SUBLATTICES AND STANDARD CONGRUENCES

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Abstract. In an earlier paper, the authors and H. Lakser proved that, for every lattice K and nontrivial congruence Φ of K, there is an extension L of K such that Φ is the restriction to K of a *standard congruence* on L.

In this note, we give a very short proof of this result in a stronger form: the L we construct is sectionally complemented and it has only one nontrivial congruence, the standard congruence.

1. Introduction

In this short note, we prove the following result:

Theorem. *Let K be a lattice and let Φ be a nontrivial congruence of K. Then there is an extension L of K satisfying the following conditions:*

(i) *L has a standard ideal S such that the restriction of $\Theta[S]$ (the smallest congruence under which S is in a single congruence class) to K is Φ.*
(ii) *L is sectionally complemented;*
(iii) *the congruence lattice of L is the three-element chain.*

The existence of the extension L satisfying (i) was proved in G. Grätzer, H. Lakser, and E. T. Schmidt [2].

2. Construction

Let A and B be simple extensions of K and K/Φ with zeros 0_A and 0_B, and atoms p_A and p_B, respectively. Let N_6 be the six-element lattice defined on the set $\{0_{N_6}, p, q_1, q_2, r, s\}$ with the covering relations $0_{N_6} \prec p,\ q_1,\ q_2;\quad q_1,\ q_2 \prec r;$ $p,\ r \prec s$. Let M be the disjoint union of A, B, and N_6, with the zeros identified and denoted by 0, p_A and p identified, p_B and q_2 identified, see Figure 1. Then M is a chopped lattice. (For some general results on chopped lattices, see [3].) Any two elements of M have a meet and two elements have a join iff they belong to A, or B, or N_6, in which case the join is formed in the appropriate lattice.

An *ideal* I of M is a non-empty subset $I \subseteq M$ such that $I \cap A$ is an ideal in A, $I \cap N_6$ is an ideal in N_6, and $I \cap B$ is an ideal in B. The *finitely generated ideals* of M form a lattice $\mathrm{Id_{fg}}\, M$. By identifying $a \in M$ with $(a]$, we regard $\mathrm{Id_{fg}}\, M$ as an extension of M. An easy modification of a result of G. Grätzer and H. Lakser

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(published in G. Grätzer [1]) yields that $\mathrm{Id_{fg}}\, M$ is a congruence-preserving extension of the chopped lattice M.

Define $L = \mathrm{Id_{fg}}\, M$.

3. Proof

Since A and B are simple, it is clear that M has only one nontrivial congruence (whose nontrivial congruence classes are $B \cup \{r, q_1\}$ and $\{p, s\}$), so (iii) holds for L.

It is easy to see (Lemma 11 of [4]) that L can be constructed, alternatively, as follows: take the chopped sublattice N of M on the set $A \cup N_6$. Form $\mathrm{Id_{fg}}\, N$ (which has the atom q_2) and form M' as the disjoint union of $\mathrm{Id_{fg}}\, N$ and B with the zeros identified and the atoms q_2 and p_B identified. Then $\mathrm{Id_{fg}}\, M'$ is naturally isomorphic to L.

By Theorem 5 of [4], $\mathrm{Id_{fg}}\, N$ is sectionally complemented; again by the same result, $\mathrm{Id_{fg}}\, M'$ is sectionally complemented. Since $\mathrm{Id_{fg}}\, M' \cong L$, it follows that L is sectionally complemented, verifying (ii).

Since in a sectionally complemented lattice all congruences are standard, (i) follows trivially.

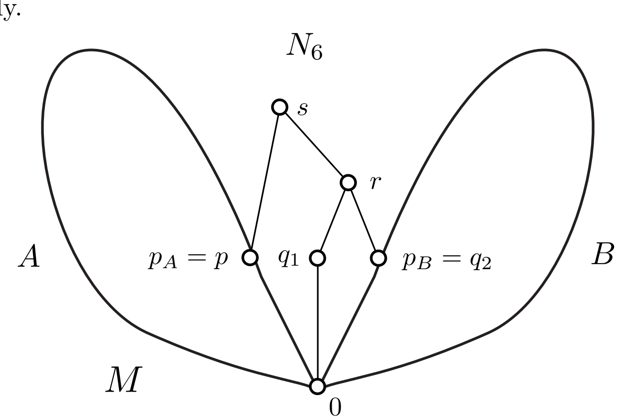

Figure 1

References

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[2] G. Grätzer, H. Lakser, and E. T. Schmidt, *Restriction of standard congruences on lattices*, manuscript. Submitted for publication to Contributions to General Algebra 10, Proceedings of the Klagenfurt Conference May 29–June 1, 1997.

[3] G. Grätzer and E. T. Schmidt, *A lattice construction and congruence-preserving extensions,* Acta Math. Acad. Sci. Hungar. **66** (**4**) (1995), 275–288.

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