ON AUTOMORPHISM GROUPS OF SIMPLE ARGUESIAN LATTICES

E. T. SCHMIDT

Abstract. Let G be a group. In this paper we prove that there exists a simple arguesian lattice M whose automorphism group is isomorphic to G.

A lattice L is called interval finite, if every interval of L is finite. In this note we give a new proof of a theorem of Christian Herrmann [3]. This theorem was proved by G. Grätzer and E. T. Schmidt [2] for finite groups and later by Christian Herrmann [3] in the present form.

Theorem. *Every group G can be represented as the automorphism group of an interval finite, simple, arguesian lattice M.*

Let G be a given group. By R. Frucht [1], there exists an undirected graph $\langle V; E\rangle$ with no loops whose automorphism group is isomorphic to G (that is, V is a set and the set E of edges is a subset of two-elements subsets of V). We begin our construction with this graph.

We consider first a vector space $\mathbf{V}$ over the two element field Z_2 with a basis V'. We assume that V and V' have the same cardinality, i.e. $|V| = |V'|$. Then we can identify the vertices of the graph with the basis elements of this vector space, that means, we can consider the elements $v_0; v_1; v_2; \ldots$ of V as the basis elements of the vector space $\mathbf{V}$. Let A be the lattice of all finitely generated subspaces of the vector space $\mathbf{V}$. This lattice A is obviously a simple, atomistic, arguesian lattice. The vector space $\mathbf{V}$ is over the two element field Z_2, consequently every line contains three points. The subspace generated by v_i will be denoted by the same letter v_i. The lattice A has the following three types of atoms:

1. The atoms v_i, $i \in I$ (i.e. the elements of the basis), these form the set V and I an arbitrary index set;
2. Consider the third point $v_i + v_j$ ($i; j \in I$) of the line $\overline{v_i; v_j}$ spanned by v_i and v_j. Some of these $v_i + v_j$-s correspond to edges of the graph (i.e. $\{v_i; v_j\}$ is a edge), in this case $v_i + v_j$ will be denoted by v_{ij}. All these atoms form a subset W;
3. All other atoms.

We consider the given G as a subgroup of the automorphism group of A To the vertices of the Frucht graph correspond the atoms $v_i \in V; i \in I$ and to the edges $\{v_i; v_j\}$ correspond the atoms v_{ij}, these determine the edges in V. Obviously, every permutation of the v_i-s can be extended to an automorphism of A and every

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automorphism of A is determined by its restriction to the basis V. Indeed, if ® and ¯ are two automorphisms of A such that their restrictions to V are the same, then the restriction of ° = ®¯ ¡ 1 is the identity map " of V. By any extension of " (i.e. automorphism with the property that its restriction to V is ") the atoms v_i and v_j are …xed elements, conseqently $v_i + v_j$ must be …xed. Similarly, $(v_i + v_j) + v_k$ must be a …xd element. In this way we get that by an extension of " all atoms are …xed elements which means that this extension is the identity mapping of A. It follows that all automorphisms with the property that V and W are invariant form a group isomorphic to G. To ensure that we have no more automorphisms than the graph we must label the vertices and the edges, i.e the atoms v_i 2 V and v_{ij} 2 W. This will be done by lattices which are glued to A. The idea of the gluing is the following. The ideal $(v_i]$ of A has two elements. We will de…ne a special lattice F_1 with a two element dual ideal D_1 which is therefore isomorphic to $(v_i]$. Similarly, for every v_{ij} 2 W we use a lattice F_2 with the dual ideal D_2. For every i 2 I we consider an isomorphic copy ${F_1}^i$ of F_1 with the dual ideal ${D_1}^i$ and similarly the lattices ${F_2}^{ij} \cong F_2$ with the dual ideal ${D_2}^{ij}$. We can apply the gluing construction for the lattices A, ${F_1}^i$ and ${F_2}^{ij}$ simultaneously, identifying the ideal $(v_i]$ with ${D_1}^i$ and $(v_{ij}]$ with ${D_2}^{ij}$. On this way we get a join-semilattice and M is the arguesian lattice generated by this con…guration. First we de…ne the lattices F_1, F_2. We give the description of M as a sublattice of a vectorspace lattice and prove that this is a simple arguesian lattice with the given automorphism group.

N is the chain of all nonnegative integers and N¤ denotes the chain of the nonpositive integers. Take the direct product C_2 £ N¤, (where C_2 denotes the two element lattice). In this direct product for every i 2 N the elements (0; ¡ i ¡ 1); (1; ¡ i ¡ 1); (0; ¡ i); (1; ¡ i) form a “covering square” (isomorphc to C_2 £ C_2). Into these “covering squares”, for i = 0; 1; : : : we insert one more elemwnt z_i so that a copy of M_3, the …ve element non distributive modular lattice, is obtained. The resulting lattice is F_1, see Figure 1a. The lattice F_2 is similar but we don’t insert z_0, into the …rst “covering square” , see Figure 1b. The dual ideal consisting of (0; 0) and (1; 0) etc. of F_1 is D_1. We use isomorphic copies of F_1 and F_2 to label the v_i-s and the v_{ij}-s.

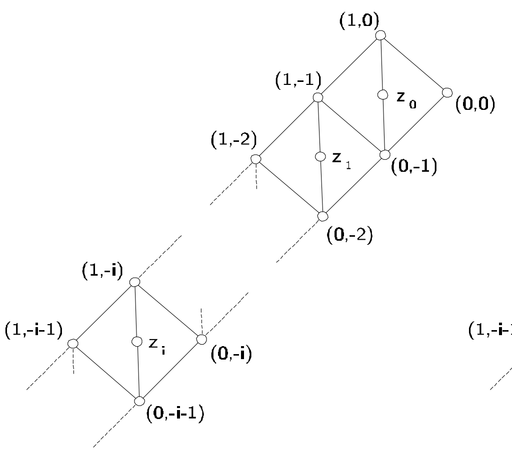

Figure 1a

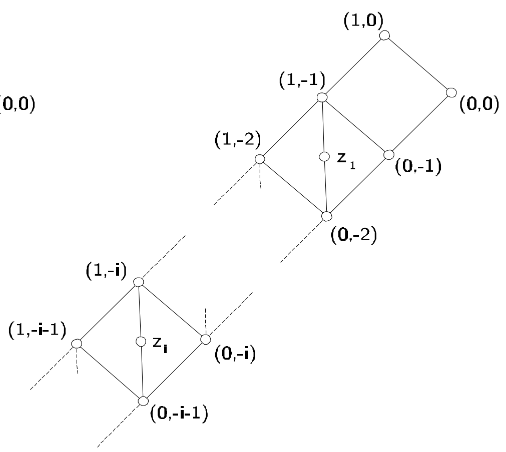

Figure 1b

F_1 is a simple arguesian lattice and it has exactly one nontrivial automorphism ®, where ®(z_0) = (0; 0) and ®(0; 0) = (z_0). F_2 is a rigid (has no nontrivial automorphism) arguesian lattice, its congruence lattice is the four element Boolean lattice.

We de…ne our lattice M as a sublattice of a vectorspace lattice K = L(W) of a vectorspace W over Z_2. Take the set $fu_j{}^k; v_j; j\ 2\ I; k\ 2\ Ng$ as a basis of W. Let $z_j{}^k$ be the third point of the line spanned by $u_j{}^k$ and v_j. De…ne the following subspaces, (where [X] denotes the subspace spanned by the set X): $o = [u_j{}^k; j\ 2\ I; k\ 2\ N]; v_i = [v_i; u_j{}^k; j\ 2\ I; k\ 2\ N] = [v_i; o]$. The convex sublattice of K, generated by (as lattice) v_i-s form a sublattice isomorphic to A, we identify A with this sublattice.

Set $u_i{}^0 = o; u_i{}^1 = [u_j{}^k; j\ 2\ I; k\ 2\ N; u_j{}^k \neq u_i{}^0]; u_i{}^2 = [u_j{}^k; i\ 2\ I; k\ 2\ N; u_j{}^k \neq u_i{}^0; u_i{}^1] :::$. Then $u_i{}^0 > u_i{}^1 > u_i{}^2 > :::$ is a a chain of type $!^¤$. The convex sublattice generated by these chains will be denoted by C. Take the sublattice A [C, then A is a dual ideal and C is an ideal of this lattice. We adjoin further elements $w_i{}^0; w_i{}^1; w_i{}^2; :::$ and $z_i{}^1; z_i{}^2; z_i{}^3 :::$, which are de…ned as follows:.

$w_i{}^1 = [u_i{}^1; v_i]; w_i{}^2 = [u_k{}^2; v_i]; w_i{}^3 = [u_k{}^3; v_i] :::$

and

$z_i{}^1 = [u_i{}^1; z_i{}^1]; z_i{}^2 = [u_i{}^2; z_i{}^2]; z_i{}^3 = [u_i{}^3; z_i{}^3] :::$.

Then the join of the chains $u_i{}^0 > u_i{}^1 > u_i{}^2 > :::$ and $w_i{}^0 > w_i{}^1 > w_i{}^2 > :::$ form a sublattice isomorphic to $C_2 £ N^¤$. For every j, $u_i{}^j; z_i{}^{j+1}$ and $w_i{}^{j+1}$ generete M_3. For every i 2 I all these elements form a sublattice, the ‡ap

$F_1{}^i = fu_i{}^0; u_i{}^1; u_i{}^2 :::g\ [\ fw_i{}^0; w_i{}^1; w_i{}^2 :::g\ [\ fz_i{}^1; z_i{}^2; z_i{}^3 :::g$ isomorphic to the lattice F_1.

Similarly, we de…ne for the elements v_{ij} the ‡aps

$F_2{}^{ij} = fu_{ij}{}^0; u_{ij}{}^1; u_{ij}{}^2 :::g\ [\ fw_{ij}{}^0; w_{ij}{}^1; w_{ij}{}^2 :::g\ [\ fz_{ij}{}^2; z_{ij}{}^3; z_{ij}{}^4 :::g$ isomorphic to F_2.

Let M be A [C [$\bigcup (F_1{}^i; F_2{}^{ij}\ j\ i; j\ 2\ I)$.

M can be vizulised as follows:

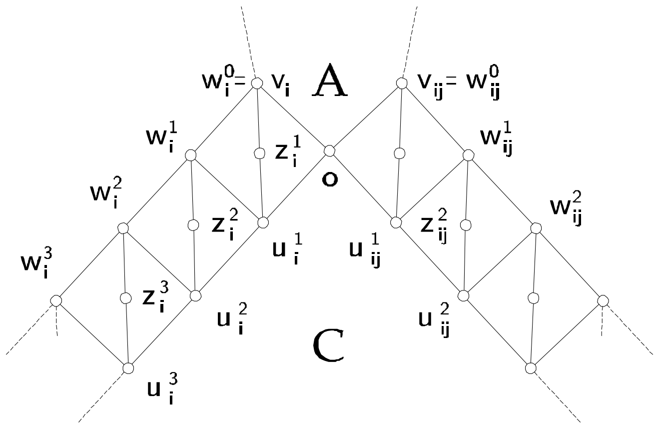

Figure 2

It is easy to see that M is a sublattice of K. The lattice K is an arguesian lattice, consequently M is again arguesian. We prove that.M is simple. We know that A and the $F_1^{\,i}$-s are simple lattices and the intervals $[u_i^k, u_i^{k+1}]$ and $[u_j^k, u_j^{k+1}]$ resp. $[u_{ij}^k, u_{ij}^{k+1}]$ and $[u_j^k, u_j^{k+1}]$ are projective in C. These imply that any two prime intervals are projective, which proves that M is a simple lattice.

M contains the chains $w_i^1 > w_i^2 > w_i^3 > \dots$ and $w_{ij}^0 > w_{ij}^1 > w_{ij}^2 \dots$, where $w_i^1, w_i^2, \dots$ resp. $w_{ij}^1, w_{ij}^2 \dots$, $(i, j \in I)$ are meet irreducible elements.and M has no other chains of this type. Then for any automorphism the image of w_i^1 must be w_j^1 for some j and similarly the image of u_{ij}^1 is some u_{kl}^1. This yields that the restriction of an automorphism to the atoms of the dual ideal A of M is a permutation, where V and W are invariant. This proves that the automorphim group of M is isomorphic to G.

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Mathematical Institute of the Technical University of Budapest, Műegyetem rkp. 3, H-1521 Budapest, Hungary

E-mail address: schmidt@math.bme.hu

URL: http://www.bme.math/~schmidt/