CONGRUENCE REPRESENTATIONS OF JOIN-HOMOMORPHISMS OF FINITE DISTRIBUTIVE LATTICES: SIZE AND BREADTH

G. GRÄTZER, H. LAKSER, and E. T. SCHMIDT

Abstract.
Let K and L be lattices, and let φ be a homomorphism of K into L. Then φ induces a natural 0-preserving join-homomorphism of $\operatorname{Con} K$ into $\operatorname{Con} L$.

Extending a result of A. Huhn, the authors proved that if D and E are finite distributive lattices and ψ is a 0-preserving join-homomorphism from D into E, then D and E can be represented as the congruence lattices of the finite lattices K and L, respectively, such that ψ is the natural 0-preserving join-homomorphism induced by a suitable homomorphism $\varphi\colon K \to L$. Let m and n denote the number of join-irreducible elements of D and E, respectively, and let $k = \max(m, n)$. The lattice L constructed was of size $O(2^{2(n+m)})$ and of breadth $n + m$.

We prove that K and L can be constructed as 'small' lattices of size $O(k^5)$ and of breadth three.

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1. Introduction

The congruence lattice, $\operatorname{Con} L$, of a finite lattice L is a finite distributive lattice (Funayama and Nakayama [2]). The converse is a result of Dilworth, first published in Grätzer and Schmidt [9].

For a distributive lattice D with n join-irreducible elements, the original constructions (Dilworth's and also the one in Grätzer and Schmidt [9]) produced lattices of size $O(2^{2n})$ and of order dimension $O(2n)$. In Grätzer and Lakser [4], this was improved to size $O(n^3)$ and order dimension 2 (therefore, planar and breadth 2). Finally, in Grätzer, Lakser, and Schmidt [5], a size $O(n^2)$ planar lattice was constructed:

Theorem 1. *Let D be a finite distributive lattice with n join-irreducible elements. Then there exists a planar lattice L of $O(n^2)$ elements with $\operatorname{Con} L \cong D$.*

Let K and L be lattices, and let φ be a homomorphism of K into L. Then φ induces a map $\operatorname{Con}\varphi$ of $\operatorname{Con} K$ into $\operatorname{Con} L$: for a congruence relation Θ of K, let the image Θ under $\operatorname{Con}\varphi$ be the congruence relation of L generated by the set $\Theta\varphi = \{\, \langle a\varphi, b\varphi\rangle \mid a \equiv b \ (\Theta) \,\}$.

The following result was proved by Huhn in [11] for embeddings and for arbitrary ψ in Grätzer, Lakser, and Schmidt [7]:

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Theorem 2. *Let D and E be finite distributive lattices, and let*

$$\psi\colon D \to E$$

be a 0-preserving join-homomorphism. Then there are finite lattices K and L, a lattice homomorphism $\varphi\colon K \to L$, and isomorphisms

$$\alpha\colon D \to \operatorname{Con} K, \qquad \beta\colon E \to \operatorname{Con} L$$

with

$$\psi\beta = \alpha(\operatorname{Con}\varphi).$$

Furthermore, φ is an embedding iff ψ separates 0.

Theorem 2 concludes that the following diagram is commutative:

$$\begin{array}{ccc} D & \xrightarrow{\ \psi\ } & E \\ \cong\big\downarrow\alpha & & \cong\big\downarrow\beta \\ \operatorname{Con} K & \xrightarrow{\operatorname{Con}\varphi} & \operatorname{Con} L \end{array}$$

See Grätzer, Lakser, and Schmidt [6] for a short proof.

A lattice L is said to be of *breadth* p, if p is the smallest integer with the property that for every finite $X \subseteq L$, there exists a $Y \subseteq X$ such that $|Y| \leq p$ and $\bigwedge X = \bigwedge Y$. Note that this concept is self-dual. If L is of *breadth* p, then for every finite $X \subseteq L$, there exists a $Y \subseteq X$ such that $|Y| \leq p$ and $\bigvee X = \bigvee Y$. If a finite lattice L is of breadth p, then there is an element $a \in L$ with at least p covers. The breadth of the Boolean lattice C_2^n is n.

In this paper, we prove the following improvement of Theorem 2 along the lines of Theorem 1:

Theorem. *Let D be a finite distributive lattice with n join-irreducible elements, let E be a finite distributive lattice with m join-irreducible elements, let $k = \max(m, n)$, and let*

$$\psi\colon D \to E$$

be a 0-preserving join-homomorphism. Then there is a finite lattice L of breadth 3 with $O(k^5)$ elements, a planar lattice K with $O(n^2)$ elements, a lattice homomorphism $\varphi\colon K \to L$, and isomorphisms

$$\alpha\colon E \to \operatorname{Con} L, \qquad \beta\colon D \to \operatorname{Con} K$$

with

$$\psi\alpha = \beta(\operatorname{Con}\varphi),$$

that is, such that the diagram

$$\begin{array}{ccc} D & \xrightarrow{\ \psi\ } & E \\ \cong\big\downarrow\beta & & \cong\big\downarrow\alpha \\ \operatorname{Con} K & \xrightarrow{\operatorname{Con}\varphi} & \operatorname{Con} L \end{array}$$

is commutative. Furthermore, φ is an embedding iff ψ separates 0.

In the last sentence of the Theorem, 'ψ separates 0' means that only the zero of D is mapped under ψ to the zero of E.

Outline. Function lattices play a crucial role in the construction. Section 2 deals with funtion lattices, in general, while Section 3 discusses function lattices over M_3 and N_5. Actually, we need a somewhat more general construction, which we name *generalized function lattices*; these are discussed in Section 4.

Coloring is useful for the presentation of the first construction; it is introduced in Section 5.

The first construction produces the planar lattice K of the Theorem; it is borrowed from Grätzer, Lakser, and Schmidt [5] and briefly described in Section 6.

The second construction is based on *multi-coloring*, introduced in Section 7; given a finite lattice M and a multi-coloring κ, we construct a generalized function lattice $M[\kappa]$.

The main construction is given, in four steps, in Section 8. The verification is presented in Section 9.

Section 10 discusses the Theorem and the related open problems.

Notation. We use the notation of Grätzer [3].

C_n denotes the n-element chain with $0 < 1 < \cdots < n-1$. Let $N_5 = \{o, a, b, c, i\}$, where $a < b$, denote the five-element nonmodular lattice and let $M_3 = \{o, a, b, c, i\}$ be the five-element modular nondistributive lattice, both with zero o and unit i.

2. Function lattices, general observations

For a lattice M, let M^{C_n} denote the set of all order-preserving maps of C_n to M, partially ordered by

$$\alpha \leq \beta \quad \text{iff} \quad x\alpha \leq x\beta, \text{ for all } x \in C_n.$$

Then M^{C_n} is a lattice; it is called a *function lattice*. (In general, a function lattice M^P is defined for any poset P.) The lattice M^{C_n} is a subdirect product of n copies of M; we shall use vector notation for the (isotone) maps.

As illustrations, Figure 1 shows $N_5^{C_3}$ and Figure 2 depicts $M_3^{C_2}$.

In this section, we prove some general properties of function lattices.

Lemma 1. *$\langle a_1, \ldots, a_n \rangle \prec \langle b_1, \ldots, b_n \rangle$ in M^{C_n} iff there exists a k with $1 \leq k \leq n$ such that $a_k \prec b_k$ in M and $a_i = b_i$, for $i \neq k$.*

Proof. Let $\langle a_1, \ldots, a_n \rangle \prec \langle b_1, \ldots, b_n \rangle$ in M^{C_n}. If there are $1 \leq k < l \leq n$ such that $a_k < b_k$ and $a_l < b_l$, then define $c_i = a_i$, for $i < l$ and $c_i = b_i$, for $i \geq l$. Obviously, $\langle a_1, \ldots, a_n \rangle < \langle c_1, \ldots, c_n \rangle$, since $a_l < b_l = c_l$, and $\langle c_1, \ldots, c_n \rangle < \langle b_1, \ldots, b_n \rangle$, since $c_k = a_k < b_k$. The lemma now easily follows. □

A sublattice of a finite lattice is called *cover-preserving*, if a prime interval of the sublattice is a prime interval of the whole lattice.

Lemma 2. *M^{C_n} is a cover-preserving sublattice of M^n.*

Proof. Indeed, if $\langle a_1, \ldots, a_n \rangle \prec \langle b_1, \ldots, b_n \rangle$ in M^{C_n}, then by Lemma 1, there exists a k with $1 \leq k \leq n$ such that $a_k \prec b_k$ and $a_i = b_i$, for $i \neq k$. But then $\langle a_1, \ldots, a_n \rangle \prec \langle b_1, \ldots, b_n \rangle$ in M^n is clear. □

For $x \in M$, let $\mathbf{x}_n$ denote the constant function $\langle x, \ldots, x \rangle$ in M^{C_n}; if n is clear from the context, it will be dropped. The constant maps form a sublattice of M^{C_n}; we identify M with this sublattice. In Figure 1 and Figure 2, the elements of the form $\mathbf{x}$ are black filled.

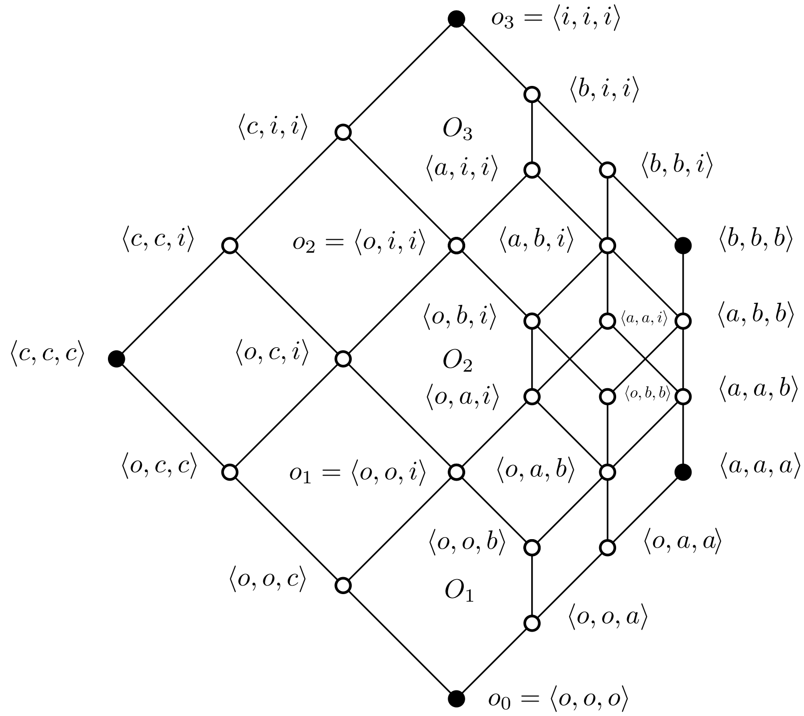

FIGURE 1. The lattice $N_5^{C_3}$.

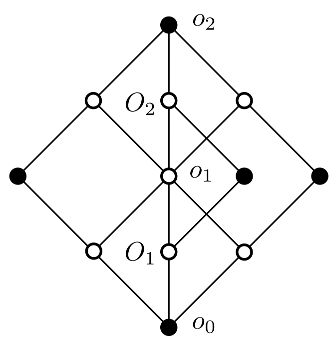

FIGURE 2. The lattice $M_3^{C_2}$.

Lemma 3. *If $\mathfrak{p} = [u, v]$ is a prime interval of M, then the corresponding interval $[\mathbf{u}, \mathbf{v}]$ of M^{C_n} is isomorphic to C_{n+1}.*

Proof. The interval $[\mathbf{u}, \mathbf{v}]$ in M^{C_n} consists of the elements

$$\begin{aligned}\mathbf{u} = \langle u, u, \dots, u, u\rangle,\ \langle u, u, \dots, u, v\rangle,\ \langle u, u, \dots, v, v\rangle,\ \dots,\\ \langle u, v, \dots, v, v\rangle,\ \langle v, v, \dots, v, v\rangle = \mathbf{v},\end{aligned}$$

and the coverings

$$\mathbf{u} = \langle u, u, \dots, u, u\rangle \prec \langle u, u, \dots, u, v\rangle \prec \langle u, u, \dots, v, v\rangle \prec \\ \dots \prec \langle u, v, \dots, v, v\rangle \prec \langle v, v, \dots, v, v\rangle = \mathbf{v}$$

are clear from Lemma 1. ☐

Take the following elements of M^n:

$$o_i = \langle 0, \dots, 0, \underbrace{1, \dots, 1}_{i}\rangle,$$

for $0 \leq i \leq n$, where 0 and 1 is the zero and unit of M, respectively. Then M is naturally isomorphic to the interval $O_i = [o_{i-1}, o_i] \subseteq M^n$, for $1 \leq i \leq n$, under the isomorphism

$$x \longrightarrow \langle 0, \dots, 0, x, \underbrace{1, \dots, 1}_{i}\rangle, \quad x \in M.$$

Observe that all these elements belong to M^{C_n}, hence the intervals $O_i = [o_{i-1}, o_i] \subseteq M^{C_n}$, for $1 \leq i \leq n$. (These elements and intervals are marked in Figures 1 and 2.) So we can consider M^{C_n} as a subdirect product of the O_i, $1 \leq i \leq n$, that is, a sublattice of $\prod(O_i \mid 1 \leq i \leq n) \cong M^n$. Let $O(M)$ denote the sublattice $\bigcup(O_i \mid i \leq n)$ of M^n; note that $O(M)$ is a sublattice of M^{C_n}.

A finite lattice K is a *congruence-preserving extension* of L, if L is a sublattice of K and every congruence of L has *exactly one* extension to K. Of course, then the congruence lattice of L is isomorphic to the congruence lattice of K.

Lemma 4. *Let E be a sublattice of M^n containing $O(M)$. Then M^n is a congruence-preserving extension of E. In particular, M^n is a congruence-preserving extension of M^{C_n}.*

Proof. Let Θ be a congruence relation of E. Since $O_i \subseteq E$, for $1 \leq i \leq n$, we can restrict Θ to O_i, to obtain the congruence Θ_i. Then $\prod(\Theta_i \mid 1 \leq i \leq n)$ is (up to isomorphism) a congruence of M^n that extends Θ. To show the uniqueness of the extension, let Φ be a congruence of M^n that extends Θ. Then Φ restricted to any O_i will agree with Θ restricted to O_i, hence $\Phi = \Theta$. ☐

Observe that this proof holds for function lattices (with finite exponents, P), in general, so we obtain a result of Duffus, Jónsson, and Rival [1]: $\operatorname{Con} M^P \cong (\operatorname{Con} M)^n$, where $n = |P|$.

3. Function lattices over M_3 and N_5

In this section, we investigate, in detail, the cases $M = M_3$ and $M = N_5$. See Figure 1 and Figure 2, for illustration. Note that $N_5^{C_2}$ is planar, that is why we show $N_5^{C_3}$.

The structure of $M_3^{C_n}$ is rather well known (Schmidt [12]):

Lemma 5. *$M_3^{C_n}$ is a modular lattice containing $\{\mathbf{o}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{i}\}$ as a $\{0, 1\}$-sublattice isomorphic to M_3. The interval $[\mathbf{o}, \mathbf{a}]$ is isomorphic to C_{n+1} and $M_3^{C_n}$ is a congruence-preserving extension of the chain $[\mathbf{o}, \mathbf{a}] \cong C_{n+1}$. In particular, every prime interval of $M_3^{C_n}$ is projective to a prime interval of $[\mathbf{o}, \mathbf{a}]$.*

Now we proceed to describe the structure of $N_5^{C_n}$.

Lemma 6. *Let Θ be the kernel of the n-th projection on $N_5^{C_n}$, that is, of the homomorphism $\langle x_1, \dots, x_n\rangle \to x_n$ of $N_5^{C_n}$ to N_5. Let A_x, $x \in \{o, a, b, c, i\}$, denote the five congruence classes (A_x contains $\mathbf{x}$). Then*

(i) $A_o = \{\mathbf{o}\}$.
(ii) $A_a = [\langle o, \dots, o, a\rangle, \mathbf{a}] \cong C_n$.
(iii) $A_b = [\langle o, \dots, o, b\rangle, \mathbf{b}] \cong C_3^{C_{n-1}}$.
(iv) $A_c = [\langle o, \dots, o, c\rangle, \mathbf{c}] \cong C_n$.
(v) $A_i = [\langle o, \dots, o, i\rangle, \mathbf{i}] \cong N_5^{C_{n-1}}$.
(vi) A_a *is isomorphic to* $[\mathbf{o}_{n-1}, \mathbf{a}_{n-1}] \subseteq N_5^{C_{n-1}}$.
(vii) A_b *is isomorphic to* $[\mathbf{o}_{n-1}, \mathbf{b}_{n-1}] \subseteq N_5^{C_{n-1}}$.
(viii) A_c *is isomorphic to* $[\mathbf{o}_{n-1}, \mathbf{c}_{n-1}] \subseteq N_5^{C_{n-1}}$.
(ix) $A_o \cup A_a \cup A_b \cong C_3^{C_n}$.

Moreover,

(x)
$$\begin{aligned} &A_a \cup [\langle o, \dots, o, b\rangle, \langle a, \dots, a, b\rangle] \cup [\langle o, \dots, o, i\rangle, \langle a, \dots, a, i\rangle] \\ &\qquad \cong A_a \times C_3 \cong [\mathbf{o}_{n-1}, \mathbf{a}_{n-1}] \times C_3. \end{aligned}$$

(xi) $A_b \cup [\langle o, \dots, o, i\rangle, \langle b, \dots, b, i\rangle] \cong A_b \times C_2 \cong [\mathbf{o}_{n-1}, \mathbf{b}_{n-1}] \times C_2$.
(xii) $A_c \cup [\langle o, \dots, o, i\rangle, \langle c, \dots, c, i\rangle] \cong A_c \times C_2 \cong [\mathbf{o}_{n-1}, \mathbf{c}_{n-1}] \times C_2$.

Proof. Obvious, by direct computation. ☐

For a finite lattice M, an *edge* $E_\mathfrak{p}$ of M^{C_n} is an interval $[\mathbf{u}, \mathbf{v}]$ of M^{C_n}, where $\mathfrak{p} = [u, v]$ is a prime interval of M.

Lemma 7. *Every prime interval $\mathfrak{p}$ of $N_5^{C_n}$ is projective to a prime interval $\mathfrak{q}$ in one of the edges $[\mathbf{o}, \mathbf{a}]$, $[\mathbf{a}, \mathbf{b}]$, $[\mathbf{b}, \mathbf{i}]$ of $N_5^{C_n}$.*

Proof. We prove this by induction on n.

If $n = 1$, then every prime interval is either one of the edges listed or it is projective to one of the edges listed by Lemma 6.

Let us assume that the statement is proved for $n - 1$. Let $\mathfrak{p}$ be a prime interval of $N_5^{C_n}$. We partition $N_5^{C_n}$ as in Lemma 6 into the sets A_x, $x \in \{o, a, b, c, i\}$. Then $A_i \cong N_5^{C_{n-1}}$, so the statement of this lemma is assumed for A_i.

Let $\mathfrak{p} = [u, v]$ be a prime interval of $N_5^{C_n}$.

First, let $\mathfrak{p} \subseteq A_i$.

For a prime interval $\mathfrak{q}$ of N_5, let $E_{\mathfrak{q},i-1}$ and $E_{\mathfrak{q},i}$ be the corresponding edges of A_i and $N_5^{C_n}$, respectively. By Lemma 6, either $E_{\mathfrak{q},i-1} \subseteq E_{\mathfrak{q},i}$, or $E_{\mathfrak{q},i-1}$ and $E_{\mathfrak{q},i}$ are contained in a distributive sublattice of $N_5^{C_n}$, in which every prime interval of $E_{\mathfrak{q},i-1}$ is perpective to a prime interval of $E_{\mathfrak{q},i}$; so the statement follows for $\mathfrak{p}$.

Second, let $\mathfrak{p} \nsubseteq A_i$.

If $\mathfrak{p} \subseteq A_o \cup A_c$, then the statement is trivial since the prime interval is perspective to one in $[\mathbf{b}, \mathbf{i}]$.

If $\mathfrak{p} \subseteq A_o \cup A_a \cup A_b$, then the statement is trivial since the edges of $N_5^{C_n}$ in this distributive lattice form maximal chains.

Finally, if $u \in A_c$ or $u \in A_b$ and $v \in A_i$, then pick $\mathfrak{q} = [\mathbf{o}_n, w]$, where w is the least element of A_a or of A_c, respectively, and observe that $\mathfrak{q}$ is in the edge $[\mathbf{o}, \mathbf{a}]$ or it is perspective to a prime interval in the edge $[\mathbf{b}, \mathbf{i}]$. ☐

Finally, in this section, we look at size and breadth.

Lemma 8. *$N_5^{C_n}$ and $M_3^{C_n}$ are lattices of breadth 3. The lattice $N_5^{C_n}$ has $O(n^3)$ elements and $M_3^{C_n}$ has $O(n^2)$ elements.*

Proof. An arbitrary element of $N_5^{C_n}$ has either the form

$$\langle \underbrace{o, \dots, o}_{i}, \underbrace{a, \dots, a}_{j}, \underbrace{b, \dots, b}_{k}, \underbrace{i, \dots, i}_{l} \rangle, \tag{1}$$

where $i + j + k + l = n$ $(0 \leq i,\ j,\ k,\ l \leq n)$ or the form

$$\langle \underbrace{o, \dots, o}_{i}, \underbrace{c, \dots, c}_{j}, \underbrace{i, \dots, i}_{k} \rangle, \tag{2}$$

where $i + j + k = n$ $(0 \leq i,\ j,\ k \leq n)$.

To prove the first statement of the lemma, we prove the stronger statement that an element of $N_5^{C_n}$ can have at most three upper covers. We get an upper cover of u, represented as in (1), by replacing the last o by a, or the last a by b, or the last b by i, proving the statement for u. The proof for an element u represented as in (2) is similar.

The number of elements of $N_5^{C_n}$ represented as in (1) is the number of ways we can choose i, j, and k so that $i + j + k + l = n$, for some l; there are $O(n^3)$ choices. Similarly, the number of elements of $N_5^{C_n}$ represented as in (2) is $O(n^2)$, proving both statements for $N_5^{C_n}$. The proof for $M_3^{C_n}$ is similar. ☐

4. Generalized function lattices

For a lattice M, a finite chain C_n, and congruences $\Theta_1, \dots, \Theta_n$ of M, a *generalized function lattice* over M is the sublattice of $M/\Theta_1 \times \dots \times M/\Theta_n$ defined by

$$\{\, \langle [a_1]\Theta_1, \dots, [a_n]\Theta_n \rangle \mid \langle a_1, \dots, a_n \rangle \in M^{C_n} \,\}.$$

Equivalently, let Θ be a congruence of M^{C_n}; by Lemma 4, Θ can be described by the restrictions $\Theta_1, \dots, \Theta_n$ of Θ to the intervals $O_1, \dots, O_n$. The generalized function lattice defined in the previous paragraph is isomorphic to M^{C_n}/Θ.

Now we borrow the arguments of Lemma 1 and Lemma 2:

Lemma 9. *The covering relation*

$$\langle [a_1]\Theta_1, \dots, [a_n]\Theta_n \rangle \prec \langle [b_1]\Theta_1, \dots, [b_n]\Theta_n \rangle$$

holds in the generalized function lattice iff there exists a k with $1 \leq k \leq n$ such that $[a_k]\Theta_k \prec [b_k]\Theta_k$ in M/Θ_k and $[a_i]\Theta_i = [b_i]\Theta_i$, for $i \neq k$.

Lemma 10. *The generalized function lattice is a cover-preserving sublattice of $M/\Theta_1 \times \dots \times M/\Theta_n$.*

To prove these two lemmas, observe that if

$$\langle [a_1]\Theta_1, \dots, [a_n]\Theta_n \rangle \leq \langle [b_1]\Theta_1, \dots, [b_n]\Theta_n \rangle,$$

then

$$\langle [b_1]\Theta_1, \dots, [b_n]\Theta_n \rangle = \langle [a_1 \vee b_1]\Theta_1, \dots, [a_n \vee b_n]\Theta_n \rangle,$$

and $\langle a_1 \vee b_1, \dots, a_n \vee b_n \rangle \in M^{C_n}$, so we can assume without the loss of generality that $a_i \leq b_i$, $1 \leq i \leq n$. Now we can follow the arguments of Lemmas 1 and 2, *mutatis mutandis*.

Similarly, we can borrow the argument of Lemma 8:

Lemma 11. *A generalized function lattice over N_5 is of breadth 3.*

We do not need the corresponding statement for M_3 since every generalized function lattice over M_3 is a function lattice over M_3.

5. Coloring

Let M be a finite lattice, and let Q be a finite set; the elements of Q will be called *colors*. Following Teo [13], a *coloring* μ of M over Q is a map

$$\mu\colon \mathfrak{P}(M) \to Q$$

of the *set of prime intervals* $\mathfrak{P}(M)$ of M into Q satisfying the condition: if two prime intervals generate the same congruence relation of M, then they have the same color; that is,

$$\mathfrak{p},\ \mathfrak{q} \in \mathfrak{P}(M) \text{ and } \Theta(\mathfrak{p}) = \Theta(\mathfrak{q}) \text{ imply that } \mathfrak{p}\mu = \mathfrak{q}\mu.$$

Since the join-irreducible congruences of M are exactly those that can be generated by prime intervals, equivalently, μ can be regarded as a map of the set $\mathrm{J}(\mathrm{Con}\, M)$ of join-irreducible congruences of M into Q:

$$\mu\colon\ \mathrm{J}(\mathrm{Con}\, M) \to Q.$$

If all prime intervals of M have the same color $q \in Q$, then we speak of a *monochromatic* lattice of color q.

We shall define a coloring by specifying μ on a large enough subset of $\mathfrak{P}(M)$ so that for every prime interval of M there is one in the subset that generates the same congruence.

Let M_i be a lattice colored by μ_i over Q_i, for $1 \leq i \leq n$. Then $\prod(\, M_i \mid 1 \leq i \leq n\,)$ has a natural coloring over $\bigcup(\, Q_i \mid 1 \leq i \leq n\,)$, since every prime interval of $\prod(\, M_i \mid 1 \leq i \leq n\,)$ is uniquely associated with a k, $1 \leq k \leq n$, and a prime interval of M_k.

Definition 1. We call $M \subseteq \prod(\, M_i \mid 1 \leq i \leq n\,)$ a *colored subdirect product* of the M_i, $1 \leq i \leq n$, if the following conditions are satisfied:

(i) M is a subdirect product of the M_i, $1 \leq i \leq n$;
(ii) M is a *cover-preserving sublattice* of $\prod(\, M_i \mid 1 \leq i \leq n\,)$;
(iii) the coloring of M is the coloring inherited from the coloring of $\prod(\, M_i \mid 1 \leq i \leq n\,)$.

By Lemma 2, if M is colored over Q, then M^{C_n} is also colored over Q.

6. The first construction: a planar lattice

The proof of the Theorem starts with the planar construction of Grätzer, Lakser, and Schmidt [5]. We shall outline it in a somewhat simplified form.

Let D be a finite distributive lattice, and let $J = \mathrm{J}(D) = \{d_1, \dots, d_n\}$ be the set of join-irreducible elements of D. Let S_0 be a chain of length $2n$. We color the prime intervals of S_0 over J as follows: we color the lower-most two prime intervals of S_0 with d_1, the next two with d_2, and so on. For each $d \in J$, there is a unique subchain $d^b \prec d^m \prec d^t$ of S_0 such that the prime intervals $[d^b, d^m]$ and $[d^m, d^t]$ have color d, and no other prime interval of S_0 has color d.

Let S_1 be a chain of length n. We color the prime intervals of S_1 by an arbitrary bijection. Thus, for each $d \in J$, there is in S_1 *exactly one prime interval* of color d; we denote it by $[d^o, d^i]$.

We set $K_0 = S_0 \times S_1$. We shall regard S_0 and S_1 as sublattices of K_0, in the usual manner. We extend the lattice K_0 to a lattice K: for each $d \in J$, we adjoin two new elements $m_0(d)$ and $m_1(d)$, as illustrated in Figure 3, and for each pair $a > c$ in J, we add a new element $n(a, c)$, as illustrated in Figure 4. To $d \in J$, assign the congruence of K generated by any/all prime intervals of this color. This defines an isomorphism between J and the poset of join-irreducible congruences of K; consequently, the congruence lattice of L is isomorphic to D.

Note that K is a planar lattice and $|K| < 3(n+1)^2$.

For instance, if D is the five-element distributive lattice of Figure 5, then $\mathrm{J}(D)$ is the poset $\{d_1, d_2, d_3\}$ with $d_1 < d_3$, $d_2 < d_3$, and we obtain the lattice K of Figure 5.

7. Multi-coloring and the second construction

Let M be a finite lattice, and let Q be a finite set. A *multi-coloring* of M over Q is an *isotone map* μ from $\mathfrak{P}(M)$ into $P^+(Q)$ (the set of all nonempty subsets of Q); *isotone* means that if $\mathfrak{p}$, $\mathfrak{q} \in \mathfrak{P}(M)$ and $\Theta(\mathfrak{p}) \leq \Theta(\mathfrak{q})$, then $\mathfrak{p}\mu \subseteq \mathfrak{q}\mu$. Equivalently, a multi-coloring is an isotone map of the poset $\mathrm{J}(\mathrm{Con}\, M)$ into the poset $P^+(Q)$.

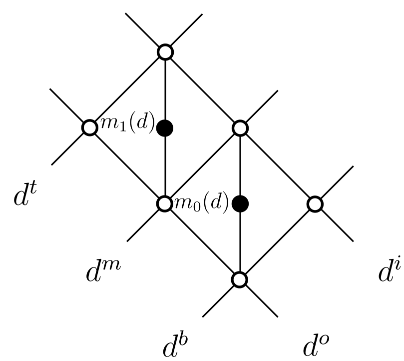

FIGURE 3. Adding the elements $m_0(d)$ and $m_1(d)$.

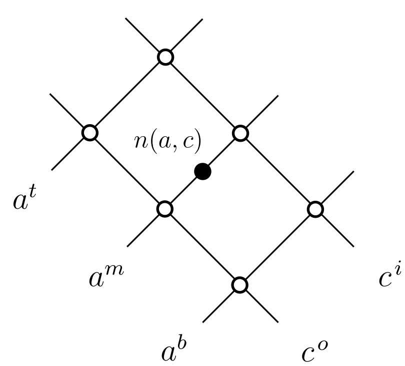

FIGURE 4. Adding the element $n(a, c)$.

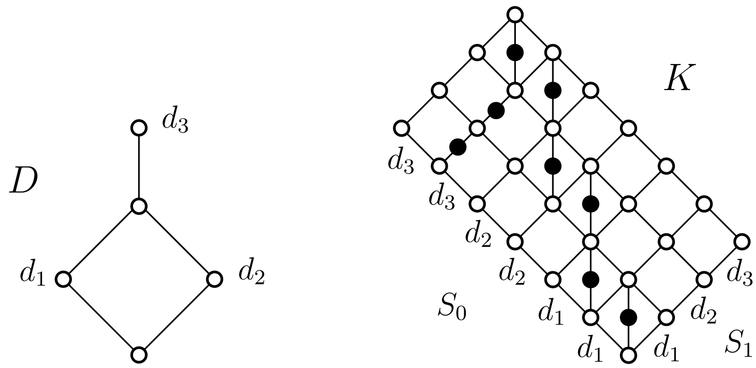

FIGURE 5. The lattice K constructed from D.

The second construction starts with a lattice M multi-colored by κ and constructs a generalized function lattice $M[\kappa]$ with *coloring* $\mu[\kappa]$. The lattice M embeds into $M[\kappa]$ such that the congruence structure of $M[\kappa]$ is easy to work with and κ is determined by $\mu[\kappa]$. We construct $M[\kappa]$ as a generalized function lattice.

Let M be a finite lattice with a multi-coloring κ over the n-element set $Q = \{q_1, q_2, \dots, q_n\}$. For any k with $1 \leq k \leq n$, define the binary relation Φ_k on M as follows:

$$u \equiv v \ (\Phi_k) \text{ iff } q_k \notin \mathfrak{p}\kappa, \text{ for any prime interval } \mathfrak{p} \subseteq [u \wedge v, u \vee v].$$

Lemma 12. *Φ_k is a congruence relation on M.*

Proof. The relation Φ_k is obviously reflexive and symmetric. To show the transitivity of Φ_k, assume that $u \equiv v \ (\Phi_k)$ and $v \equiv w \ (\Phi_k)$, and let $\mathfrak{q}$ be a prime interval in $[u \wedge w, u \vee w]$. Then $\mathfrak{q}$ is collapsed by $\Theta(u, v) \vee \Theta(v, w)$, hence there is a prime interval $\mathfrak{p}$ in $[u \wedge v, u \vee v]$ or in $[v \wedge w, v \vee w]$ satisfying $\Theta(\mathfrak{q}) \leq \Theta(\mathfrak{p})$. It follows from the definition of multi-coloring that $\mathfrak{q}\kappa \subseteq \mathfrak{p}\kappa$; since $q_k \notin \mathfrak{p}\kappa$, it follows that $q_k \notin \mathfrak{q}\kappa$, hence $u \equiv w \ (\Phi_k)$. The proof of the Substitution Property is similar. □

We define $M[\kappa]$ as the generalized function lattice over M determined by the congruences $\Phi_1, \dots, \Phi_n$. Set $M_i = M/\Phi_i$, for $1 \leq i \leq n$.

For $a \in M$, define

$$a[\kappa] = \langle [a]\Phi_1, \dots, [a]\Phi_n \rangle.$$

Then the map $a \to a[\kappa]$ maps M into $M[\kappa]$.

For $1 \leq i \leq n$, the lattice M_i is colored over Q; in fact, it is monochromatic. So we can regard $M_1 \times \cdots \times M_n$ as colored over Q. By Lemma 10, $M[\kappa]$ is a cover-preserving sublattice of $M_1 \times \cdots \times M_n$, so $M[\kappa]$ inherits the coloring, which we shall denote by $\mu[\kappa]$.

Let us color the chain C_{n+1} by Q as follows: the color of the prime interval $[i-1, i]$ of C_{n+1} is q_i, for $1 \leq i \leq n$. For a prime interval $\mathfrak{p} = [a, b]$ in M, we denote by $C_{n+1,\mathfrak{p}}$ the homomorphic image of C_{n+1} obtained by collapsing all prime intervals of color *not in* $\mathfrak{p}\kappa$.

The following lemma states the most important properties of $M[\kappa]$:

Lemma 13. *$M[\kappa]$ with the* coloring *$\mu[\kappa]$ over Q has the following properties:*

(i) *$M[\kappa]$ with the coloring $\mu[\kappa]$ is a colored subdirect product of the monochromatic lattices M_i, of color q_i, $1 \leq i \leq n$.*

(ii) *The map $a \to a[\kappa]$ is a lattice embedding of M into $M[\kappa]$.*
(iii) *For any prime interval $\mathfrak{p} = [a, b]$ in M, the interval $[a[\kappa], b[\kappa]]$ is isomorphic to $C_{n+1,\mathfrak{p}}$.*
(iv) *The coloring $\mu[\kappa]$ of $M[\kappa]$ determines the multi-coloring μ of M, namely, for every prime interval $\mathfrak{p} = [a, b]$ of M,*

$$\mathfrak{p}\mu = \{\, \mathfrak{q}\mu[\kappa] \mid \mathfrak{q} \in \mathfrak{P}(M[\kappa]) \text{ and } \mathfrak{q} \subseteq [a[\kappa], b[\kappa]]\,\}.$$

(v) *For any prime interval $\mathfrak{p} = [a, b]$ in $M[\kappa]$, there is a unique k with $1 \leq k \leq n$, and a prime interval $\mathfrak{q}$ in M_k such that $\mathfrak{p}$ is projective to $\mathfrak{q}$. Define a congruence relation*

$$\Phi_{\mathfrak{p}} = \omega_1 \times \cdots \times \omega_{k-1} \times \Theta(\mathfrak{q}) \times \omega_{k+1} \times \cdots \times \omega_n$$

of $\prod(\, M_i \mid 1 \leq i \leq n\,)$, where ω_j is the trivial congruence ω on M_j, for $j \neq k$. Then $\Theta(\mathfrak{p})$ is the restriction of $\Phi_{\mathfrak{p}}$ to $M[\kappa]$.
(vi) *The congruence lattice of $M[\kappa]$ is described by the following formula:*

$$\operatorname{Con} M[\kappa] \cong \prod(\, \operatorname{Con} M_i \mid 1 \leq i \leq n\,).$$

Proof. (i) and (iii) obviously hold.

(ii) The map $a \to a[\kappa]$ is obviously a lattice homomorphism. We have to prove that it is one-to-one. Let a, $b \in M$ and $a \neq b$; we have to prove that $a[\kappa] \neq b[\kappa]$. Let $\mathfrak{p}$ be a prime interval in $[a \wedge b, a \vee b]$. Since $\mu[\kappa]$ is a multi-coloring, there is a $q_i \in \mathfrak{p}\mu[\kappa]$. Obviously, then $a \not\equiv b \pmod{\Phi_i}$, from which the statement follows.

(iv) Let $a \prec b$ in M. Then $[\mathbf{a}, \mathbf{b}]$ in M^{C_n} is isomorphic to C_{n+1} ($\cong C_2^{C_n}$). By the definition of Φ, we get the fourth statement.

(v) and (vii) are also trivial. ☐

Let $A = [a, b]$ be an interval of M. Then the multi-coloring κ of M defines a multi-coloring κ_A on A; so the lattice $A[\kappa_A]$ is defined. On the other hand, A is a sublattice of $M[\kappa]$ (by identifying $x \in A$ with $x[\kappa]$), so it defines an interval $(A)_{M[\kappa]} = [a[\kappa], b[\kappa]]$ of $M[\kappa]$.

Lemma 14. *The lattices, $A[\kappa_A]$ and $(A)_{M[\kappa]}$ are isomorphic.*

Proof. Let $\mathbf{A}$ denote the interval $[\mathbf{a}, \mathbf{b}]$ of M^{C_n}. Then obviously $\mathbf{A}$ is isomorphic to A^{C_n}. The lattice $A[\kappa_A]$ is $\mathbf{A}/\Phi_A$, where Φ_A is the congruence defined on $\mathbf{A}$ by the multi-coloring κ_A. It is obvious from the definition of Φ that Φ_A is the restriction of Φ to $\mathbf{A}$, from which the isomorphism follows. ☐

8. The main construction

Let D and E be finite distributive lattices, and let

$$\psi\colon D \to E$$

be a 0-preserving join-homomorphism. We can trivially assume that ψ separates 0 (see [7]). Let $n = |\operatorname{J}(D)|$, $m = |\operatorname{J}(E)|$, $k = \max(n, m)$. We proceed in several steps.

We suggest that the reader follow the construction with the example shown on Figure 6. Note that the lattice D of Figure 6 is the same as the lattice D of Figure 5, for which the small planar lattice K satisfying $\operatorname{Con} K \cong D$ is already shown in Figure 5.

We do the construction in four steps.

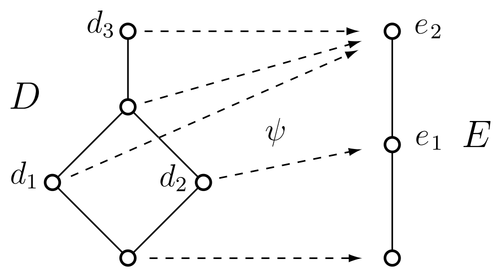

FIGURE 6. A simple example of a join-homomorphism ψ.

Step 1. We represent D as the congruence lattice of a planar lattice K as described in Section 6. To simplify the notation, we identify D with $\operatorname{Con} K$.

Step 2. We color C_{m+1} with $\mathrm{J}(E)$ so that there is a bijection between the prime intervals of C_{m+1} and $\mathrm{J}(E)$.

We define a map κ of $\mathfrak{P}(K)$ to subsets of $\mathrm{J}(E)$:

$$\mathfrak{p}\kappa = \mathrm{J}(E) \cap (\Theta(\mathfrak{p})\psi]_E = \{\, x \mid x \in \mathrm{J}(E),\ x \leq \Theta(\mathfrak{p})\psi \,\}.$$

κ is obviously isotone. ψ separates 0, so $\mathfrak{p}\kappa \neq \varnothing$. Therefore, κ is a multi-coloring of K over $\mathrm{J}(E)$. (Figure 7 shows the lattice K of Figure 5 multi-colored with subsets of $\mathrm{J}(E)$.) Now we apply the construction in Section 7 to obtain the generalized function lattice $K[\kappa]$ with the coloring $\mu[\kappa]$.

Step 3. For every $k = \langle k_0, k_1 \rangle \in K_0$, $k_0 < 1_{S_0}$, $k_1 < 1_{S_1}$, define the interval

$$B_k = [\langle k_0, k_1 \rangle, \langle k_0^{\dagger}, k_1^{\dagger} \rangle]$$

of K_0, where $k_0^{\dagger}$ is the covering element of k_0 in S_0 and $k_1^{\dagger}$ is the covering element of k_1 in S_1. Since K_0 is a sublattice of K, which in turn, is a sublattice of $K[\kappa]$, it follows that B_k defines an interval $(B_k)_K$ of K, and an interval $(B_k)_{K[\kappa]}$ of $K[\kappa]$. Observe that B_k is C_2^2; $(B_k)_K$ is C_2^2, or N_5, or M_3. Lemma 14 describes $(B_k)_{K[\kappa]}$:

Lemma 15. $(B_k)_{K[\kappa]}$ *is isomorphic to* $((B_k)_K)[\kappa]$.

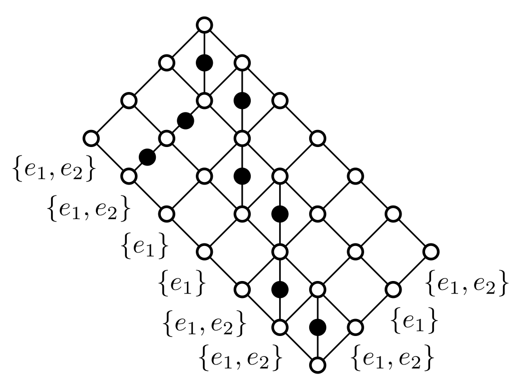

FIGURE 7. A multi-colored lattice K.

We define a subset K^+ of $K[\kappa]$ as follows, see Figure 8 (the elements of K are black-filled):

$$K^+ = \bigcup(\,(B_k)_{K[\kappa]} \mid k = \langle k_0, k_1\rangle \in K_0,\ k_0 < 1_{S_0},\ k_1 < 1_{S_1}\,).$$

Then K^+ is a sublattice of $K[\kappa]$. Note that the grid K_0 is a sublattice of K^+. The *extended grid* K_0' is $K_0[\kappa] \cap K^+$, which is of the form $S_0' \times S_1'$, where we obtain the chain S_0' from S_0 by replacing a prime interval $\mathfrak{p}$ by the chain $C_{m+1,\mathfrak{p}}$ in which the prime intervals of color not in $\mathfrak{p}\kappa$ are collapsed, and similarly for S_1.

Observe that $K_0' \cap (B_k)_{K[\kappa]}$, the extended grid restricted to a $(B_k)_{K[\kappa]}$ is a sublattice of $(B_k)_{K[\kappa]}$ of the form $C_{m+1}/\Phi_0 \times C_{m+1}/\Phi_1$, where Φ_0 factors out C_{m+1} by the colors of $[k_0, k_0^\dagger]\kappa$, and Φ_1 factors out C_{m+1} by the colors of $[k_1, k_1^\dagger]\kappa$.

We define an ideal I of K^+ as the restriction of the extended grid to

$$[\langle 0_{S_0}, 0_{S_1}\rangle, \langle 0_{S_0}, 1_{S_1}\rangle]_K^+,$$

which is a chain.

Lemma 16.

(i) *K_0 is a sublattice of K, and K is a sublattice of K^+. Moreover, K^+ is a cover-preserving sublattice of $K[\kappa]$. Therefore, K^+ is a colored lattice with the coloring $\mu[\kappa]$ restricted to it.*

(ii) *$q \in \mathrm{J}(E)$ is the color of a prime interval of I iff $q \le a\psi$, for some $a \in \mathrm{J}(D)$.*

(iii) *For every prime interval $\mathfrak{p}$ of K^+, there is a prime interval $\mathfrak{q} \subseteq I$ of the same color (that is, $\mathfrak{p}\mu[\kappa] = \mathfrak{q}\mu[\kappa]$) such that $\mathfrak{p}$ and $\mathfrak{q}$ generate the same congruence in K^+.*

(iv) *$O(|K^+|) = n^5$.*

Proof. (i) and (ii) follow from the definitions.

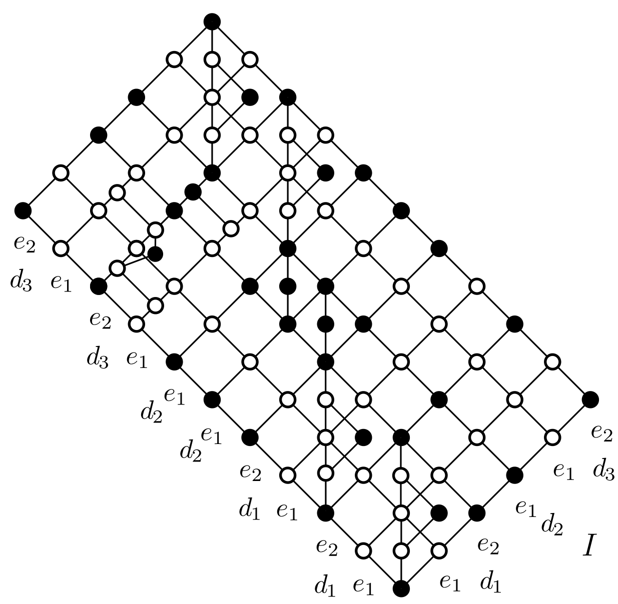

FIGURE 8. The lattice K^+.

(iii) Let $\mathfrak{p}$ be a prime interval of K^+; then $\mathfrak{p}$ is a prime interval of some $(B_k)_{K[\kappa]}$. By Lemma 7, there is a prime interval $\mathfrak{t}$ of B_k, such that $\mathfrak{p}$ is projective to a prime interval $\mathfrak{r}$ in an edge $E_\mathfrak{t}$ of $(B_k)_{K[\kappa]}$.

If $E_\mathfrak{t}$ is a prime interval of the extended grid, then $E_\mathfrak{t}$ is associated with a prime interval of S'_0 or of S'_1. In the latter case, $E_\mathfrak{t}$ is perpective to a prime interval of I. In the former case, take the prime interval of S_0 that contains the prime interval of the extended grid associated with $E_\mathfrak{t}$. By the construction of K, there is an M_3 in K that will identify this edge with one in I.

If $E_\mathfrak{t}$ is not a prime interval of the extended grid, then B_k is an N_5 and $\mathfrak{t}$ is $[o, a]$ or $[a, b]$ (or dually). By Lemma 7, $\mathfrak{r}$ is projective to a prime interval $\mathfrak{s}$ in the maximal chain containing the interval $[\mathbf{o}_n, \mathbf{b}_n]$ of $(B_k)_{K[\kappa]}$. By the construction of K, such a prime interval projects up or down in an N_5, making it projective to a prime interval of the extended grid.

(iv) is easy, since $|(B_k)_{K[\kappa]}| = O(m^3)$ by Lemma 8, and there are $O(n^2)$ such blocks by Step 1. ☐

Step 4. We represent E as the congruence lattice of a planar lattice L_0 as in Section 8 with the "grid", $T_0 \times T_1$, where T_1 is a chain of length $m = |\operatorname{J}(E)|$. We have $O(|L_0|) = m^2$. We again identify E with $\operatorname{Con} L_0$, and we regard L_0 as colored over $\operatorname{J}(E)$ by coloring the prime interval $\mathfrak{p}$ with $\Theta(\mathfrak{p}) \in \operatorname{J}(E)$.

L_0 has a dual ideal

$$D_0 = \{\, \langle x, 1_{T_1} \rangle \mid x \in T_0 \,\}$$

isomorphic to T_0.

We form the lattice

$$L_1 = T_0 \times I,$$

with the ideal

$$I_1 = \{\, \langle x, 0_I \rangle \mid x \in T_0 \,\}$$

isomorphic to T_0 and dual ideal

$$D_1 = \{\, \langle 1_{T_0}, x \rangle \mid x \in I \,\}$$

isomorphic to I. Since both T_0 and I are colored over $\operatorname{J}(E)$, there is a coloring of L_1 over $\operatorname{J}(E)$.

We glue L_0 and L_1 together over D_0 and I_1; the resulting lattice has D_1 as a dual ideal; so we can glue this lattice together with K^+ over D_1 and I, to obtain L_2. Since the gluing preserves the coloring, L_2 is colored over $\operatorname{J}(E)$.

Finally, we obtain the lattice L from L_2 as follows: take any 'prime square' of L_1 (that is, any interval of the form $[\langle a_0, a_1 \rangle, \langle b_0, b_1 \rangle]$, where $[a_0, b_0]$ is a prime interval of T_0 and $[a_1, b_1]$ is a prime interval of I) that is monochromatic (that is, $[a_0, b_0]$ in L_0 and $[a_1, b_1]$ in K^+ have the same color), and add an element to make the interval $[\langle a_0, a_1 \rangle, \langle b_0, b_1 \rangle]$ in L isomorphic to M_3.

9. Proof of the Theorem

Obviously, L has $O(k^5)$ elements.

Let φ denote the embedding of K into L.

We have to verify that $\operatorname{Con} \varphi = \psi\alpha$. It is enough to prove that $\Theta(\operatorname{Con} \varphi) = \Theta\psi\alpha$ for join-irreducible congruences Θ in K.

So let $\Theta = \Theta(\mathfrak{p})$, where $\mathfrak{p} = [a, b]$ is a prime interval of K. By Lemma 13, $\Theta(\mathfrak{p}) \operatorname{Con} \varphi = \Theta(a[\kappa], b[\kappa])$ collapses in $K[\kappa]$ the prime intervals of color $\leq \Theta\psi$; the same holds in L_0 and in L.

Let us assume that an element a of L has more than three covers. Since L is glued together over chains from three lattices, K^+, L_0, and L_1, it follows that a and its covers must be in one of these lattices. The element a and its covers cannot be in L_0 because the construction in Section 8 is planar. The lattice L_1 is a direct product of two chains with some additional elements to form M_3-s, so no element of L_1 has more than three covers. Finally, if a and its covers belong to K^+, then there is a largest grid element $k = \langle k_0, k_1 \rangle \in K_0$ ($k_0 < 1_{S_0}$, $k_1 < 1_{S_1}$) contained in a and then a and its covers belong to $(B_k)_{K[\kappa]}$, which by Lemma 15 is isomorphic to $((B_k)_K)[\kappa]$. Since $(B_k)_K$ is C_2^2, or N_5, or M_3, the lattice $((B_k)_K)[\kappa]$ is of breadth 3 by Lemma 11.

10. Discussion

Grätzer, Rival, and Zaguia [8] proved that the $O(n^2)$ result of Grätzer, Lakser, and Schmidt (see the Introduction) is 'best possible' in the sense that in Theorem 1 size $O(n^2)$ cannot be replaced by size $O(n^\alpha)$, for any $\alpha < 2$. This was improved in Zhang [14] and in Grätzer and Wang [10].

There are two crucial questions left open in this paper.

The first question is whether $O(k^5)$ is the optimal size for the lattice L in the Theorem. Can one prove (analogously to Grätzer, Rival, and Zaguia [8]) that size $O(k^5)$ cannot be replaced by size $O(k^\alpha)$, for any $\alpha < 5$? Can one find a lower bound for $|L|$ as in the result of Grätzer and Wang [10]?

The second question is whether breadth 3 is optimal for L? This is almost certainly so since a breadth 2 lattice cannot contain a C_2^3, making it very difficult to direct the congruences.

It seems to us that the lattice L we construct in this paper is of order dimension 3. It would be interesting to prove this.

Although this whole paper deals with the construction of the lattice L, it should be pointed out that we could not have started with a different K. The properties of the lattice K (borrowed from Grätzer, Lakser, and Schmidt [5]) are crucial for the construction of L. Can one construct L starting from a different lattice K?

References

[1] D. Duffus, B. Jónsson, and I. Rival, 'Stucture results for function lattices', *Canad. J. Math.* **33** (1978), 392–400.

[2] N. Funayama and T. Nakayama, 'On the distributivity of a lattice of lattice-congruences', *Proc. Imp. Acad. Tokyo* **18** (1942), 553–554.

[3] G. Grätzer, *General Lattice Theory. Second Edition* (Birkhäuser Basel, 1998).

[4] G. Grätzer and H. Lakser, 'Congruence lattices of planar lattices', *Acta Math. Hungar.* **60** (1992), 251–268.

[5] G. Grätzer, H. Lakser, and E.T. Schmidt, 'Congruence lattices of small planar lattices', *Proc. Amer. Math. Soc.* **123** (1995), 2619–2623.

[6] ———, 'Congruence representations of join homomorphisms of distributive lattices: A short proof', *Math. Slovaca* **46** (1996), 363–369.

[7] ———, 'Isotone maps as maps of congruences. I. Abstract maps', *Acta Math. Acad. Sci. Hungar.* **75** (1997), 81-111.

[8] G. Grätzer, I. Rival, and N. Zaguia, 'Small representations of finite distributive lattices as congruence lattices', *Proc. Amer. Math. Soc.* **123** (1995), 1959–1961.

[9] G. Grätzer and E.T. Schmidt, 'On congruence lattices of lattices', *Acta Math. Acad. Sci. Hungar.* **13** (1962), 179–185.
[10] G. Grätzer and D. Wang, 'A lower bound for congruence representations', *Order* **14** (1997), 67–74.
[11] A.P. Huhn, 'On the representation of distributive algebraic lattices. I–III', *Acta Sci. Math. (Szeged)* **45** (1983), 239–246; **53** (1989), 3–10, 11–18.
[12] E.T. Schmidt, 'Zur Charakterisierung der Kongruenzverbände der Verbände', *Mat. Časopis Sloven. Akad. Vied.* **18** (1968), 3–20.
[13] S.-K. Teo, 'On the length of the congruence lattice of a lattice', *Period. Math. Hungar.* **21** (1990), 179–186.
[14] Y. Zhang, 'A note on 'Small representations of finite distributive lattices as congruence lattices'', *Order* **13** (1996), 365–367.

Department of Mathematics
University of Manitoba
Winnipeg, Man. R3T 2N2
Canada

Mathematical Institute of the Technical University of Budapest
Műegyetem rkp. 3
H-1521 Budapest
Hungary