SOME COMBINATORIAL ASPECTS OF CONGRUENCE LATTICE REPRESENTATIONS

G. GRÄTZER AND E. T. SCHMIDT

Abstract. A finite distributive lattice D can be represented as the congruence lattice, $\operatorname{Con} L$, of a finite lattice L. We shall discuss the combinatorial aspects of such—and related—representations, specifically, optimal size, breadth, and degree of symmetry.

1. Introduction

The congruence lattice, $\operatorname{Con} L$, of a finite lattice L is a finite distributive lattice according to a result of N. Funayama and T. Nakayama [5]. The converse is a result of R. P. Dilworth from 1944: *Every finite distributive lattice D can be represented as the congruence lattice,* $\operatorname{Con} L$*, of a finite lattice L.*

This result was first published in 1962 in the paper [13] of the present authors, where—and, one assumes, in the original proof by Dilworth—the lattice L constructed was very large.

Since a finite distributive lattice D is determined by the poset $J(D)$ of its join-irreducible elements, it is logical to measure the size of the lattice L representing D as a function of $n = |J(D)|$. The original constructions produced lattices of size $O(2^{2n})$. This was improved to $O(n^3)$ in G. Grätzer and H. Lakser [8]; it was conjectured that $O(n^3)$ can be improved to $O(n^2)$ and that $O(n^2)$ is best possible. In Section 2, we sketch the $O(n^2)$ construction as given in G. Grätzer, H. Lakser, and E. T. Schmidt [9]. The real combinatorics is in G. Grätzer, I. Rival, and N. Zaguia [12], proving that $O(n^2)$ is, indeed, best possible; this is also outlined in Section 2.

The original construction produced a lattice L of breadth $O(2n)$. This was reduced to 2 in G. Grätzer and H. Lakser [8], and retained in G. Grätzer, H. Lakser, and E. T. Schmidt [9]. So the final result constructs a planar lattice L of size $O(n^2)$.

What is more combinatorial than symmetry? We can measure the symmetry of a lattice L with its automorphism group, $\operatorname{Aut} L$. It is a result of G. Birkhoff [4] that *every finite group is isomorphic to the automorphism group of a finite lattice.*

In [6], the first author raised the question whether every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L with a prescribed automorphism group. This problem was solved for finite lattices by V. A. Baranskiĭ [3] and A. Urquhart [22], independently.

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In [14], the present authors have developed a new approach to this problem; we shall discuss this in Section 3. For a general lattice L, it is well known that $\operatorname{Con} L$ is a distributive algebraic lattice. A long standing conjecture is that the converse also holds. One can attempt to solve this conjecture—at least for some classes of distributive algebraic lattices—by considering a finite lattice K, a sublattice of the finite lattice L, and ask how $\operatorname{Con} K$ is related to $\operatorname{Con} L$? One can then hope to represent an infinite distributive algebraic lattice as a congruence lattice by a direct limit. (In a recent manuscript [23], F. Wehrung exhibits a distributive algebraic lattice for which this method cannot be successful.)

It turns out that if K is a sublattice of L, then there is a natural 0-preserving join-homomorphism from $\operatorname{Con} K$ to $\operatorname{Con} L$. Let K and L be lattices and let $\varphi\colon K \to L$ be a lattice homomorphism (not necessarily an embedding). We define the map

$$\overline{\varphi}\colon \operatorname{Con} K \to \operatorname{Con} L$$

as follows: for each $\Theta \in \operatorname{Con} K$, define the set $\Theta\varphi$ of pairs of elements of L that are φ-images of Θ-congruent pairs of elements of K, that is,

$$\Theta\varphi = \{\, \{x\varphi, y\varphi\} \mid x,\ y \in K,\ x \equiv y\ (\Theta)\,\};$$

then the image of Θ under $\overline{\varphi}$ is the congruence of L *generated by* $\Theta\varphi$. Now the representation theorem takes the following form: for finite lattices K and L, the map $\overline{\varphi}$ *can be characterized as a* 0*-preserving join-homomorphism.* This major result is due to A. P. Huhn [17].

In Section 4 we discuss some combinatorial questions related to Huhn's theorem: how large the lattices must be and what can be said about the breadth? Recent results of G. Grätzer, H. Lakser, and E. T. Schmidt prove that if ψ is a 0-preserving join-homomorphism of D into E (D and E are finite distributive lattices) and $n = \max(|J(D), |J(E)|)$, then there are finite lattices K and L and there is a lattice homomorphism $\varphi\colon K \to L$ that represent ψ so that the size of K and L is $O(n^5)$ and K and L are of breadth 3. We conjecture that this result is the best.

For the basic concepts and notations, we refer the reader to G. Grätzer [6], as reviewed in the lecture "Congruences lattices 101".

2. Planar lattices, small lattices

We start this section with the following result (G. Grätzer, H. Lakser, and E. T. Schmidt [9]):

$O(n^2)$ Theorem. *Let D be a finite distributive lattice with n join-irreducible elements. Then there exists a* planar lattice of $O(n^2)$ elements *with* $\operatorname{Con} L \cong D$.

In the lecture "Congruences lattices 101", we have reviewed how to construct a finite lattice L *representing* the finite distributive lattice D, that is, satisfying $\operatorname{Con} L \cong D$. If $P = J(D)$ (or equivalently, $D = H(P)$) and $|P| = n$, then L has $O(2^{2n})$ elements. We improve on this result by making L planar and "small".

To construct L from $P = \{p_1, p_2, \dots, p_n\}$, we take a chain

$$C = \{c_0, c_1, \dots, c_{2n}\}, \quad c_0 \prec c_1 \prec \dots \prec c_{2n}.$$

We assign to every prime interval $[c_i, c_{i+1}]$ an element of P (we shall call it the "color") so that each element of P is the color of two adjacent prime intervals: let the color of $[c_0, c_1]$ and $[c_1, c_2]$ be p_1; of $[c_2, c_3]$ and $[c_3, c_4]$ be p_2, and so on, of $[c_{2n-2}, c_{2n-1}]$ and $[c_{2n-1}, c_{2n}]$ be p_n.

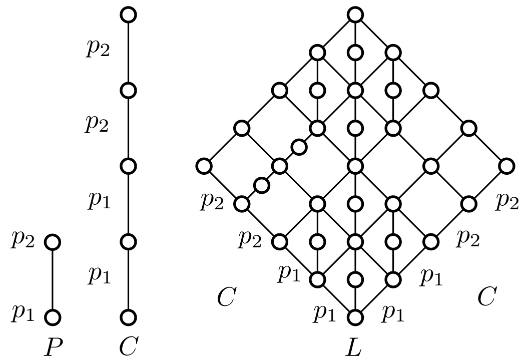

Figure 1

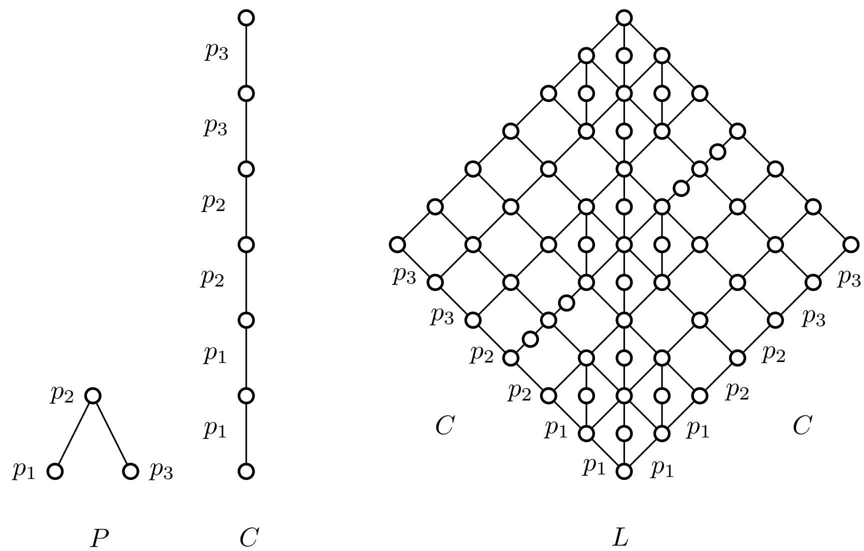

Figure 2

Follow this on the two examples in Figures 1 and 2; in Figure 1, $P = \{p_1, p_2\}$ and $p_1 < p_2$, while in Figure 2, $P = \{p_1, p_2, p_3\}$ and $p_1 < p_2$, $p_3 < p_2$. The color of a prime interval is indicated on the diagrams.

Do you see the pattern? In C^2, we fill in a "covering square" ($\mathfrak{C}_2 \times \mathfrak{C}_2$) with one more element so that we obtain an $\mathfrak{M}_3$ if the two sides have the same color, see Figure 3. Moreover, if p, $q \in P$ and $p < q$, then we take the "double covering square" ($\mathfrak{C}_3 \times \mathfrak{C}_2$) where the longer side has two prime intervals of color q and the shorter side is of color p, and we add one more element to obtain $\mathfrak{N}_{5,5}$—the "domino"—as illustrated in Figure 3.

It is an easy computation to show that $|L| = kn^2$ for some constant k, and that $D \cong \operatorname{Con} L$; this isomorphism is established by assigning to $p \in P$ the congruence of L generated by collapsing any (all) prime intervals of color p.

Now we argue that the $O(n^2)$ result is the best (G. Grätzer, I. Rival, and N. Zaguia [12]).

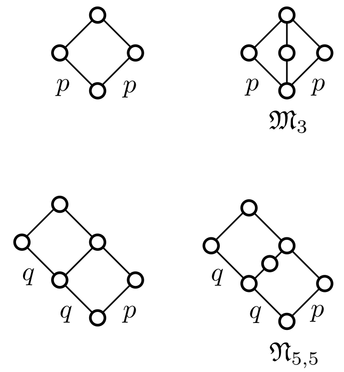

Figure 3

***$O(n^2)$* Best Theorem.** *Let α be a real number satisfying the following condition: Every distributive lattice D with n join-irreducible elements can be represented as the congruence lattice of a lattice L with $O(n^\alpha)$ elements. Then $\alpha \geq 2$.*

Let n be a natural number. Let D_n be a distributive lattice whose partially ordered set P_n of join-irreducible elements is bipartite and about half the elements are maximal and about half are minimal; in P_n there are $k_1 n^2$ covering edges for some constant k_1. Let L_n be a lattice satisfying $\operatorname{Con} L_n \cong D_n$, and assume that L_n has $k_2 n^\alpha$ elements, for some constant k_2, where $\alpha < 2$.

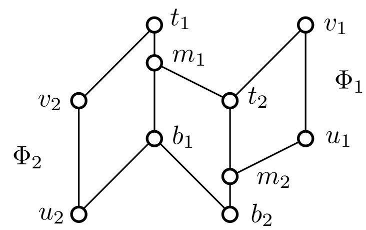

Figure 4

For a covering pair $\Phi_1 \prec \Phi_2$ of join-irreducible congruences of L_n, there are prime intervals $[u_1, v_1]$, $[u_2, v_2]$ of L satisfying $\Phi_1 = \Theta(u_1, v_1)$ and $\Phi_2 = \Theta(u_2, v_2)$. Since $u_1 \equiv v_1 \ (\Theta(u_2, v_2))$, we can get (as discussed in Section 6 of [7]) from $[u_2, v_2]$ to $[u_1, v_1]$ by a series of up and down steps as exemplified in Figure 4; we denote by b_1, m_1, t_1 the bottom, middle, and top elements of the first step, and by b_2, m_2, t_2 the bottom, middle, and top elements of the second step (in general, there are any—finite—number of steps), so

$$\begin{aligned} b_1 &= u_2 \vee x_1, & t_1 &= v_2 \vee x_1,\\ b_2 &= b_1 \wedge x_2, & t_2 &= t_1 \wedge x_2,\\ u_1 &= b_2 \vee x_3, & v_1 &= t_2 \vee x_3, \end{aligned}$$

(for some elements x_1, x_2, and x_3, not shown in the diagram) and we name $m_1 = b_1 \vee t_2$ and $m_2 = t_2 \wedge u_1$.

It is easy to see that either $\Phi_1 = \Theta(b_1, m_1)$ and $\Phi_2 = \Theta(m_1, t_1)$ or $\Phi_1 = \Theta(b_2, m_2)$ and $\Phi_2 = \Theta(m_2, t_2)$. In general, if $\Phi_1 \prec \Phi_2$ in $J(\operatorname{Con} L)$, then there is a three-element chain $\{e_1, h, e_2\}$ in L such that $\Phi_1 = \Theta(h, e_1)$, $\Phi_2 = \Theta(h, e_2)$, and $e_1 < h < e_2$ or $e_2 < h < e_1$.

There are $k_1 n^2$ such pairs of congruences Φ_1, Φ_2 of L_n; to each pair corresponds a three-element chain $e_1 < h < e_2$. Since there are $k_1 n^2$ such chains and there are $k_2 n^\alpha$ elements in L_n, there must be an element $h \in L_n$ that appears as the middle element of $\frac{k_1 n^2}{k_2 n^\alpha} = k_3 n^{2-\alpha}$ three-element chains. For half of these chains, $k_4 n^{2-\alpha}$ in number, for some constant k_4, the top interval (or dually, the bottom interval) defines the larger congruence. So we obtain in L_n an element h and a set A of elements, $k_4 n^{2-\alpha}$ in number, so that all the $\Theta(h, x)$ are maximal (or all are minimal) join-irreducible congruences of L_n. Obviously, these congruences are pairwise incomparable. Now it is easy to see that A—under join—generates a free join-semilattice F (or in the dual case, a free meet-semilattice). The set $F \subseteq L_n$ has $k_4 2^{n^{2-\alpha}}$ elements. But this is a contradiction since L_n has only $k_2 n^\alpha$ elements.

Yong Zhang has recently obtained a stronger form of the $O(n^2)$ Best Theorem.

3. Symmetry

In the Introduction, we have described the Independence Theorem of the automorphism group and the congruence lattice of a finite lattice, due to V. A. Baranskiĭ [3] and A. Urquhart [22]. Let us now state it formally:

Independence Theorem. *Let G be a finite group and let D be a finite distributive lattice. Then there exists a finite lattice L with* $\operatorname{Aut} L \cong G$ *and* $\operatorname{Con} L \cong D$.

Next we describe a stronger form of independence introduced in [14]; it does not use the characterization theorems.

Let L be a lattice. The lattice K is a *congruence-preserving extension* of L, if K is an extension and every congruence of L has exactly one extension to K; then the congruence lattice of K is isomorphic to the congruence lattice of L. Since K is a congruence-preserving extension of L, it follows that $D = \operatorname{Con} L$, as an abstract lattice, is also the congruence lattice of K. We would like to argue that the "algebraic reasons why D is the congruence lattice of L" have been retained in K. For instance, let a_i, $b_i \in L$, $i = 1, 2, 3$, and let

$$\Theta(a_1, b_1) \vee \Theta(a_2, b_2) = \Theta(a_3, b_3)$$

in L. Then L has a finite partial sublattice $\mathfrak{H}$ forcing this equation. Since K is an extension of L, $\mathfrak{H}$ is a partial sublattice of K, hence this equation holds also in K; and it holds for the same "algebraic reason"—namely, the existence of $\mathfrak{H}$.

There is an analogous concept for automorphisms. Let L be a lattice. K is an *automorphism-preserving extension* of L if K is an extension and every automorphism of L has exactly one extension to K; moreover, every automorphism of K is the extension from an automorphism of L. Then the automorphism group of L is isomorphic to the automorphism group of K.

The automorphism group of L is the group of all permutations of L that are not excluded for some algebraic reason. An "algebraic reason" can always be formulated in the form of the existence of a finite partial sublattice $\mathfrak{H}$. Thus we can again argue that the "algebraic reasons why $G = \operatorname{Aut} L$ is the automorphism group of L" have been retained in K.

Now we are ready to state the main result of G. Grätzer and E. T. Schmidt [14]:

Strong Independence Theorem. *Let L_C and L_A be finite lattices with more than one element satisfying $L_C \cap L_A = \varnothing$. Then there exists a lattice K that is a congruence-preserving extension of L_C and an automorphism-preserving extension of L_A.*

Of course, then the congruence lattice of K is isomorphic to the congruence lattice of L_C, and the automorphism group of K is isomorphic to the automorphism group of L_A.

Arguing intuitively, as above, this theorem states that any set of algebraic conditions forcing the shape of a congruence lattice are compatible with any other set of algebraic conditions forcing the shape of an automorphism group.

It is logical to ask whether the results of this section could be combined with the results of the previous section. For instance, in the Independence Theorem could we require that the lattice L be planar? The answer to this is in the negative: the class PG of automorphism groups of planar lattices is characterized in L. Babai [1] (see also L. Babai and D. Duffus [2]). PG is a small subclass of the class of finite groups; for instance the three element cyclic group is not in PG. However, if $G \in \mathrm{PG}$, then L can be chosen planar, see G. Grätzer and H. Lakser [8].

Of special interest to combinatorics are lattices of vector spaces. Given a field F and a vector space V over F of dimension n, we can form the lattice $L(F, n)$ of all subspaces of V. Let us say that L is a *vector-space lattice over F*, if L is isomorphic to a sublattice of $L(F, n)$ for some n. We raise the question: does the Independence Theorem hold vector-space lattices?

Since a vector-space lattice is modular, and the congruence lattice of a finite modular lattice is Boolean, obviously a vector-space lattice L satisfying the conditions of the Independence Theorem cannot be finite. We have proved the following result ([15] and [16]):

Independence Theorem for $GF(2)$. *Let G be a finite group and let D be a finite distributive lattice. Then there exists a vector-space lattice L over $GF(2)$ with $\mathrm{Aut}\, L \cong G$ and $\mathrm{Con}\, L \cong D$.*

In this result, $GF(2)$ is the two-element field. We do not know whether this result holds for any other field.

4. Join homomorphisms of finite distributive lattices

When we consider the congruence lattices of a finite lattice L and of a sublattice, of special interest is the case when the sublattice I is an ideal of L. In this case, the restriction map is a $\{0, 1\}$-*homomorphism* of $\mathrm{Con}\, L$ into $\mathrm{Con}\, I$. For instance, we can have $\mathrm{Con}\, L$ the three-element chain (with elements 0, a, 1) and $\mathrm{Con}\, I$ the four-element Boolean lattice (with elements 0, b, c, 1), so that the restriction map represents the homomorphism φ mapping 0 into 0, and a and 1 into 1. Using the techniques of Section 2 and of G. Grätzer and H. Lakser [8], this can be realized; see Figure 5, where the black-filled elements form I. Notice that the top black-filled element and the gray-filed elements form the lattice shown in Figure 1, whose congruence lattice is the three-element chain.

The general result is the following (G. Grätzer and H. Lakser [8]):

Theorem. *Let D, E be finite distributive lattices, and let $\psi \colon D \to E$ be a $\{0, 1\}$-homomorphism (lattice homomorphism). Then there exist a finite lattice L, an ideal*

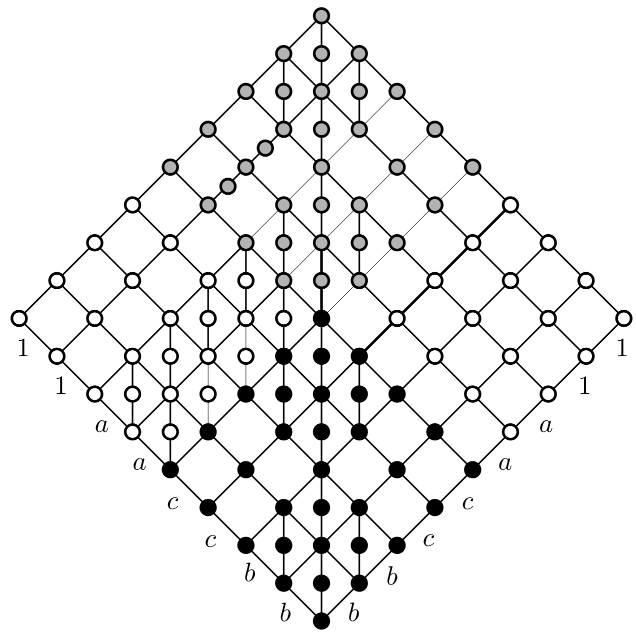

Figure 5

I of L, and lattice isomorphisms

$$\alpha\colon D \to \operatorname{Con} L, \qquad \beta\colon E \to \operatorname{Con} I$$

such that $\psi\beta$ is the composition of α with the restriction of $\operatorname{Con} L$ to $\operatorname{Con} I$, that is, such that the diagram

$$\begin{array}{ccc} D & \xrightarrow{\ \psi\ } & E \\ \cong\big\downarrow\alpha & & \cong\big\downarrow\beta \\ \operatorname{Con} L & \xrightarrow{\ \textit{restriction}\ } & \operatorname{Con} I \end{array}$$

is commutative.

This construction can be made optimal from a combinatorial point of view: if $n = \max(|J(D)|, |J(E)|)$, then L can be constructed as a planar lattice of size $O(n^2)$.

The first indication that direct limits of finite lattices may be useful for the congruence lattice representation problem is a lemma of P. Pudlák [18].

Lemma. *Every distributive lattice is a direct limit of all the finite distributive lattices contained in it as distributive join-semilattices with zero.*

And the corresponding result for 0-preserving join-homomorphisms is the following (A. P. Huhn [17], see also G. Grätzer, H. Lakser, and E. T. Schmidt [10, 11]):

Represenation Theorem for Pairs. *Let D and E be finite distributive lattices, and let*

$$\psi\colon E \to D$$

be a 0*-preserving join-homomorphism. Then there are finite lattices* K, L, *a lattice homomorphism* $\varphi\colon K \to L$, *and isomorphisms*

$$\alpha\colon D \to \operatorname{Con} L, \qquad \beta\colon E \to \operatorname{Con} K$$

with

$$\psi\alpha = \beta(\overline{\varphi}),$$

that is, such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\ \psi\ } & D \\ \cong\Big\downarrow\beta & & \cong\Big\downarrow\alpha \\ \operatorname{Con} K & \xrightarrow{\ \overline{\varphi}\ } & \operatorname{Con} L \end{array}$$

is commutative. Furthermore, φ *is an embedding iff* ψ *separates* 0.

Note that the map $\overline{\varphi}$ was defined in the Introduction.

An immediate application of this result is the following (G. Grätzer, H. Lakser, and E. T. Schmidt [10]):

Theorem. *Let* D_1 *and* D_2 *be finite distributive lattices, and let*

$$\psi\colon D_1 \to D_2$$

be an isotone map *that preserves* 0. *Then there is a finite lattice* L *with sublattices* L_1 *and* L_2 *and there are isomorphisms*

$$\alpha_1\colon D_1 \to \operatorname{Con} L_1, \qquad \alpha_2\colon D_2 \to \operatorname{Con} L_2$$

such that the diagram

$$\begin{array}{ccccc} D_1 & & \xrightarrow{\ \psi\ } & & D_2 \\ \cong\Big\downarrow\alpha_1 & & & & \cong\Big\downarrow\alpha_2 \\ \operatorname{Con} L_1 & \xrightarrow{\ \textit{extension}\ } & \operatorname{Con} L & \xrightarrow{\ \textit{restriction}\ } & \operatorname{Con} L_2 \end{array}$$

is commutative.

We would like to examine the Representation Theorem for Pairs from the combinatorial point of view of this paper: what can be said about the size and breadth of K and L?

Let $n = \max(|J(K)|, |J(L)|)$. To represent D and E (separately), we need lattices of size $O(n^2)$. We have to place in the image of every prime interval $\mathfrak{p}$ of K copies of all the prime intervals $\mathfrak{q}$ of L that generate a congruence that maps below the ψ-image of the congruence generated by $\mathfrak{p}$; therefore we need size $O(n^4)$. To represent the map ψ, we need an additional order of magnitude, so $O(n^5)$ seems like the right size. The following is an unpublished result of G. Grätzer, H. Lakser, and E. T. Schmidt:

Theorem. *The lattices* K *and* L *in the Representation Theorem for Pairs can be constructed of size* $O(n^5)$ *and of breadth* 3.

We could not prove, however, the analogue of the G. Grätzer, I. Rival, and N. Zaguia result (see Section 2); we cannot even prove that the size $O(n^4)$ is unattainable, nor can we prove that, in general, K and L cannot be planar. We conjecture that K and L can be constructed to have order dimension 3, which is very much stronger than breadth 3.

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