CONGRUENCE LATTICES OF FINITE SEMIMODULAR LATTICES

G. GRÄTZER, H. LAKSER, AND E. T. SCHMIDT

Abstract. We prove that every finite distributive lattice can be represented as the congruence lattice of a finite (planar) *semimodular* lattice.

1. Introduction

A classical result of R. P. Dilworth (*circa* 1940, unpublished, see [1], pp. 455–457) states that a finite distributive lattice D can be represented as the congruence lattice of a finite lattice L.

There are a number of papers strengthening this result by requiring that the lattice L representing D have special properties. The lattice L constructed by Dilworth is *atomistic*. A *sectionally complemented* lattice L is constructed in G. Grätzer and E. T. Schmidt [7], while a *planar* lattice is constructed in G. Grätzer and H. Lakser [4]. A "small" lattice L is constructed in G. Grätzer, H. Lakser, and E. T. Schmidt [5]: if D has n join-irreducible elements, the lattice L is of size $O(n^2)$. (This is "best possible", according to G. Grätzer, I. Rival, and N. Zaguia [6].)

In this paper, we construct a *semimodular* lattice L:

Theorem. *Every finite distributive lattice D can be represented as the congruence lattice of a finite* semimodular *lattice S. In fact, S can be constructed as a* planar *lattice of size $O(n^3)$, where n is the number of join-irreducible elements of D.*

This result, with size $O(n^4)$, was announced in [9]; the present paper contains an improved construction, due to the second author, yielding size $O(n^3)$. It would be interesting to decide whether the size $O(n^2)$ is possible for (planar) semimodular lattices.

2. Preliminaries

We use the basic concepts and notations as in [2]; in particular, for a finite distributive lattice D, $\mathrm{J}(D)$ denotes the poset of join-irreducible elements. $\mathrm{Con}\, L$ denotes the congruence lattice of the lattice L. For a prime interval $\mathfrak{p} = [a, b]$, $\Theta(\mathfrak{p}) = \Theta(a, b)$ is the smallest congruence collapsing a and b. $\mathfrak{C}_2$ denotes the two-element chain.

It is convenient to describe congruences of a finite lattice using coloring:

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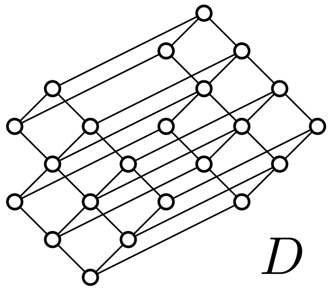

Figure 1

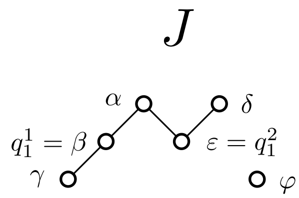

Figure 2

Let L be a finite lattice and let Γ be a finite set; the elements of Γ will be called *colors*. A *coloring* μ of L over Γ is a map

$$\mu\colon \mathfrak{P}(L) \to \Gamma$$

of the set of prime intervals $\mathfrak{P}(L)$ of L into Γ satisfying the condition: if two prime intervals generate the same congruence relation of L, then they have the same color; that is,

$$\mathfrak{p},\ \mathfrak{q} \in \mathfrak{P}(L) \text{ and } \Theta(\mathfrak{p}) = \Theta(\mathfrak{q}) \quad \text{imply that} \quad \mathfrak{p}\mu = \mathfrak{q}\mu.$$

Since the join-irreducible congruences of L are exactly those that can be generated by prime intervals, equivalently, μ can be regarded as a map of the set $\mathrm{J}(\mathrm{Con}\, L)$ of join-irreducible congruences of L into Γ:

$$\mu\colon \mathrm{J}(\mathrm{Con}\, L) \to \Gamma.$$

In view of this condition, it is enough to define μ on sufficiently many prime intervals so that every prime interval is projective to one on which μ is defined.

Let A and B be lattices, D_A a dual ideal of A, I_B an ideal of B, and D_B a dual ideal of B. Let us assume that D_A, I_B, and D_B are isomorphic. We now define what it means that we obtain C *by gluing B to A k-times*. For $k = 1$, let C be the gluing of A and B over D_A and I_B with the dual ideal D_B regarded as a dual ideal D_C of C. Now if C_{k-1} with the dual ideal $D_{C_{k-1}}$ is the gluing of B to A $k-1$-times, then we glue C_{k-1} and B over $D_{C_{k-1}}$ and I_B to obtain C the gluing of B to A k-times with the dual ideal D_B regarded as a dual ideal D_C of C. Observe that if A and B are semimodular, then so is C. Since we construct the lattice S of the Theorem from semimodular components using gluing, the semimodularity of S follows.

3. The construction

We construct the semimodular lattice S of the Theorem in several steps. The construction is easy to follow on pictures but somewhat notational in a formal presentation. So we suggest that the reader follow it on the example we present; the example is the smallest one that illustrates various aspects of the construction. This example represents the 22-element distributive lattice D of Figure 1 as the congruence lattice of a semimodular lattice. The poset J of join-irreducibles has six elements, and it is shown in Figure 2.

Take the eight-element, nonmodular, semimodular lattice S_8 of Figure 3. S_8 has an ideal, $I_{S_8} = (b]$, and a dual ideal, $D_{S_8} = [c)$, both isomorphic to $\mathfrak{C}_2$; we shall utilize these for repeated gluings. The elements of I_{S_8} are black filled and the

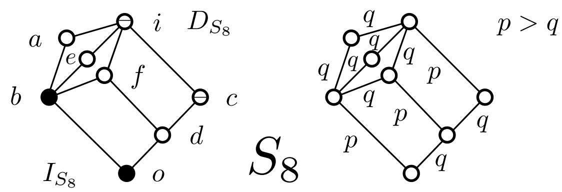

Figure 3

elements of D_{S_8} are striped on Figure 3. It is easy to see that the congruence lattice of S_8 is the three-element chain. Using the notation $\mathrm{J}(\mathrm{Con}\,\mathfrak{C}_3) = \{p, q\}$, with $p > q$, we also show the colored S_8 in Figure 1.

Let D be a finite distributive lattice, and let $J = \mathrm{J}(D)$ be the poset of its join-irreducible elements, $n = |J|$. We enumerate

$$p_1,\ p_2,\ \ldots,\ p_m$$

the non-minimal elements of J. For every p_i, $i = 1, 2, \ldots, m$, let

$$v(p_i) = \{q_i^1, q_i^2, \ldots, q_i^{k_i}\}$$

denote the set of all lower covers of p_i in J; since p_i is non-minimal, it follows that $k_i > 0$. Let

$$r_1,\ r_2,\ \ldots,\ r_t$$

enumerate all elements of J that are incomparable with all other elements.

In the example, $m = 3$, $t = 1$. Let

$$\begin{aligned} p_1 &= \alpha, & v(\alpha) &= \{\beta, \varepsilon\}, & q_1^1 &= \beta, & q_1^2 &= \varepsilon, \\ p_2 &= \beta, & v(\beta) &= \{\gamma\}, \\ p_3 &= \delta, & v(\delta) &= \{\varepsilon\}. \end{aligned}$$

So $k_1 = 2$, $k_2 = k_3 = 1$.

Step 1.

For every i, with $1 \leq i \leq m$, we construct a lattice A_i with an ideal I_i and a dual ideal D_i, where I_i is a chain of length $2(k_i + \cdots + k_m)$ and D_i is a chain of length $2(k_{i+1} + \cdots + k_m)$.

Now we shall use twice the construction, *gluing k-times*, described in Section 2. To form A_i, glue S_8 to itself $(k_i - 1)$-times with the ideal I_{S_8} and the dual ideal D_{S_8}, to obtain the lattice A_i^1 with a dual ideal $D_{A_i^1}$. Now take

$$\mathfrak{C}_2^2 = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$$

with the ideal

$$I_{\mathfrak{C}_2^2} = \{\langle 0,0\rangle, \langle 1,0\rangle\}$$

and the dual ideal

$$D_{\mathfrak{C}_2^2} = \{\langle 0,1\rangle, \langle 1,1\rangle\},$$

and glue $2(k_{i+1} + \cdots + k_m)$-times $\mathfrak{C}_2^2$ to A_i^1. The ideal I_i is generated by the element $\langle 0,1\rangle$ of the top $\mathfrak{C}_2^2$, while D_i is generated by the unit element of A_i^1.

We define a coloring μ_i of A_i as follows. On any copy of S_8, $[o, b]\mu_i = p_i$ and on the j-th copy of S_8,

$$[o, d]\mu_i = [d, c]\mu_i = q_i^j;$$

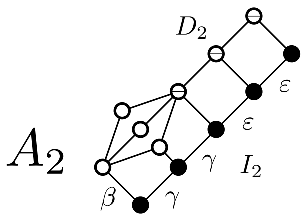

Figure 4

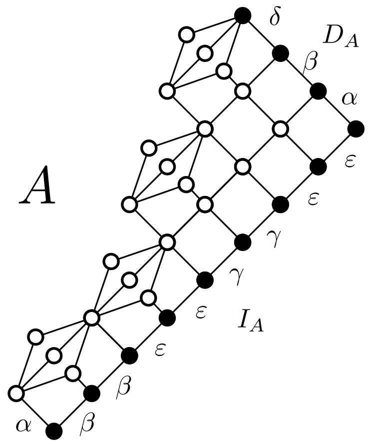

Figure 5

on the first two copies of $\mathfrak{C}_2^2$,

$$[\langle 0,1\rangle,\langle 1,1\rangle]\mu_i = q_{i+1}^1,$$

on the next two copies,

$$[\langle 0,1\rangle,\langle 1,1\rangle]\mu_i = q_{i+1}^2,$$

after k_{i+1} pairs, the next two satisfies

$$[\langle 0,1\rangle,\langle 1,1\rangle]\mu_i = q_{i+2}^1,$$

and so on.

Figure 4 shows A_2 for the example. The elements forming I_2 are black filled; the elements forming D_2 are striped. Note that I_2 is of length $2(k_2 + k_3) = 4$, while D_2 is of length $2k_3 = 2$.

Lemma 1. *μ_i is a coloring of A_i. The join-irreducible congruences of A_i are generated by prime intervals of I_i and by $[o,b]$ of the bottom S_8 in A_i. If $\mathfrak{p}$ and $\mathfrak{q}$ are $[o,b]$ or a prime interval $[o,d]$ or $[d,c]$ of a copy of S_8 in A_i, then $\Theta(\mathfrak{p}) \geq \Theta(\mathfrak{q})$ iff $\mathfrak{p}\mu_i \geq \mathfrak{q}\mu_i$. In particular, $\Theta(o,b) \succ \Theta(o,d)$ in $\mathrm{J}(\mathrm{Con}\, A_i)$, where o, b, d are in a copy of S_8 in A_i. If $\mathfrak{p}$ is a prime interval $[\langle 0,1\rangle,\langle 1,1\rangle]$ in a copy of $\mathfrak{C}_2^2$, then $\Theta(\mathfrak{p})$ is incomparable to any $\Theta(\mathfrak{q})$, where $\mathfrak{q}$ is $[o,b]$ or a prime interval of I_i different from $\mathfrak{p}$.*

Proof. This is trivial since every prime interval of S_8 is projective to one of $[o,b]$, $[o,d]$, $[d,c]$. ☐

Step 2.

We define the lattice A by gluing together the (colored) lattices A_i, $1 \leq i \leq m$.

For $1 \leq i \leq m$, we define, by induction, the lattice $\overline{A}_i$, which contains A_i, and, therefore, D_i, as a dual ideal. Let $\overline{A}_1 = A_1$. Assume that $\overline{A}_i$ with D_i as a dual ideal has been defined. Observe that both D_i and I_{i+1} are chains of length $2(k_{i+1} + \cdots + k_m)$, and so they are isomorphic; in fact, this isomorphism preserves colors. We glue $\overline{A}_i$ to A_{i+1} over D_i and I_{i+1} to obtain $\overline{A}_{i+1}$. Define $A = \overline{A}_m$ and $I_A = I_1$.

Observe that μ_i on D_i agrees with μ_{i+1} on I_{i+1}; therefore, the μ_i, $1 \leq i \leq m$, define a coloring μ_A of A.

Let D_A be the dual ideal of A generated by the element $\langle 0, 1 \rangle$ of the top $\mathfrak{C}_2^2$ in A_1. D_A is a chain of length m. The prime interval $[o, b]$ in the bottom S_8 in A_i $(1 \leq i \leq m)$ is projective to a unique prime interval $\mathfrak{p}$ of D_A; define $\mathfrak{p}\mu_A = [o, b]\mu_A$.

Figure 5 show this lattice for the example. The elements of I_A and D_A are black filled.

Lemma 2. *μ_A is a coloring of A. The join-irreducible congruences of A are generated by prime intervals of I_A and D_A. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime intervals in I_A and D_A.*

(i) *If $\mathfrak{p}$ and $\mathfrak{q}$ are prime intervals of D_A, then $\Theta(\mathfrak{p})$ and $\Theta(\mathfrak{q})$ are incomparable.*

(ii) *If $\mathfrak{p}$ is a prime interval of D_A and $\mathfrak{q}$ is a prime interval of I_A, then $\Theta(\mathfrak{p})$ and $\Theta(\mathfrak{q})$ are comparable iff $\mathfrak{p} \subseteq A_i$, for some $1 \leq i \leq m$, $\mathfrak{q}$ is perspective to some $[o, d]$ or $[d, c]$ in some S_8 in A_i; in which case, $\Theta(\mathfrak{p}) \succ \Theta(\mathfrak{q})$ in $\mathrm{J}(\mathrm{Con}\, A)$.*

(iii) *If $\mathfrak{p}$ and $\mathfrak{q}$ are prime intervals of I_A, then $\Theta(\mathfrak{p}) \geq \Theta(\mathfrak{q})$ iff $\mathfrak{p}$ and $\mathfrak{q}$ are perspective to prime intervals $\mathfrak{p}'$ and $\mathfrak{q}'$ in some A_i, respectively, for some $1 \leq i \leq m$, and $\mathfrak{p}'$ and $\mathfrak{q}'$ are adjacent edges of some S_8 in A_i; in which case, $\Theta(\mathfrak{p}) = \Theta(\mathfrak{q})$.*

Proof. This is obvious from the statement that if A and B are glued together over the dual ideal D of A and the ideal I of B, then a congruence of the glued lattice is obtained from a congruence Θ of A and a congruence Φ of B with the property that the restriction of Θ to D agrees with the restriction of Φ to I. ☐

Observe that the congruence lattice of A is still quite different from D in two ways: the congruences that correspond to the r_i are still missing; prime intervals in $I_A \cup D_A$ of the same color generate incomparable congruences with one exception: they are adjacent intervals in I_A, perspective to the two prime intervals of some S_8 in some A_i. For instance, in the example, see Figure 5, the prime interval of D_A of color β generates a congruence incomparable to the congruence generated by a prime interval of I_A of color β; also, a prime interval of color ε in the top part of I_A generates a congruence incomparable to the congruence generated by a prime interval of color ε in the lower part of I_A.

Step 3.

We extend A to a lattice B with an ideal I_B which is a chain and which has the property that every prime interval of B is projective to a prime interval of I_B.

This step is easy. We form the lattice D_A^2 with the ideal

$$I_{D_A^2} = \{\, \langle x, 0_{D_A} \rangle \mid x \in D_A \,\},$$

where 0_{D_A} is the zero of D_A. Let 1_{D_A} denote the unit element of D_A and, for $x \in D_A$, $x < 1_{D_A}$, let x^* denote the cover of x in D_A. For every $x \in D_A$, $x < 1_{D_A}$, we add an element m_x to D_A^2 so that the elements

$$\langle x, x \rangle, \langle x, x^* \rangle, \langle x^*, x \rangle, x_m, \langle x^*, x^* \rangle$$

form a sublattice isomorphic to $\mathfrak{M}_3$ with $\langle x, x \rangle$ as zero and $\langle x^*, x^* \rangle$ as unit. Let M be the resulting lattice. Obviously, M is a finite planar modular lattice whose congruence lattice is isomorphic to the congruence lattice of D_A. $I_{D_A^2}$ is also an ideal of M; we shall denote it by I_M.

Figure 6 shows M for the example. The elements of I_M are black filled.

We glue A to M over D_A and I_M to obtain B. Let I_B be defined as the ideal generated by $\langle 0, 1_{D_A} \rangle$. We define μ_B as an extension of μ_A; every prime interval $\mathfrak{p}$ of M is projective to exactly one prime interval $\overline{\mathfrak{p}}$ of I_M, we define $\mathfrak{p}\mu_B = \overline{\mathfrak{p}}\mu_A$.

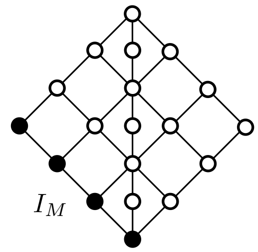

Figure 6

Lemma 3. *μ_B is a coloring of B. The join-irreducible congruences of B are generated by prime intervals of I_B. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime intervals in I_B.*

(i) *If $\mathfrak{p}$ and $\mathfrak{q}$ are prime intervals of M, then $\Theta(\mathfrak{p})$ and $\Theta(\mathfrak{q})$ are incomparable.*
(ii) *If $\mathfrak{p}$ is a prime interval of M and $\mathfrak{q}$ is a prime interval of I_A, then $\Theta(\mathfrak{p})$ and $\Theta(\mathfrak{q})$ are related exactly as $\Theta_A(\overline{\mathfrak{p}})$ and $\Theta_A(\mathfrak{q})$ are related in A.*
(iii) *If $\mathfrak{p}$ and $\mathfrak{q}$ are prime intervals of I_A, then $\Theta(\mathfrak{p})$ and $\Theta(\mathfrak{q})$ are related exactly as $\Theta_A(\mathfrak{p})$ and $\Theta_A(\mathfrak{q})$ are related in A.*

Proof. This is obvious from the congruence structure of M. ☐

Step 4.

We extend B to the lattice S of the Theorem.

This is also an easy step. We take a chain C of length n and we color C over J so that the coloring is a bijection. We form the lattice $C \times I_B$. For every pair of prime intervals, $\mathfrak{p} = [a, b]$ of C and $\mathfrak{q} = [c, d]$ of I_B, if $\mathfrak{p}$ and $\mathfrak{q}$ have the same color, then we add an element $m(\mathfrak{p}, \mathfrak{q})$ to C over J so that the elements

$$\langle a, c\rangle, \langle b, c\rangle, \langle a, d\rangle, m(\mathfrak{p}, \mathfrak{q}), \langle b, d\rangle$$

form a sublattice isomorphic to $\mathfrak{M}_3$. Let N denote the resulting lattice. N is obviously modular and planar. Set

$$I_N = \{\, \langle x, 0_{I_B}\rangle \mid x \in C \,\},$$
$$D_N = \{\, \langle 1_C, x\rangle \mid x \in I_B \,\},$$

where 0_{I_B} is the zero of I_B and 1_C is the unit of C. Then I_N is the ideal of N (isomorphic to C) and D_N is a dual ideal of N (isomorphic to I_B). Every prime interval of N is projective to a prime interval of I_N, so we have a natural coloring μ_N on N. Note that this coloring agrees with the coloring μ_B on D_N under the isomorphism with I_B.

We glue N to B over D_N and I_B to obtain S with the coloring μ_S. Set $I_S = I_N$. Figure 7 is a sketch of S.

It is clear from the construction and from the lemmas that every prime interval of S is projective to a prime interval of I_S and that distinct prime intervals of I_S generate distinct join-irreducible congruences of S.

It remains to see that if $\mathfrak{p}$ and $\mathfrak{q}$ are distinct prime intervals, then $\Theta(\mathfrak{p}) \geq \Theta(\mathfrak{q})$ iff $\mathfrak{p}\mu_S \geq \mathfrak{q}\mu_S$. Since J is finite, it is sufficient to prove that $\Theta(\mathfrak{p}) \succ \Theta(\mathfrak{q})$ in $\mathrm{J}(\mathrm{Con}\, S)$ iff $\mathfrak{p}\mu_S \succ \mathfrak{q}\mu_S$ in $\mathrm{J}(D)$. But this is clear since if $\mathfrak{p}\mu_S \succ \mathfrak{q}\mu_S$ in $\mathrm{J}(D)$, then $p\mu_S = p_i$, for some $1 \leq i \leq m$, and $q\mu_S = q_i^j$, for some $1 \leq j \leq k_i$, so $\Theta(\mathfrak{p}) \succ \Theta(\mathfrak{q})$ was guaranteed in A_i.

To establish that the size of S is $O(n^3)$, we give a very crude upper bound for $|S|$. $2n^2 + 1$ is an upper bound for $|I_i|$, $1 \leq i \leq m$, so $3(2n^2 + 1)$ is an upper bound

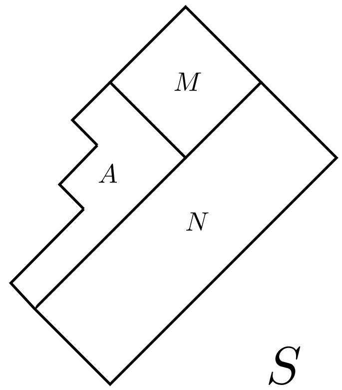

Figure 7

for $|A_i|$ and $3(2n^2+1)n$ is an upper bound for $|A|$. Since $|D_A| \leq n+1$, we get the upper bound $(n+1)^2+n+1$ for $|M|$. Finally, $|I_B| \leq 2n^2+1+n+1 = 2n^2+n+2$, so $|N| \leq 2(2n^2+n+2)(n+1)$. Therefore,

$$3(2n^2+1)n + (n+1)^2 + n + 1 + 2(2n^2+n+2)(n+1)$$

is an upper bound for S and it is a cubic polynomial in n. This completes the proof of the Theorem.

It is not difficult to find better upper bounds for $|S|$; for instance,

$$|S| \leq 3n^3 + 2n^2 - 7n + 4.$$

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