CONGRUENCE-PRESERVING EXTENSIONS OF FINITE LATTICES TO SEMIMODULAR LATTICES

G. GRÄTZER AND E.T. SCHMIDT

Abstract. We prove that every finite lattice has a congruence-preserving extension to a finite *semimodular* lattice.

1. Introduction

A classical result of R.P. Dilworth (see [1]) states:

Theorem 1. *Let D be a finite distributive lattice. Then there exists a finite lattice L such that the congruence lattice of L is isomorphic to D.*

A number of papers have appeared (see the References) strengthening this result by requiring that L have special properties. The most recent is [9]:

Theorem 2. *Let D be a finite distributive lattice. Then there exists a finite* semimodular *lattice S such that the congruence lattice of S is isomorphic to D.*

Let L be a lattice. If K is a sublattice of L, we call L an *extension* of K. We call L a *congruence-preserving extension* of K iff every congruence of K has *exactly one* extension to L. In this case, the congruence lattice of K is isomorphic to the congruence lattice of L; in formula, $\operatorname{Con} K \cong \operatorname{Con} L$.

The first important result on congruence-preserving extensions is due to M. Tischendorf [14]:

Theorem 3. *Every finite lattice has a congruence-preserving embedding to a finite* atomistic *lattice.*

In [12], the present authors proved the following much sharper result:

Theorem 4. *Every finite lattice has a congruence-preserving embedding into a finite* sectionally complemented *lattice.*

In this paper, we prove the following:

Theorem. *Every finite lattice K has a congruence-preserving embedding into a finite* semimodular *lattice L.*

We would like to thank the referee for his very thoughtful comments.

Date: January 10, 2000.

2000 *Mathematics Subject Classification.* Primary 06B10; Secondary 06C10.

Key words and phrases. Congruence lattice, congruence-preserving embedding, semimodular, finite.

The research of the first author was supported by the NSERC of Canada.

The research of the second author was supported by the Hungarian National Foundation for Scientific Research, under Grant No. T023186.

2. Preliminaries

We use the basic concepts and notations as in [3]; in particular, for a finite distributive lattice D, let $\mathrm{J}(D)$ and $\mathrm{M}(D)$ denote the poset of join-irreducible and the poset of meet-irreducible elements, respectively. $\mathfrak{M}_3$ denotes the five-element modular nondistributive lattice and $\mathfrak{C}_2$ the two-element chain.

We assume that the reader is familiar with the concept of *gluing*, first introduced by R.P. Dilworth; see, for instance, [1].

If L is an extension of K, Θ is a congruence of K, and Φ is a congruence of L, then Φ is an *extension* of Θ to L iff the restriction of Φ to K equals Θ. We say that K in L has the *Congruence Extension Property* iff every congruence of K has an extension to L. Note that if L is a congruence-preserving extension of K, then K has the Congruence Extension Property in L, but not conversely (a congruence may have many extensions).

Let D be a finite distributive lattice, and let $m \in \mathrm{M}(D)$. Then $(m]$ is a prime ideal of D, hence $D - (m]$ is a prime dual ideal; let $m^\dagger$ denote the generator of $D - (m]$. Equivalently, $m^\dagger$ is the smallest element of D not contained in m. Note that $m^\dagger \in \mathrm{J}(D)$. The following result is well-known:

Lemma 1. *The map $m \to m^\dagger$ is a natural bijection between $\mathrm{M}(D)$ and $\mathrm{J}(D)$. For any $d \in D$ and $m \in \mathrm{M}(D)$, the inequality $m^\dagger \leq d$ is equivalent to $d \nleq m$.*

We need a more detailed version of Theorem 2 (see [9]):

Theorem 5. *Let D be a finite distributive lattice. Then there exists a finite* semimodular *lattice S with the following two properties:*

(i) *The congruence lattice of S is isomorphic to D.*
(ii) *S has an ideal C, which is a chain, such that every join-irreducible congruence of L is generated by a prime interval of C.*

As an illustration, if D is the distributive lattice of Figure 1, then $\mathrm{J}(D)$ is the poset of Figure 2; S is shown in Figure 3, where the elements of the ideal C are black filled.

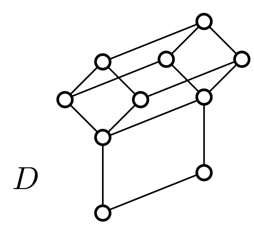

Figure 1

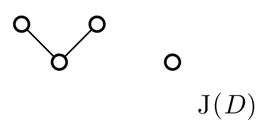

Figure 2

Observe that every prime interval of S is projective to a prime interval of C, so every congruence of S is determined by its restriction to C. However, S is not a congruence-preserving extension of C.

3. A modular lattice

In this section, we construct a modular lattice that will be used in the proof of the Theorem.

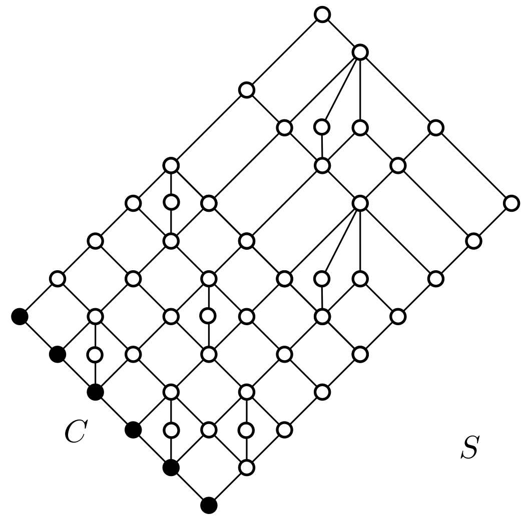

Figure 3

Let $n \geq 1$ be a natural number, and for every i with $1 \leq i \leq n$, we take a copy M_i of $\mathfrak{M}_3$, with atoms p_i, q_i, and r_i. We form $\mathfrak{M}_3^n$ and regard M_i as an ideal of $\mathfrak{M}_3^n$, so p_i, q_i, and r_i are atoms of $\mathfrak{M}_3^n$, for $1 \leq i \leq n$.

Let B be the sublattice of $\mathfrak{M}_3^n$ generated by $\{\, p_i \mid 1 \leq i \leq n \,\}$. Obviously, B is a 2^n element Boolean lattice, an ideal of $\mathfrak{M}_3^n$.

Define

$$\overline{q}_i = \bigvee (\, q_j \mid 1 \leq j \leq i \,),$$

for $1 \leq i \leq n$, and set $E = \{\, \overline{q}_i \mid 0 \leq i \leq n \,\}$, where $\overline{q}_0$ is the zero of $\mathfrak{M}_3^n$. Obviously, E is a maximal chain (of length n) in the ideal $(\overline{q}_n]$ of $\mathfrak{M}_3^n$.

Lemma 2. *The sublattice A of $\mathfrak{M}_3^n$ generated by B and E is isomorphic to $B \times E$ under the isomorphism*

$$b \vee e \to \langle b, e \rangle.$$

Proof. It is easy to prove this directly or to derive this from the results in Section IV.1 of [3] (in particular, Theorems 11–14). ☐

Let $b \in B$ and let i satisfy the conditions: $p_{i+1} \nleq b$ and $0 \leq i < n$. Define the element of $\mathfrak{M}_3^n$:

$$r(b, i) = b \vee \overline{q}_i \vee r_{i+1},$$

and the subset M of $\mathfrak{M}_3^n$:

$$M = A \cup \{\, r(b, i) \mid b \in B,\ 0 \leq i < n,\ p_{i+1} \nleq b \,\}.$$

M is a sublattice of $\mathfrak{M}_3^n$, hence M is a modular lattice. M contains B and E as ideals.

Let Θ be a congruence of B. Let Θ^E be the congruence on E satisfying: $\overline{q}_i \equiv \overline{q}_{i+1}$ (Θ^E) in E iff $p_{i+1} \equiv 0$ (Θ) in B, for $0 \leq i < n$. Then $\Theta \times \Theta^E$ is a congruence on $B \times E$. We extend this to a congruence Θ^M of M as follows: let $r(b, i)$ be defined (that is, $b \in B$ and $p_{i+1} \nleq b$); if $b \equiv b \vee p_{i+1}$ (Θ) in B, then $r(b, i) \in [b \vee \overline{q}_i]\Theta^M$, otherwise, $\{r(b, i)\}$ is a singleton congruence class.

The map $\Theta \to \Theta^M$ is an isomorphism between $\operatorname{Con} B$ and $\operatorname{Con} M$. In fact, M is a congruence-preserving extension of both B and E. Let

$$E' = \{\, 1_B \vee e \mid e \in E \,\},$$

where 1_B is the unit element of B. Obviously, E and E' are isomorphic chains and E' is a dual ideal of M.

To summarize:

Lemma 3. *For each $n \geq 1$, $\mathfrak{M}_3^n$ has sublattices B, E', and M satisfying the following conditions:*

(i) *B is an ideal of M and it is isomorphic to the Boolean lattice $\mathfrak{C}_2^n$.*
(ii) *E' is a dual ideal of M and it is a chain of length n.*
(iii) *M is a congruence-preserving extension of both B and E'.*

Note that $\operatorname{Con} M$ is a Boolean lattice and $\operatorname{Con} M \cong \operatorname{Con} B \cong \operatorname{Con} E'$.

4. Proving the Theorem

We are given a finite lattice K. In this section, let k denote the number of join-irreducible congruences of K, that is, $k = |\mathrm{J}(\operatorname{Con} K)|$. Of course, we also have that $k = |\mathrm{M}(\operatorname{Con} K)|$.

To prove the Theorem, we have to construct a semimodular congruence-preserving extension L. We glue L together from three lattices, sketched in Figure 4.

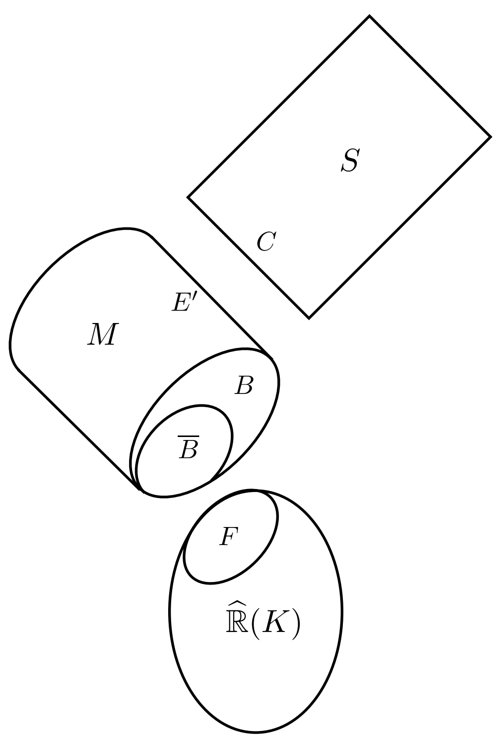

Figure 4

The first lattice is S of Section 2 with the ideal C (which is a chain), constructed so that $\operatorname{Con} S$ be isomorphic to $\operatorname{Con} K$. In this section, let n denote the length of the chain C; obviously, $k \leq n$.

The second lattice is M of Section 3, with the ideal B (which is Boolean) and the dual ideal E' (which is a chain), constructed with the integer n (the length of the chain C).

The third lattice is $\widehat{\mathbb{R}}(K)$ that was constructed in [12]:

Lemma 4. *Let K be a finite lattice. Then K has an extension to a finite, semimodular, sectionally complemented lattice $\widehat{\mathbb{R}}(K)$ with the following properties:*

(i) *Each congruence Θ of K has an extension to a congruence $\widehat{\Theta}$ of $\widehat{\mathbb{R}}(K)$.*

(ii) *$\widehat{\mathbb{R}}(K)$ has a dual ideal F with the set of dual atoms*

$$V = \{\, v_\Psi \mid \Psi \in \mathrm{M}(\operatorname{Con} K)\,\}.$$

F is a Boolean lattice isomorphic to $\mathfrak{C}_2^k$.

(iii) *For each congruence Θ of K, the extension $\widehat{\Theta}$ is generated by collapsing*

$$V_\Theta = \{\, v_\Psi \mid \Theta \nleq \Psi\,\} \subseteq V$$

to the unit 1_F of F.

(iv) *For each congruence Θ of K, the extension $\widehat{\Theta}$ is generated by collapsing the set of atoms $\{\, v'_\Psi \mid \Psi^\dagger \leq \Theta\,\}$ to the zero 0_F of F.*

(v) *$\mathbb{R}(K)$ is a congruence-preserving extension of F.*

To prove this lemma, argue as in Sections 3 and 4 of [12] (using P. Pudlák and J. Tůma [13] to embed each factor of $\mathbb{R}(K)$ into a finite partition lattice, which is semimodular.[1] Note that $\widehat{\mathbb{R}}(K)$ is a direct product of k simple (partition) lattices, so $\operatorname{Con}\widehat{\mathbb{R}}(K) \cong \mathfrak{C}_2^k$; moreover, V picks out one element from each "direct factor" (in a dual sense), and so it is clear that $\mathbb{R}(K)$ is a congruence-preserving extension of F, which is stated as (v).

First step. We glue M and S together over E' and C to obtain the lattice T. Note that the chains E' and C are of the same length, hence there is a unique isomorphism between them.

Let $\Theta \to \Theta^S$ be an isomorphism between $\operatorname{Con} K$ and $\operatorname{Con} S$. Since M is a congruence-preserving extension of E' and every congruence of S is determined by its restriction to C, we conclude that Θ^S has a unique extension Θ^T to T, for every $\Theta \in \operatorname{Con} K$. So $\Theta \to \Theta^T$ is an isomorphism between $\operatorname{Con} K$ and $\operatorname{Con} S$.

Since M is a congruence-preserving extension of B, every congruence Θ^T is determined by its restriction to B. Since B is Boolean, the join-irreducible congruences of T are exactly the congruences of the form $\Theta(0_B, p)$, for an atom p of B, where 0_B denotes the zero of B. Thus for every $\Phi \in \mathrm{J}(\operatorname{Con} K)$, we can pick an atom p_Φ of B so that there is a bijection $\Phi \mapsto p_\Phi$, $\Phi \in \mathrm{J}(\operatorname{Con} K)$, between $\mathrm{J}(\operatorname{Con} K)$ and the set of atoms:

$$U = \{\, p_\Phi \mid \Phi \in \mathrm{J}(\operatorname{Con} K)\,\}.$$

Let $\overline{B}$ denote the sublattice of B generated by U; of course, $\overline{B} \cong \mathfrak{C}_2^k$ and $\overline{B}$ is an ideal of B.

[1] We would like to point out that instead of the very complicated Pudlák-Tůma result, one can use a more accessible result of R.P. Dilworth to obtain a semimodular $\widehat{\mathbb{R}}(K)$; see [1] and [2].

In T, there is a one-to-one correspondence between congruences and certain subsets of U: If Θ is a congruence of K, then the subset that corresponds to the congruence Θ^T of T is

$$U_\Theta = \{\, p_\Phi \mid \Phi \in \mathrm{J}(\operatorname{Con} K) \text{ and } \Phi \leq \Theta \,\}.$$

T is obtained by gluing together two semimodular lattices, hence it is semimodular.

Second step. The lattice $\overline{B}$ of the first step and the lattice F of Lemma 4.(ii) are isomorphic; to do the second gluing, we have to find an appropriate isomorphism.

Lemma 5. *The map*

$$p_\Phi \to v'_{\Phi^\dagger}$$

(*that is, the image of* p_Φ *is the complement of* $v_{\Phi^\dagger}$ *in* F) *defines an isomorphism* α *between* $\overline{B}$ *and* F*. For every congruence* Θ *of* K*, the restriction of* Θ^T *to* $\overline{B}$ *maps by* α *to the restriction of* $\widehat{\Theta}$ *to* F*.*

Proof. By Lemma 1, α is a bijection between the atoms of $\overline{B}$ and the atoms of F, hence, α defines an isomorphism between $\overline{B}$ and F. For a congruence Θ of K, the restriction of Θ^T to $\overline{B}$ is the congruence of $\overline{B}$ obtained by collapsing U_Θ to $0_{\overline{B}}$ $(= 0_B)$. So the α image of this restriction is the congruence of F obtained by collapsing $U_\Theta\alpha$ to 0_F. Now compute:

$$\begin{aligned}
U_\Theta\alpha &= \{\, p_\Phi \mid \Phi \in \mathrm{J}(\operatorname{Con} K) \text{ and } \Phi \leq \Theta \,\}\alpha \\
&= \{\, v'_{\Phi^\dagger} \mid \Phi \in \mathrm{J}(\operatorname{Con} K) \text{ and } \Phi \leq \Theta \,\} \\
&= \{\, v'_\Psi \mid \Psi \in \mathrm{M}(\operatorname{Con} K) \text{ and } \Theta \nleq \Psi \,\} \\
&= V'_\Theta.
\end{aligned}$$

So the α image of the restriction of Θ^T to $\overline{B}$ is the congruence of F collapsing v'_Θ to 0_F, or equivalently, collapsing v_Θ to 1_F, which is the restriction of the congruence $\widehat{\Theta}$ of $\widehat{\mathbb{R}}(K)$ to F. $\square$

Now we glue T and $\widehat{\mathbb{R}}(K)$ together over $\overline{B}$ and F, as identified by the isomorphism α. L is obtained by gluing together two semimodular lattices, hence it is semimodular.

Let Θ be a congruence relation of K. Then Θ^T is a congruence relation of T and $\widehat{\Theta}$ is a congruence of $\widehat{\mathbb{R}}(K)$. By Lemma 5, the restriction of Θ^T to $\overline{B}$ maps by α to the restriction of $\widehat{\Theta}$ to F. Therefore, Θ^T and $\widehat{\Theta}$ define a congruence Θ^L on L. Every congruence of T is of the form Θ^T, for some congruence Θ of K; moreover, by Lemma 4.(v), every congruence of $\widehat{\mathbb{R}}(K)$ is a unique extension of a congruence of F. These two facts combine to show that $\Theta \to \Theta^L$ is an isomorphism between $\operatorname{Con} K$ and $\operatorname{Con} L$. Moreover, Θ^L is an extension of the congruence Θ of K regarded as a sublattice of $\widehat{\mathbb{R}}(K)$, so K in L has the Congruence Extension Property. Finally, since every congruence of L is of the form Θ^L, for some congruence Θ of K, we conclude that L is a congruence-preserving extension of K, completing the proof of the Theorem.

References

[1] K.P. Bogart, R. Freese, and J.P.S. Kung (editors), *The Dilworth Theorems. Selected papers of Robert P. Dilworth*, Birkhäuser Verlag, Basel-Boston, 1990, pp. 460–464.

[2] P. Crawley and R.P. Dilworth, *Algebraic Theory of Lattices.* Prentice-Hall, Englewood Cliffs, N. J., 1973.

[3] G. Grätzer, *General Lattice Theory. Second edition*, Birkhäuser Verlag, Basel. 1998. xix+663 pp.

[4] G. Grätzer and H. Lakser, *Homomorphisms of distributive lattices as restrictions of congruences*, Canad. J. Math. **38** (1986), 1122–1134.

[5] ———, *Congruence lattices of planar lattices*, Acta Math. Hungar. **60** (1992), 251–268.

[6] G. Grätzer, H. Lakser, and E.T. Schmidt, *Congruence lattices of small planar lattices*, Proc. Amer. Math. Soc. **123** (1995), 2619–2623.

[7] ———, *Isotone maps as maps of congruences. I. Abstract maps*, Acta Math. Acad. Sci. Hungar. **75** (1997), 105–135.

[8] ———, *Congruence representations of join homomorphisms of distributive lattices: A short proof*, Math. Slovaca **46** (1996), 363–369.

[9] ———, *Congruence lattices of finite semimodular lattices*, Canad. Math. Bull. **41** (1998), 290–297.

[10] G. Grätzer, I. Rival, and N. Zaguia, *Small representations of finite distributive lattices as congruence lattices*, Proc. Amer. Math. Soc. **123** (1995), 1959–1961.

[11] G. Grätzer and E.T. Schmidt, *On congruence lattices of lattices*, Acta Math. Acad. Sci. Hungar. **13** (1962), 179–185.

[12] ———, *Congruence-preserving extensions of finite lattices to sectionally complemented lattices*, Proc. Amer. Math. Soc. **127** (1999), 1903–1915.

[13] P. Pudlák and J. Tůma, *Every finite lattice can be embedded into a finite partition lattice,* Algebra Universalis **10** (1980), 74–95.

[14] M. Tischendorf, *The representation problem for algebraic distributive lattices*, Ph.D. thesis, Fachbereich Mathematik der Technischen Hochschule Darmstadt, Darmstadt, 1992.

Department of Mathematics, University of Manitoba, Winnipeg, MN R3T 2N2, Canada
E-mail address: `gratzer@cc.umanitoba.ca`
URL: `http://www.maths.umanitoba.ca/homepages/gratzer/`

Mathematical Institute of the Technical University of Budapest, Műegyetem rkp. 3, H-1521 Budapest, Hungary
E-mail address: `schmidt@math.bme.hu`
URL: `http://www.bme.math/~schmidt/`