ON THE INDEPENDENCE THEOREM OF RELATED STRUCTURES FOR MODULAR (ARGUESIAN) LATTICES

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ABSTRACT. Let $\mathfrak{D}$ be a finite distributive lattice with more than one element and let $\mathfrak{G}$ be a finite group. We prove that there exists a modular (arguesian) lattice M such that the congruence lattice of M is isomorphic to $\mathfrak{D}$ and the automorphism group of M is isomorphic to $\mathfrak{G}$.

1. INTRODUCTION

The “related structures” of the title are the congruence lattice and the automorphism group of a lattice. In [4, Problem II. 18], the first author raised the problem whether the congruence lattice and the automorphism group of a finite lattice are independent. This problem was solved affirmatively by V. A. Baranskiĭ [1], [2] and A. Urquhart [11].

Both proofs are based on two characterization theorems:

- (i) Every finite distributive lattice is isomorphic to the congruence lattice of a finite lattice (R. P. Dilworth, 1945, first published in [6]).
- (ii) Every finite group is isomorphic to the automorphism group of a finite (distributive) lattice (G. Birkhoff [3]).

(There is a different approach to the Independence Theorem in [7] that does not rely on characterization theorems.)

In [10], the second author proved that every finite distributive lattice is isomorphic to the congruence lattice of a modular lattice (of course, no longer finite since the congruence lattice of a finite modular lattice is always Boolean). So it is natural to ask whether we could prove the Independence Theorem for modular lattices.

Independence Theorem for Modular Lattices. *Let $\mathfrak{D}$ be a finite distributive lattice with more than one element and let $\mathfrak{G}$ be a finite group. Then there exists a modular lattice M such that the congruence lattice of M is isomorphic to $\mathfrak{D}$ and the automorphism group of M is isomorphic to $\mathfrak{G}$.*

The proof uses two constructions:

Theorem 1. *Let $\mathfrak{G}$ be a finite group. Then there exists a simple modular lattice S with an atom p such that the automorphism group of S is isomorphic to $\mathfrak{G}$ and every automorphism keeps p fixed.*

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Theorem 2. *Let $\mathfrak{D}$ be a finite distributive lattice with more than one element. There exists a rigid modular lattice R with zero 0 and a dual atom q such that the congruence lattice of R is isomorphic to $\mathfrak{D}$.*

Recall that a lattice R is *rigid*, if it does not have a nontrivial automorphism.

Theorem 1 was proved in [8] without the statement about the atom; in Section 2, we modify the construction of [8] to obtain the form of the result we need in this paper. In Section 3, we prove Theorem 2 and, in Section 4, the Independence Theorem for Modular Lattices.

In Section 5, we prove a much stronger form of the Independence Theorem: We construct an *arguesian* lattice A with a given finite congruence lattice and finite automorphism group. To accomplish that, we need Theorems 1 and 2 for arguesian lattices. In fact, in [8], we did that for Theorem 1. In Section 5, we prove this stronger form of Theorem 2 and show how the two arguesian theorems can be combined to get the Arguesian Independence Theorem.

2. SIMPLE MODULAR LATTICES

In [8], the construction starts by representing the finite group $\mathfrak{G}$ as the automorphism group of a finite graph $\langle V, E \rangle$. The reader should find it evident that if $\langle V, E \rangle$ has the property stated in the next lemma, then the modular lattice we construct does have the atom required in Theorem 1.

Lemma 1. *Every finite group $\mathfrak{G}$ can be represented as the automorphism group of a finite graph $\langle V, E \rangle$ with the property:*

There is a non-isolated vertex $f \in V$ fixed by all the automorphisms of $\langle V, E \rangle$.

Proof. Let $\langle V_1, E_1 \rangle$ be a finite graph whose automorphism group is isomorphic to $\mathfrak{G}$. Form $\langle V, E \rangle$ as the disjoint union of $\langle V_1, E_1 \rangle$ with a connected rigid finite graph $\langle R, S \rangle$ with a vertex $f \in R$ of degree $> |V_1|$. An automorphism α of $\langle V, E \rangle$ cannot take f into V_1 since the degree of a vertex of $\langle V_1, E_1 \rangle$ is $\leq |V_1|$. Hence α takes f into R . Since $\langle R, S \rangle$ is connected, α takes R into R . By the rigidity of $\langle R, S \rangle$, α is the identity map on R . So $\langle V, E \rangle$ has the same automorphism group as $\langle V_1, E_1 \rangle$, and f is fixed by all the automorphisms. $\square$

3. RIGID MODULAR LATTICES

In this section, we prove Theorem 2; the proof is based on the construction in [10]. For every finite poset P , we have to construct a rigid modular lattice R_P with a dual atom q such that $\text{Con } R_P \cong 2^P$.

We shall use the following lemma (the first six statements are due to the second author, see [10]; the last statement is due to A. Mitchke and R. Wille [9]).

Lemma 2. *Let E be a chain with bounds 0_E and 1_E , and let $M_3 = \{0, a, b, c, 1\}$ be the five-element nondistributive modular lattice. Let $M_3[E]$ denote the subposet of E^3 consisting of all $\langle x, y, z \rangle$ satisfying $x \wedge y = y \wedge z = z \wedge x$. Then the following statements hold:*

- (i) $M_3[E]$ is a modular lattice.
- (ii) The subset

$$\overline{M}_3 = \{\langle 0, 0, 0 \rangle, \langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 0, 1 \rangle, \langle 1, 1, 1 \rangle\}$$

of $M_3[E]$ is a sublattice of $M_3[E]$ and it is isomorphic to M_3 .

- (iii) The subposet $\overline{E} = \{ \langle x, 0, 0 \rangle \mid x \in E \}$ of $M_3[E]$ is a chain and it is isomorphic to E ; we identify E with $\overline{E}$.
- (iv) $\overline{M}_3$ and E generate $M_3[E]$.
- (v) The subset

$$\{ \langle x, x \wedge y, y \rangle \mid x, y \in E \}$$

of $M_3[E]$ is a sublattice and it is isomorphic to E^2 .

- (vi) Let Θ be a congruence relation of $E = \overline{E}$; then there is a unique congruence $\overline{\Theta}$ of $M_3[E]$ such that $\overline{\Theta}$ restricted to $\overline{E}$ is Θ ; therefore, $\text{Con } M_3[E] \cong \text{Con } E$.
- (vii) Let M be a modular lattice; let $A = \{o, a, b, c, i\}$ be a sublattice of M isomorphic to M_3 ; let B be a chain in $[o, a]$ (including o and a) isomorphic to E . Then the sublattice of M generated by A and B is isomorphic to $M_3[E]$.

The lattice R_P will be put together by gluing together lattices of the form $M_3[E]$ with distributive lattices. The following lemma states some well-known facts about gluing and congruences.

Lemma 3. *Let A be a lattice with dual ideal D with more than one element, and let B be a lattice with ideal I with more than one element. Let D and I be isomorphic under the isomorphism φ . Let L denote the gluing of A and B over D and I with respect to φ .*

- (i) *If Θ is a congruence of L , then Θ_A (the restriction of Θ to A) is a congruence of A , and Θ_B (the restriction of Θ to B) is a congruence of B , and Θ_A restricted to D is the same as Θ_B restricted to I under φ .*
- (ii) *If Φ_A is a congruence of A and Φ_B is a congruence of B such that Φ_A restricted to D is the same as Φ_B restricted to I under φ , then there is a unique congruence Φ of L such that the restriction of Φ to A is Φ_A , and the restriction of Φ to B is Φ_B .*
- (iii) *If every congruence Φ of B has the property that Φ_I (the restriction of Φ to I) has a unique extension to B (namely, Φ) and Φ_I (as a congruence of D) has an extension to a congruence of A , then $\text{Con } L \cong \text{Con } A$.*
- (iv) *If B and I are simple, then $\text{Con } L \cong \text{Con } A$.*

We now combine Lemmas 2 and 3 to construct the basic building block, see Figure 1.

Lemma 4. *Let U be a chain with zero, 0_U , and unit, 1_U . Let $0_U \leq u < v \leq 1_U$, and let V be the interval $[u, v]$ of U . Construct the lattice L as follows:*

- (i) *Form the direct product $U \times V$.*
- (ii) *Glue the interval $[\langle 0_U, u \rangle, \langle u, v \rangle]$ of $U \times V$ with the dual ideal $[\langle u, u \rangle, \langle u, v \rangle]$ ($\cong V$) to $M_3[V]$ with the ideal $\{ \langle 0, 0, v \rangle \mid v \in V \}$ ($\cong V$).*
- (iii) *Glue the lattice we obtained with the dual ideal $\{ \langle 1, x, x \rangle \mid x \in V \}$ ($\cong V$) to the interval $[\langle v, u \rangle, \langle 1_U, v \rangle]$ with the ideal $[\langle v, u \rangle, \langle v, v \rangle]$ ($\cong V$); the resulting lattice is denoted by $U \circledast V$.*
- (iv) *Consider U as a sublattice of $U \circledast V$ by identifying $x \in U$ with $\langle x, u \rangle$ if $x \leq u$ or if $v \leq x$; otherwise, $x \in [u, v]$, and we identify x with $\langle x, 0, 0 \rangle$. This identifies U with a principal ideal of $U \circledast V$.*

Then every congruence Θ of U has a unique extension to a congruence $\overline{\Theta}$ of $U \circledast V$; therefore, $\text{Con } U \cong \text{Con}(U \circledast V)$.

FIGURE 1. The lattice U .

We use induction on $n = |P|$, the size of P , to construct a rigid modular lattice R_P with the following properties:

- (I_1) R_P is a rigid modular lattice with zero 0_P , unit 1_P , and a dual atom q_P ;
- (I_2) every principal congruence of R_P is a join of join-irreducible congruences and the join-irreducible congruences of R_P form a poset isomorphic to P ;
- (I_3) R_P has an element a_P with a unique complement a'_P such that the ideals $(a_P]$ and $(a'_P]$ are isomorphic to a countable successor ordinal α_P .
- (I_4) every congruence of R_P is determined by its restriction to $(a'_P]$.

Observe that (I_4) is equivalent to the following condition:

- (I'_4) R_P contains a chain

$$0_P \leq b_1 < c_1 \leq b_2 < c_2 \leq \cdots \leq b_n < c_n \leq a'_P$$

such that the join-irreducible congruences of R_P are exactly the principal congruences $\Theta(b_1, c_1), \Theta(b_2, c_2), \dots, \Theta(b_n, c_n)$.

If $P = \mathbf{1}$ (that is, $|P| = 1$), then the lattice of Figure 2 satisfies these requirements with $\alpha_1 = 4$; this lattice is also simple and rigid.

Now let P be a finite poset, and let us assume that R_S has been constructed for all finite posets S satisfying $|S| < |P|$. The reader may find it useful to follow the induction step with Figure 3, which illustrates the construction of $R_{\mathbf{2}}$ ($P = \mathbf{2}$, the two-element chain); the congruence lattice of $R_{\mathbf{2}}$ is, of course, the three-element chain, $\mathbf{2}^2$.

FIGURE 2. The lattice R_1 .

FIGURE 3. The lattice R_2 .

Let p be a maximal element of P . Then for the poset $Q = P - \{p\}$, there exists a rigid modular lattice R_Q with a dual atom q_Q such that (I_1) – (I_4) hold; in particular, there is a chain

$$0_Q \leq b_1 < c_1 \leq b_2 < c_2 \leq \cdots \leq b_{n-1} < c_{n-1} \leq a_Q'$$

such that the join-irreducible congruences of R_Q are the principal congruences $\Theta(b_1, c_1), \Theta(b_2, c_2), \dots, \Theta(b_{n-1}, c_{n-1})$. Let $C = [a_Q]$; by (I_3) (observing that, by modularity, $(a_Q') \cong [a_Q]$), C is a well-ordered chain; it is isomorphic to α_Q .

Let $q_1, q_2, \dots, q_r$ be the elements of P covered by p . Let $\Theta(b_{k_1}, c_{k_1}), \Theta(b_{k_2}, c_{k_2}), \dots, \Theta(b_{k_r}, c_{k_r})$ be the join-irreducible congruences corresponding to $q_1, q_2, \dots, q_r$, respectively. Without loss of generality, we can assume that

$$b_{k_1} < c_{k_1} \leq b_{k_2} < c_{k_2} \leq \cdots \leq b_{k_r} < c_{k_r}.$$

Let D be the chain

$$[b_{k_1}, c_{k_1}] \oplus [b_{k_2}, c_{k_2}] \oplus \cdots \oplus [b_{k_r}, c_{k_r}]$$

(which is the disjoint union of the intervals in which the top element of each interval is identified with the bottom element of the next interval). For every natural number i , let D_i be a chain isomorphic to D , and let x^i denote the image of x in D_i under this isomorphism. Finally, we consider the ordinal sum $\mathcal{C}$ of these chains with two elements adjoined, q^* and 1^* , where 1^* is the unit element of $\mathcal{C}$ and q^* is

a dual atom,

$$\mathcal{C} = D_1 \oplus D_2 \oplus \cdots \oplus \{q^*, 1^*\}.$$

We apply Lemma 4 first to $U_1 = C$ and $V_1 = [b_{k_1}, c_{k_1}]$, to obtain the lattice $C \circledast [b_{k_1}, c_{k_1}]$ and dual ideal $\{c_{k_1}\} \times C (\cong C)$, and second to $U_2 = C$ and $V_2 = [b_{k_2}, c_{k_2}]$, to obtain the lattice $C \circledast [b_{k_2}, c_{k_2}]$ and ideal $\{b_{k_2}\} \times C (\cong C)$, and we glue these two lattices together over the given ideal and dual ideal. We proceed similarly and glue $C \circledast [b_{k_3}, c_{k_3}]$ to the resulting lattice, and so on. In r steps, we obtain a lattice L with an ideal and a dual ideal both isomorphic to C .

Now for each $i < \omega$, we take a copy L_i of L , and glue L_2 to L_1 , L_3 to the resulting lattice, and so on, in the last step, we glue L_i . Call the lattice we obtained $\bar{L}_i$. Obviously, $\bar{L}_1 \subseteq \bar{L}_2 \subseteq \cdots \subseteq \bar{L}_n$, so we can take $\bigcup (\bar{L}_i \mid i < \omega)$. We adjoin to this lattice $\{q^*, 1^*\} \times C$ (where $\{q^*, 1^*\}$ is regarded as the two-element chain with $q^* < 1^*$) so that

$$\begin{aligned} \langle u, c \rangle \wedge \langle v, d \rangle &= \langle u \wedge v, c \wedge d \rangle, \\ \langle u, c \rangle \vee \langle v, d \rangle &= \langle u \vee v, c \vee d \rangle, \end{aligned}$$

for u and/or $v \in \{q^*, 1^*\}$ and $c, d \in C$.

We add one more element. C has a dual atom q_C ; the lattice $\{q^*, 1^*\} \times \{q_C, 1_Q\}$ is a cover-preserving four-element Boolean sublattice of the lattice we have constructed. We add an element w so that $\{q^*, 1^*\} \times \{q_C, 1_Q\}$ with w form a cover-preserving M_3 . Let R_Q^1 denote the lattice we have just obtained.

Finally, let us glue R_Q and R_Q^1 together over the dual ideal C and ideal $\{b_{k_1}\} \times C (\cong C)$ in L_0 ; call the resulting lattice R'_Q .

To investigate R'_Q , let us present a more intuitive description. Form the direct product of $\mathcal{C}$ and C ; we shall call this the *base* of R_Q^1 . Consider the interval $[b_{k_j}^i, c_{k_j}^i]$ in $\mathcal{C}$ and the corresponding interval $[b_{k_j}, c_{k_j}]$ in C . In the direct product $\mathcal{C} \times C$, replace

$$[b_{k_j}^i, c_{k_j}^i] \times [b_{k_j}, c_{k_j}]$$

with $M_3[b_{k_j}, c_{k_j}]$ so that

$$\langle x, y \rangle \in [b_{k_j}^i, c_{k_j}^i] \times [b_{k_j}, c_{k_j}]$$

is replaced by $\langle x, x \wedge y, y \rangle$ (see statement 5 of Lemma 2) and we add the element w in the top prime square so that it form an M_3 . Of course, with this definition it is not clear whether R_Q^1 is a lattice, whether it is modular, and what are the congruences of R_Q^1 . It is clear, however, that this definition is the same as the more complicated one given above, so we get all these properties of the construction from Lemmas 2–4.

Take a congruence Ψ of R_Q^1 ; then the restriction of Ψ to the base of R_Q^1 is of the form $\Phi \times \Theta$, where Φ is a congruence of $\mathcal{C}$ and Θ is a congruence of C . It is clear from Lemmas 2–4 that Φ and Θ uniquely determine Ψ . Moreover, Θ “almost” determines Φ . In fact, $a \equiv b$ (Φ) is fully determined by Θ , for $a, b \in D_1 \oplus D_2 \oplus \cdots$. The same is true for $a, b \in \{q^*, 1^*\}$. Finally, let $a \in D_1 \oplus D_2 \oplus \cdots$ and $b \in \{q^*, 1^*\}$. Then $a \equiv b$ (Φ) implies that all D_i are collapsed by Φ from some i on, and so all of $D_1 \oplus D_2 \oplus \cdots$ is collapsed by Φ . We conclude that R_Q^1 has exactly one join-irreducible congruence that is not a minimal extension of a congruence from C , namely, $\Theta(\langle 0_C, 0_C \rangle, \langle q^*, 0_C \rangle)$, and this congruence majorizes all the join-irreducible

congruences that are minimal extensions of the congruences $\Theta(b_1, c_1), \Theta(b_2, c_2), \dots, \Theta(b_{n-1}, c_{n-1})$ of C to R_Q^1 .

By the inductive assumption (I_3) , every congruence of R_Q is determined by its action on C , and so we obtain that every congruence of R_Q can be extended to R'_Q . Therefore, the join-irreducible congruences of R'_Q can be described as follows: they are the minimal extensions of the join-irreducible congruences of R_Q to R'_Q and the congruence $\Theta(\langle 0_C, 0_C \rangle, \langle q^*, 0_C \rangle)$. Hence they form a poset isomorphic to P , and so $\text{Con } R_Q \cong \mathfrak{D}$, by (I_2) .

The lattice R'_Q does not satisfy all the inductive assumptions, so we shall define R_P as an appropriate extension of R'_Q .

Define the lattice R_Q^2 as $M_3[\mathcal{C}]$ and let $\mathcal{C}_1 (\cong \mathcal{C})$ be the ideal of $M_3[\mathcal{C}]$ as defined in statement 3 of Lemma 2. Define the lattice R_Q^3 as $[a'_Q, 1_Q] \times \mathcal{C}_1$ and let $\mathcal{C}_2 (\cong \mathcal{C})$ be the dual ideal generated by $\langle 1_Q, 0_{\mathcal{C}_1} \rangle$ in R_Q^3 . Let R''_Q be the gluing of R_Q^2 and R_Q^3 over $\mathcal{C}_1$ and $\mathcal{C}_2$. Obviously, the ideal of R''_Q generated by $\langle 1, 0, 0 \rangle \in R_Q^2$ is isomorphic with the dual ideal $[a'_Q]$ of R'_Q . So we can glue R'_Q and R''_Q together to obtain R_P .

Define $a_P = \langle 1^*, 0_C \rangle (\in R_Q^1)$ and let a'_P be its unique complement, $a'_P = \langle a'_Q, 0_{\mathcal{C}_1} \rangle (\in R_Q^3)$. The dual atom $\langle 1^*, q^*, q^* \rangle$ of R_Q^2 can serve as the dual atom of R_P . Now we verify that R_P satisfies the conditions of Theorem 2 as well as the inductive assumptions (I_1) – (I_4) .

$(a_P]$ is a countable well-ordered chain, namely, $\alpha_Q \oplus \mathcal{C} \oplus \mathbf{2}$, which is a successor ordinal; the same is true of $(a'_P]$.

Let α be an automorphism of R_P . Since 1_Q is the smallest element $x \in R_P$ such that there is a sublattice isomorphic to M_3 from x to 1_P , it follows that $1_Q\alpha = 1_Q$. Since R_Q is rigid, it follows that $x\alpha = x$, for all $x \in R_Q \subseteq R_P$.

R_Q^1 is built on $\mathcal{C} \times C$, and C is kept fixed by α . Therefore, α maps $\mathcal{C}$ into itself. But $\mathcal{C}$ is well-ordered, so α is the identity map on $\mathcal{C}$, and so on all of R'_Q .

$R_Q^2 = [1_Q, 1_P]$ and both these elements are fixed by α . Next consider the three atoms in the sublattice isomorphic to M_3 stretching from 1_Q to 1_P : $\langle 1, 0, 0 \rangle$ is fixed because it is in R'_Q ; $\langle 0, 1, 0 \rangle$ and $\langle 0, 0, 1 \rangle$ cannot be interchanged because 1_Q has a complement in $[a'_Q, \langle 0, 0, 1 \rangle]$ but not in $[a'_Q, \langle 0, 1, 0 \rangle]$. So α fixes $\langle 1, 0, 0 \rangle$, $\langle 0, 1, 0 \rangle$, and $\langle 0, 0, 1 \rangle$ and also the chain $\mathcal{C}$, hence α fixes all of R_Q^2 . Since we already know that in R_Q^3 , α fixes $(1_Q]$ and $[1_Q)$, it follows that α fixes every element of R_Q^3 . We conclude that R_P is rigid.

It remains to verify (I_3) . In R_Q , we were given the chain

$$0_Q \leq b_1 < c_1 \leq b_2 < c_2 \leq \dots \leq b_{n-1} < c_{n-1} \leq a'_Q$$

such that the join-irreducible congruences of R_Q are the principal congruences $\Theta(b_1, c_1), \Theta(b_2, c_2), \dots, \Theta(b_{n-1}, c_{n-1})$. In R_P , we extend this chain with $b_n = a'_Q$ and $c_n = a'_P$. Then condition (I'_4) is obvious.

This completes the proof of Theorem 2.

4. PROOF OF THE INDEPENDENCE THEOREM FOR MODULAR LATTICES

Now it is easy to prove the Independence Theorem for Modular Lattices. Let $\mathfrak{G}$ and $\mathfrak{D}$ be given as in the Independence Theorem. Let S be constructed as in Theorem 1 for $\mathfrak{G}$ and let R be the lattice in Theorem 2 constructed for $\mathfrak{D}$. We take

the ideal $(p]$ in S and the dual ideal $[q]$ in R ; both are two-element lattices, so we can glue S and R together over $(p]$ and $[q]$. Let M be the resulting lattice.

M is a modular lattice. By statement 4 of Lemma 3, $\text{Con } M \cong \text{Con } R$, so $\text{Con } M$ is isomorphic to $\mathfrak{D}$.

Now let α be an automorphism of S , and define a map $\alpha^\dagger$ of M into itself as follows:

$$x\alpha^\dagger = \begin{cases} x\alpha, & \text{if } x \in S; \\ x, & \text{if } x \in R. \end{cases}$$

Since p and 0_S are kept fixed by α , it is evident that $\alpha^\dagger$ is an automorphism of M . Moreover, q and p is the only pair of elements in M satisfying $q \prec p$ and $M = (p] \cup [q]$; therefore, every automorphism of M acts as an automorphism on S and R . It is now clear that every automorphism of M is of the form $\alpha^\dagger$ for some automorphism α of S , hence, $\alpha \rightarrow \alpha^\dagger$ is an isomorphism between the automorphism group of S and the automorphism group of M . It follows that the automorphism group of M is isomorphic to $\mathfrak{G}$.

This completes the proof of the Independence Theorem for Modular Lattices.

5. ARGUESIAN LATTICES

The arguesian identity (see, e.g., [5, p. 260]) is very much stronger than the modular identity; it is—for projective geometries—the lattice theoretic form of Desargues' Theorem. So it is not surprising that it is more complicated to construct an arguesian lattice than a modular lattice. In this section, we outline how to modify the lattice constructions of this paper to obtain an arguesian lattice.

Let $\mathbf{V}_2$ denote the quasivariety of lattices that can be embedded in a projective geometry (identified with its subspace lattice) over the two-element field. Of course, every lattice in $\mathbf{V}_2$ is arguesian but not conversely.

We shall prove the following:

Independence Theorem for $\mathbf{V}_2$. *Let $\mathfrak{D}$ be a finite distributive lattice with more than one element and let $\mathfrak{G}$ be a finite group. Then there exists a lattice $A \in \mathbf{V}_2$ such that the congruence lattice of A is isomorphic to $\mathfrak{D}$ and the automorphism group of A is isomorphic to $\mathfrak{G}$.*

First, we state a stronger form of Theorem 1:

Theorem 1'. *Let $\mathfrak{G}$ be a finite group. Then there exists a simple lattice $S \in \mathbf{V}_2$ with zero 0_S and an atom p such that the automorphism group of S is isomorphic to $\mathfrak{G}$ and every automorphism keeps p fixed.*

This result was proved in [8].

Later in this section, we shall need a variant of this result, namely that there is an $S \in \mathbf{V}_2$ satisfying the changed requirement that the dimension of p over 0_S is $\aleph_0$. This is easily done. Let S be a sublattice of the projective geometry V over the two-element field; just take another projective geometry V' over the two-element field that contains $\aleph_0$ copies of every point in V . Then V is a sublattice of V' in the natural way; therefore, so is S , and the additional requirement for this embedding is obvious.

Second, we need the following stronger form of Theorem 2:

Theorem 2'. *Let $\mathfrak{D}$ be a finite distributive lattice with more than one element. There exists a rigid lattice $R \in \mathbf{V}_2$, with unit 1_R and a dual atom q such that the congruence lattice of R is isomorphic to $\mathfrak{D}$ and the dimension of 1_R over q is $\aleph_0$.*

Proof. The lattice R constructed in Theorem 3 could be described as follows: R is built on the direct square of a countable well-ordered chain in which some “squares” $[\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle]$ satisfying $[a_1, a_2] \cong [b_1, b_2]$ are “replaced” by $M_3[[a_1, a_2]]$.

The embedding of R into $\mathbf{V}_2$ is based on the following lemma:

Lemma 5. *Let V be a vector space, let $a, b \in V$ be of dimension $\aleph_0$ over $a \wedge b$. Then there is a $c \in V$ such that $\{a \wedge b, a, b, c, a \vee b\}$ form a sublattice isomorphic to M_3 . Let C be a chain in $[a \wedge b, a]$ with $a \wedge b, a \in C$. Then $\{a \wedge b, a, b, c, a \vee b\}$ and C generate a sublattice in V isomorphic to $M_3[C]$.*

Proof. Let $\{u_1, u_2, \dots\}$ be a basis of a over $a \wedge b$, and $\{v_1, v_2, \dots\}$ be a basis of b over $a \wedge b$. We define $c \in V$ as the subspace generated by $a \wedge b$ and

$$\{u_1 + v_1, u_2 + v_2, \dots\}.$$

It is now clear that $\{a \wedge b, a, b, c, a \vee b\}$ forms an M_3 . The second statement follows now by a reference to statement 7 of Lemma 2. $\square$

Now to embed R into $\mathbf{V}_2$, take a projective geometry V with a countably infinite base B over the two-element field. Split B into two disjoint countably infinite sets $B = B_1 \cup B_2$, and map 0_R into the subspace generated by B_1 . Now we split B_2 into two collections of disjoint countably infinite sets $\{X_i \mid i \in \alpha_P\}$ and $\{Y_i \mid i \in \alpha_P\}$ and represent α_P with the two chains: $x \in \alpha_P$ is mapped into the subspace generated by $B_1 \cup \bigcup (X_i \mid i < x)$ in the first chain, and $x \in \alpha_P$ is mapped into the subspace generated by $B_1 \cup \bigcup (Y_i \mid i < y)$ in the second chain.

Whenever we have to replace $[\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle]$ satisfying $[a_1, a_2] \cong [b_1, b_2]$ by $M_3[[a_1, a_2]]$, we apply Lemma 5. This embeds R into V . $\square$

To prove the Independence Theorem for $\mathbf{V}_2$, represent the lattice R of Theorem 2' with the additional requirement that the dimension of 1_R over q is $\aleph_0$. Now represent the lattice S of Theorem 1' by mapping 0_S into the subspace representing q and by mapping p into the subspace representing 1_R , and then proceed to represent S as in the proof of Theorem 1'.

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