\mathfrak{p}) = \Theta(\mathfrak{q}) \quad \text{imply that} \quad \mathfrak{p}\mu = \mathfrak{q}\mu.$$

Since the join-irreducible congruences of M are exactly those that can be generated by prime intervals, equivalently, μ can be regarded as a map of the set $\mathrm{J}(\mathrm{Con}\, M)$ of join-irreducible congruences of M into $\mathfrak{C}$:

$$\mu\colon \mathrm{J}(\mathrm{Con}\, M) \to \mathfrak{C}.$$

A *multi-coloring* over $\mathfrak{C}$ is an *isotone map* μ from $\mathfrak{P}(M)$ into $\mathrm{P}^+(\mathfrak{C})$ (the set of all nonempty subsets of $\mathfrak{C}$); *isotone* means that if $\mathfrak{p}$, $\mathfrak{q} \in \mathfrak{P}(M)$ and $\Theta(\mathfrak{p}) \leq \Theta(\mathfrak{q})$, then $\mathfrak{p}\mu \subseteq \mathfrak{q}\mu$. Equivalently, a multi-coloring is an isotone map of the poset $\mathrm{J}(\mathrm{Con}\, M)$ into the poset $\mathrm{P}^+(\mathfrak{C})$.

The extension lemma states that a multi-colored lattice has a natural extension to a colored lattice:

Lemma 1. *Let M be a finite lattice with a multi-coloring μ over the set $\mathfrak{C}$. Then there exist a lattice M^* with a coloring μ^* over $\mathfrak{C}$ such that the following conditions holds:*

(i) *M^* is the colored direct product of the lattices M_c, $c \in \mathfrak{C}$, where M_c is a homomorphic image of M colored by $\{c\}$.*
(ii) *There is a lattice embedding $a \mapsto a^*$ of M into M^*.*
(iii) *For every prime interval $\mathfrak{p} = [a, b]$ of M,*

$$\mathfrak{p}\mu = \{\, \mathfrak{q}\mu^* \mid \mathfrak{q} \in \mathfrak{P}(M^*) \text{ and } \mathfrak{q} \subseteq [a^*, b^*] \,\}$$

and the minimal extension of $\Theta(\mathfrak{p})$ under this embedding of M into M^* is of the form

$$\prod(\Theta(\mathfrak{p}_c) \mid c \in \mathfrak{C}),$$

where
(a) *$\mathfrak{p}_c$ is a prime interval of M_c iff $c \in \mathfrak{p}\mu$;*
(b) *$\mathfrak{p}_c$ is a trivial interval otherwise (in which case, $\Theta(\mathfrak{p}_c) = \omega_{M_c}$).*

3. The planar construction

For a finite distributive lattice D, the authors and H. Lakser constructed in [3] a finite planar lattice K with $\mathrm{Con}\, K \cong D$. To construct K, we take two finite chains, G_0 and G_1 (whose length depends from the number of the join-irreducible elements of D) colored by $\mathfrak{C} = \mathrm{J}(K)$; we assume that the coloring of G_0: $\mu_0\colon \mathfrak{P}(G_0) \to \mathfrak{C}$ is an onto map. We form the colored direct product, $G = G_0 \times G_1$; we call G the *grid.* We adjoin some elements to this grid:

(i) for some prime intervals $[a, b]$ of G, we adjoin the new element $n(a, b)$ such that $a \prec n(a, b) \prec b$ in the extended lattice;
(ii) for some prime squares $[a, b]$ of G (that is, $[a, b]$ is isomorphic to C_2^2), we adjoin the new element $m(a, b)$ such that $[a, b]$ extends to an M_3.

K is constructed as such an augmented grid; for the details, see [3].

The following property of this construction is important in this paper.

Lemma 2. *Let K be the planar lattice constructed for D. Let Θ be a congruence of K and let $d \in D$ be the element corresponding to Θ under the isomorphism $\operatorname{Con} K \cong D$. Then $K_1 = K/\Theta$ is isomorphic to the lattice constructed for the distributive lattice $D_1 = [d, 1] \subseteq D$.*

Another obvious property of this construction is the following:

Lemma 3. *Let K be the planar lattice constructed for D. Let $\overline{K}$ be the lattice we obtain from K by adjoining a new element $m(a, b)$ for all prime squares $[a, b]$ of K so that $[a, b]$ extends in $\overline{K}$ to an M_3. Then $\overline{K}$ is a finite, planar, simple lattice.*

4. The 2-distributive construction

In this section, we construct the lattice L for Theorem 2 and verify its congruence properties.

Let D, E be finite distributive lattices and let

$$\psi \colon D \to E$$

be a $\{0, \vee\}$-homomorphism.

In [3], we have observed that we can assume, without loss of generality, that ψ *separates* 0.

Step 1: the lattice K. We represent D as the congruence lattice of a planar lattice K as constructed in [3], see Section 3. We identify D with $\operatorname{Con} K$, so we view ψ a $\{0, \vee\}$-homomorphism of $\operatorname{Con} K$ into E.

Step 2: the lattice L_0. We define a map μ of $\mathfrak{P}(K)$ to subsets of $\mathrm{J}(E)$:

$$\mathfrak{p}\mu = \mathrm{J}(E) \cap (\Theta(\mathfrak{p})\psi].$$

μ is obviously isotone. ψ separates 0, so $\mathfrak{p}\mu \neq \varnothing$. Therefore, μ is a multi-coloring of K over $\mathrm{J}(E)$. We apply Lemma 1 to obtain the lattice

$$K^* = \prod(K_c \mid c \in \mathrm{J}(E)).$$

K_c is a homomorphic image of K, so by Lemma 2, K_c is also a lattice of the type described in Section 3; in particular, K_c is planar.

By Lemma 3, we can extend K_c to a finite, planar, simple lattice $\overline{K}_c$ of color $\{c\}$. Define

$$L_0 = \prod(\overline{K}_c \mid c \in \mathrm{J}(E)).$$

Since L_0 is a direct product of simple lattices, it follows that $\mathrm{J}(\operatorname{Con} L_0)$ is unordered; the congruence lattice of L_0 is a Boolean lattice with $|\mathrm{J}(E)|$ atoms. K is a sublattice of K^* and K^* is a sublattice of L_0 (under $x \mapsto x^*$), so we obtain an embedding $\varphi \colon K \to L_0$.

Let p_c be an arbitrary atom of the direct component $\overline{K}_c$; then the prime interval $[0, p_c]$ of L_0 has color c. The atoms p_c, $c \in \mathrm{J}(E)$, generate an ideal B_0 of L_0 that is a Boolean lattice with the following two properties:

(i) for any two distinct atoms, p and q, the prime intervals $[0,p]$ and $[0,q]$ have distinct colors;
(ii) every color $c \in \mathrm{J}(E)$ is the color of some prime intervals $[0,p]$.

Step 3: the lattice L_1. We represent E as the congruence lattice of a finite planar lattice L_1 as in [3], see Section 3. Then $\mathrm{Con}\, L_1$, is isomorphic to E. The grid H of E is the direct product of chains H_0 and H_1 colored by $\mathrm{J}(E)$. The grid $H = H_0 \times H_1$ inherits the coloring.

Step 4: the lattice M. Let n be the cardinality of $\mathrm{J}(E)$, and let B a Boolean lattice with 2^n elements; color this Boolean lattice by $\mathrm{J}(E)$ such that conditions (i) and (ii) above hold.

Take the colored direct product $B \times H_0$. The elements $\langle b, 0\rangle$, $b \in B$, form an ideal isomorphic to B; we identify B with this ideal. Similarly, H_0 is identified with the ideal $\{\langle 0, x\rangle \mid x \in H_0\}$. In $B \times H_0$, a prime square is of the form $\mathfrak{p} \times \mathfrak{q}$, where $\mathfrak{p}$ is a prime interval in B and $\mathfrak{q}$ is a prime interval in H_0. We call the prime square $\mathfrak{p} \times \mathfrak{q}$ *monochromatic*, if $\mathfrak{p}$ in B and $\mathfrak{q}$ in H_0 have the same color.

Represent B as a sublattice of M_3^n and H_0 as a sublattice of some M_3^k; then $B \times H_0$ is a sublattice of M_3^{n+k}. Let M denote the sublattice of M_3^{n+k} containing $B \times H_0$ with the property that, for every monochromatic prime square in $B \times H_0$, the corresponding sublattice of M is an M_3. M is a modular lattice in which $B \times \mathfrak{p}$ is isomorphic to $C_2^{n-1} \times M_3$, for any prime interval $\mathfrak{p}$ of H_0. The dual ideal H_0' of M generated by $\langle 1, 0\rangle$ is isomorphic to H_0.

Step 5: the lattice L_2. Glue together L_1 and M by identifying the ideal H_0 of L_1 with the dual ideal H_0' of M. Denote by L_2 the lattice we obtain. $\mathrm{Con}\, L_2$ is isomorphic to E and L_2 contains an ideal which is a Boolean lattice colored by the different colors.

We consider on L_2 the natural coloring over $\mathrm{J}(E)$ (a prime interval $\mathfrak{p}$ is colored by $\Theta(\mathfrak{p})\beta_1^{-1} \in \mathrm{J}(E)$). Note that L_0 and L_2 are colored over the same set, namely, $\mathrm{J}(E)$. Then B is an ideal of $B \times H_0$ and $B \times H_0$ is an ideal of L_2, consequently, B is an ideal of L_2. This ideal B is a Boolean lattice satisfying the conditions (i) and (ii).

Step 6: the lattice L. Now we have the lattice L_0 with the ideal B_0 and L_2 with an ideal B. Note that B_0 and B are isomorphic finite Boolean lattices with the same coloring. Take the dual of L_2, denote it by L_3. In L_3, the ideal B corresponds to a dual ideal B'. Again, note that B_0 and B' are isomorphic finite Boolean lattices with the same coloring. Glue together L_0 and L_3 by a color preserving identification of B_0 and B'. The resulting lattice is L. The prime intervals of L are colored by $\mathrm{J}(E)$, and we have the isomorphism $\beta\colon E \to \mathrm{Con}\, L$. Since L_0 is a sublattice of L, we may view φ as an embedding of K into L.

This completes the lattice constructions. As in [3], we have to verify that $\mathrm{ext}\, \varphi = \psi\beta$. It is enough to prove that $\Theta(\mathrm{ext}\, \varphi) = \Theta\psi\beta$, for join-irreducible congruences Θ in K.

So let $\Theta = \Theta(\mathfrak{p})$, where $\mathfrak{p} = [a, b]$ is a prime interval of K. By Lemma 1, $\Theta(\mathfrak{p})\, \mathrm{ext}\, \varphi = \Theta(a^*, b^*)$ collapses in K^* the prime intervals of color $\leq \Theta\psi$; the same holds in L_0 and in L.

Computing $\Theta\psi\beta$ we get the same result, hence $\Theta(\mathrm{ext}\, \varphi) = \Theta\psi\beta$, completing the proof.

5. Verifying the 2-distributive identity

In this section, we verify that L is 2-distributive. We start with some general statements about 2-distributivity.

Lemma 4. *If the* 2*-distributive identity fails in the lattice* L *with* x *and* y_1, y_2, y_3, *then both* $\{y_1, y_2, y_3\}$ *and* $\{y_1 \vee y_2, y_1 \vee y_3, y_2 \vee y_3\}$ *are antichains; therefore,* $\{y_1 \vee y_2, y_1 \vee y_3, y_2 \vee y_3\}$ *generates an eight-element Boolean sublattice.*

Proof. This is obvious. If, say, $y_1 \leq y_2$ or $y_1 \vee y_2 \leq y_1 \vee y_3$, then the right side of (2D) equals the left side, trivially. The last statement is Lemma I.5.9 of [2]. ☐

Corollary 5. *A planar lattice is* 2*-distributive.*

Proof. By Lemma 4, if it was not, then it would contain an eight-element Boolean lattice, a contradiction. ☐

Lemma 6. *If the* 2*-distributive identity fails in the lattice* L, *then* L *contains an element* $a \in L$ *and a sublattice* B *that is an eight-element Boolean lattice with dual atoms (in* B*)* d_1, d_2, d_3 *and unit element* u_B, *satisfying*

(1) $$a < u_B,$$

(2) $$a \nleq d_i, \quad i = 1,\ 2,\ 3,$$

(3) $$(a \wedge d_1) \vee (a \wedge d_2) \vee (a \wedge d_2) < a.$$

Proof. If the 2-distributive identity fails with x and y_1, y_2, y_3, then set $d_1 = y_1 \vee y_2$, $d_2 = y_1 \vee y_3$, $d_3 = y_2 \vee y_3$. By Lemma 4, d_1, d_2, d_3 generate an eight-element Boolean lattice B with dual atoms d_1, d_2, d_3. Set $u_B = d_1 \vee d_2$ $(= d_1 \vee d_3 = d_2 \vee d_3)$ and $a = x \wedge u_B$. The statement of the lemma is now clear. ☐

The following technical lemma will allow us to dispense with some trivial cases in the computations of this section and the next.

Lemma 7. *Let* K *be a lattice with the dual ideal* D *and let* L *be a lattice with the ideal* I. *Let* D *and* I *be isomorphic under a fixed isomorphism and let* N *be obtained from* K *and* L *by gluing them together over* D *and* I.

If both K *and* L *are* 2*-distributive, then* N *is* 2*-distributive provided that the following condition holds:*

(SC) *Let* d_1, d_2, d_3, $a \in N$ *satisfy*

(SC$_1$) $$d_1,\ d_2,\ d_3 \in L,\ a \in K - L$$

or

(SC$_2$) $$d_1,\ d_2 \in L - K,\ d_3 \in K - L.$$

If d_1, d_2, d_3 *generate an eight-element Boolean lattice* B *(with unit element* u_B*) as the dual atoms of* B *and* $a \in N$ *satisfies* $a < u_B$, *then*

(4) $$a = (a \wedge d_1) \vee (a \wedge d_2) \vee (a \wedge d_3)$$

holds in N.

Proof. Let us assume that N satisfies (SC) and N is not 2-distributive. We shall get a contradiction.

By Lemma 6, N contains an element a and a sublattice B that is an eight-element Boolean lattice with dual atoms d_1, d_2, d_3 and unit element u_B, satisfying (1)–(3). We distinguish several cases.

Case 1. $B \subseteq L$. If $a \in L$, then (4) holds because it follows from (2D) by applying it to the three atoms of B and to a, contradicting (3). So we can assume that $a \notin L$, that is, $a \in K - L$ and so condition (SC_1) is satisfied. Thus (4) contradicts (3).

Case 2. $B \subseteq K$. Then also $a \in K$, so (3) contradicts the 2-distributivity of L by applying it to the three atoms of B and to a.

Case 3. $B \nsubseteq L$ *and* $B \nsubseteq K$. One of the dual atoms, say d_1, must be in $L - K$ and one of the dual atoms, say d_3, must be in $K - L$. Therefore, two of the three atoms of B (the ones under d_3) are in $K - L$ and one (the one under d_1) is in $L - K$. Hence $d_2 \in L - K$. So B and a satisfy the assumptions of (SC_2), therefore, (4) holds, contradicting (3). ☐

As a first consequence of Lemma 7, we obtain the following:

Lemma 8. *Let N be a lattice we obtain by gluing two 2-distributive lattices over a chain. Then N is a 2-distributive lattice.*

Proof. Let K, D, L, I, N, B, d_1, d_2, d_3 be given as in Lemma 7 and let us assume that $D = I$ is a chain.

If (SC_1) holds, take $o \in I$ with $o \leq v_B$, the zero of B. Let $\overline{a} = o \vee a$. Then $a \leq \overline{a} \leq u_B$. So by the 2-distributivity of L (by applying it to the three atoms of B and to a), we have

$$\overline{a} = (\overline{a} \wedge d_1) \vee (\overline{a} \wedge d_2) \vee (\overline{a} \wedge d_3). \tag{5}$$

Since $\overline{a} \wedge d_1$, $\overline{a} \wedge d_2$, $\overline{a} \wedge d_3 \in I$, one of them, say, $\overline{a} \wedge d_1$ is the largest, so by (5), $\overline{a} = \overline{a} \wedge d_1$. It follows that $a = a \wedge d_1$, verifying (4).

If (SC_2) holds, choose $o \in I$ with $o \leq d_1 \wedge d_2$. Then $d_1 = (d_1 \wedge d_2) \vee (d_1 \wedge d_3) = (d_1 \wedge d_2) \vee (o \vee (d_1 \wedge d_3))$ and, similarly, $d_2 = (d_1 \wedge d_2) \vee (o \vee (d_2 \wedge d_3))$. Since $o \vee (d_1 \wedge d_3)$ and $o \vee (d_2 \wedge d_3)$ are in the chain $D = I$, they must be comparable. Thus d_1 and d_2 are comparable, a contradiction, so (SC_2) does not apply. ☐

Now we are ready to prove the crucial gluing lemma:

Lemma 9. *Let K be a lattice with the dual ideal D and let L be a lattice with the ideal I. Let D and I be isomorphic under a fixed isomorphism and let N be obtained from K and L by gluing them together over D and I.*

Let us further assume that $L = C \times I$, where C is a chain with zero, 0_C.

If K is 2-distributive, then so is N.

Proof. The conditions of Lemma 7 are all assumed except that L be 2-distributive. This trivially holds since $L = C \times I$ and $I = D$ is a sublattice of the 2-distributive lattice K, and C is a chain. We write any $x \in L$ as $\langle x^C, x^I \rangle$, where $x^C \in C$ and $x^I \in I$. By Lemma 7, we only have to verify that (SC) holds in N.

So let B, a, d_1, d_2, d_3, and u_B be given as in (SC); we have to compute that (4) holds.

Let d_1, d_2, d_3, a satisfy (SC_1). Let us choose $o \leq i$ in I satisfying $o \leq v_B$ (the zero of B) and $a \leq i$, $u_B^I \leq i$. Define $\overline{a} = o \vee a \in I$.

Note that the projection of B into C is either a one- or a two-element chain. If it is a one-element chain, then $\{\, b^I \mid b \in B \,\}$ is an eight-element Boolean sublattice of $I \subseteq K$ with unit $u_B^I \geq a$ and $b^I = b \wedge i$, for all $b \in B$. Apply the 2-distributivity of K to the atoms of B^I and a, and compute:

$$\begin{aligned} a &= (a \wedge d_1^I) \vee (a \wedge d_2^I) \vee (a \wedge d_3^I) \\ &= ((a \wedge i) \wedge d_1^I) \vee ((a \wedge i) \wedge d_2^I) \vee ((a \wedge i) \wedge d_3^I) \\ &= (a \wedge (i \wedge d_1^I)) \vee (a \wedge (i \wedge d_2^I)) \vee (a \wedge (i \wedge d_3^I)) \\ &= (a \wedge (i \wedge d_1)) \vee (a \wedge (i \wedge d_2)) \vee (a \wedge (i \wedge d_3)) \\ &= ((a \wedge i) \wedge d_1) \vee ((a \wedge i) \wedge d_2) \vee ((a \wedge i) \wedge d_3) \\ &= (a \wedge d_1) \vee (a \wedge d_2) \vee (a \wedge d_3), \end{aligned}$$

verifying (4).

If the projection of B into C is a two-element chain $B^C = \{c_1, c_2\}$ with $c_2 \leq c_1$, but $\{\, b^I \mid b \in B \,\}$ is still an eight-element Boolean sublattice of $I \subseteq K$, we proceed as in the previous paragraph. The alternative is that $B^I = \{\, b^I \mid b \in B \,\}$ is a four-element Boolean sublattice of $I \subseteq K$, so $B = B^C \times B^I$. Then $u_B^I = d_i^I$, for some i, say, $u_B^I = d_3^I$. This implies that $a \leq d_3$, in which case (4) is trivial.

Let d_1, d_2, d_3, a satisfy (SC_2). Let

$$\begin{aligned} B_1 &= \{d_1 \wedge d_2, d_1, d_2, u_B\}, \\ B_2 &= \{v_B, d_1 \wedge d_3, d_2 \wedge d_3, d_3\}. \end{aligned}$$

Then B_1^I is a Boolean sublattice of I. If B_1^I has one or two elements, then we can argue as in Lemma 8 that this contradicts that $d_1 \parallel d_2$. So B_1^I has four elements.

We distinguish two cases.

Case 1: $a \in L$. Then $a = \langle a^C, a^I \rangle$, where $a^C \leq u_B^C$, and $a^I \leq d_1^I \vee d_2^I$. $B^I \cup B_2$ is an eight-element Boolean sublattice of K; apply the 2-distributivity of K to the three atoms of $B^I \cup B_2$ and to a to obtain

$$a = (a \wedge d_1^I) \vee (a \wedge d_2^I) \vee (a \wedge d_3). \tag{6}$$

Note that $a \wedge d_3 = a^I \wedge d_3$, so we can rewrite (6):

$$a^I = (a^I \wedge d_1^I) \vee (a^I \wedge d_2^I) \vee (a^I \wedge d_3). \tag{7}$$

Since the first coordinate of both a and $(a \wedge d_1) \vee (a \wedge d_2) \vee (a \wedge d_3)$ is a^C and the second coordinate of a is a^I and of $(a \wedge d_1) \vee (a \wedge d_2) \vee (a \wedge d_3)$ is the right side of (7), we conclude that (4) holds.

Case 2: $a \in K$. This is easier: $a \wedge d_1 = a \wedge d_1^I$, $a \wedge d_2 = a \wedge d_2^I$, and $a = a \wedge i \leq u_B \wedge i = d_1^I \vee d_2^I$, so we obtain (4) by applying the 2-distributive identity in K to the atoms of the Boolean lattice generated by d_1^I, d_2^I, d_3 and to a. □

Now we are ready to prove that the lattices K and L are 2-distributive. K was constructed as a planar lattice, so by Corollary 5, K is 2-distributive.

In Step 2, the lattices $\overline{K}_c$, $c \in \mathfrak{C}$, are planar, hence L_0, their direct product, is also 2-distributive.

In Step 3, L_1 is planar, so it is 2-distributive. In Step 4, the modular lattice M is in the variety generated by M_3; since M_3 as a planar lattice is 2-distributive, so is M.

In Step 5, we obtain L_2 by gluing together L_1 and M over a chain; hence, by Lemma 8, L_2 is 2-distributive.

Finally, in Step 6, we glue together L_0 and L_3 over B_0 and B', where L_3 is the dual of L_2. We could do this in two steps. First glue to L_0 the dual of $B \times H_0$. This lattice is 2-distributive by Lemma 9. Then we extend the dual of $B \times H_0$ to the dual of M. This retains 2-distributivity by Lemma 6. Indeed, if we take a B and a as in the lemma, the only way it could fail in the extended lattice if a is one of the new elements $m(b_1, b_2)$. But then $b_2 \wedge d_1$, $b_2 \wedge d_2$, $b_2 \wedge d_3 \in B \times H_0$, so they generate a Boolean sublattice. Apply the 2-distributivity of the dual of M to obtain that the 2-distributive identity holds for a and the atoms of B.

The resulting lattice is L.

6. Verifying the dual 2-distributive identity

In this section we verify that K and L are dually 2-distributive.

Lemma 10. *Let K be a lattice with the dual ideal D and let L be a lattice with the ideal I. Let D and I be isomorphic under a fixed isomorphism and let N be obtained from K and L by gluing them together over D and I.*

Let us further assume that $K = C \times D$, where C is a chain with a unit element.

If L is 2-distributive, then so is N.

Proof. The conditions of Lemma 7 are all assumed except that K be 2-distributive. This trivially holds since $K = C \times D$,where $I = D$ is a sublattice of the 2-distributive lattice L and C is a chain. We write any $x \in K$ as $\langle x^C, x^D \rangle$, where $x^C \in C$ and $x^D \in D$. By Lemma 7, we only have to verify that (SC) holds in N.

So let B, a, d_1, d_2, d_3, and u_B be given as in (SC); we have to compute that (4) holds.

Let d_1, d_2, d_3, a satisfy (SC_1). Then

$$a^D = (a^D \wedge d_1) \vee (a^D \wedge d_2) \vee (a^D \wedge d_3).$$

since L is 2-distributive (we apply the 2-distributive identity in L to the atoms of B and to a^D), which immediately implies (4).

Let d_1, d_2, d_3, a satisfy (SC_2). Let us choose $o \leq i$ in $D = I$ satisfying $o \leq d_1 \wedge d_2$ and $d_3 \leq i$.

We distinguish two cases.

Case 1: $a \in L$. Set $\underline{a} = a \wedge i$. Then $a \wedge d_3 = \underline{a} \wedge d_3 = \langle d_3^C, d_3^D \wedge \underline{a} \rangle$. So

$$\begin{aligned}
&(a \wedge d_1) \vee (a \wedge d_2) \vee (a \wedge d_3)\\
&= (a \wedge d_1) \vee (a \wedge d_2) \vee \langle d_3^C, d_3^D \wedge \underline{a} \rangle\\
&= (a \wedge d_1) \vee (a \wedge d_2) \vee (o \vee \langle d_3^C, d_3^D \wedge \underline{a} \rangle)\\
&= (a \wedge d_1) \vee (a \wedge d_2) \vee (d_3^D \wedge \underline{a}),
\end{aligned}$$

which is the right side of (4) for d_1, d_2, d_3^D, and a.

Now observe that d_1, d_2, d_3^D generate an eight-element Boolean lattice in L with atoms $d_1 \wedge d_2$, $d_1 \wedge d_3^D$, $d_2 \wedge d_3^D$ (because $K = C \times D$). Since $a \leq d_1 \vee d_2 \vee d_3^D$ and L is 2-distributive, we have

$$a = (a \wedge d_1) \vee (a \wedge d_2) \vee (d_3^D \wedge a)$$

in L and so by (8), equation (4) holds, verifying (SC).

Case 2: $a \in K$. Set $a = \langle c, g \rangle$. It is easy to argue that if p_1, p_2, p_3 are the atoms of B (so the pairwise joins give d_1, d_2, d_3) and p_1, $p_2 \leq d_3$, then $p_1 \vee o$, $p_2 \vee o$, and p_3 also generate an eight element Boolean lattice B' with dual atoms d_1, d_2, and g_3. Moreover, $B' \subseteq L$, so the 2-distributive identity applies to p_1, p_2, p_3, and

g, and we obtain that

$$g = (g \wedge d_1) \vee (g \wedge d_2) \vee (g \wedge g_3). \tag{8}$$

It is now clear that (4) holds for d_1, d_2, d_3, and a. □

Now we prove that L is dually 2-distributive as we did it in Section 5 except that we use Lemma 10 in place of Lemma 9.

7. Discussion

We justify proving the main theorem for doubly 2-distributive lattices with the following:

Lemma 11. *The* 2*-distributive identity is not selfdual.*

Proof. Let L be the nine-element lattice constructed from the eight-element Boolean lattice with atoms b_1, b_2, b_3 by adding an element a between the zero element and b_1. Then L is 2-distributive by Lemma 6 and fails the dual identity with a and b_1, b_2, b_3. □

Many lemmas in this paper prove that 2-distributivity is preserved under gluing under special circumstances. The following example shows that this does not hold, in general.

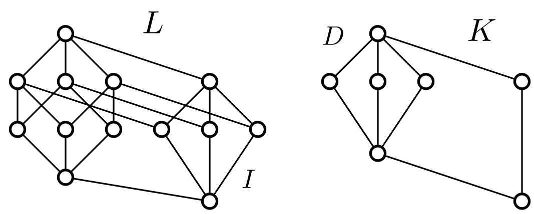

Figure 1

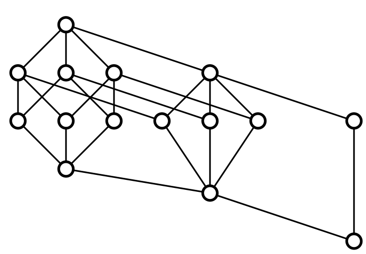

Figure 2

Figure 1 show the lattice L with the ideal I ($\cong M_3$) and the lattice K with the dual ideal D ($\cong M_3$). Both L and K are 2-distributive. Figure 2 show the lattice we obtain from L and K by gluing them together over I and D; this lattice is not 2-distributive.

We raise the following problem: can this construction be continued, that is, can we construct for Theorem 2 a lattice L that can serve as the starting lattice for the representation of a $\{0, \vee\}$-homomorphism $\psi \colon E \to F$. If this could be done, then we could represent distributive algebraic lattice with countably many compact elements as congruence lattice of 2-distributive lattices.

References

[1] N. Funayama and T. Nakayama, *On the congruence relations on lattices*, Proc. Imp. Acad. Tokyo **18** (1942), 530–531.

[2] G. Grätzer, *General Lattice Theory. Second Edition*, Birkhäuser Verlag, Basel. 1998. xix+663 pp.

[3] G. Grätzer, H. Lakser, and E. T. Schmidt, *Congruence lattices of small planar lattices*, Proc. Amer. Math. Soc. **123** (1995), 2619–2623.

[4] ———, *Congruence representations of join homomorphisms of distributive lattices: A short proof*, Math. Slovaca **46** (1996), 363–369.

[5] ———, *Isotone maps as maps of congruences. I. Abstract maps*, Acta Math. Acad. Sci. Hungar. **75** (1997), 105–135.

[6] G. Grätzer and E. T. Schmidt, *On congruence lattices of lattices*, Acta Math. Acad. Sci. Hungar. **13** (1962), 179–185.

[7] ———, *Congruence-preserving extensions of finite lattices into sectionally complemented lattices*. Accepted for publication in Proc. Amer. Math. Soc.

[8] A. P. Huhn, Schwach distributive Verbände. I. Acta Sci. Math. (Szeged) **33** (1972), 297–305.

[9] ———, *On the representation of distributive algebraic lattices. I*, Acta Sci. Math. (Szeged) **45** (1983), 239–246.

[10] ———, *On nonmodular n-distributive lattices. I*, Acta Sci. Math. (Szeged) **52** (1988), 35–45.

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