A SHORT PROOF OF A THEOREM OF BIRKHOFF

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ABSTRACT. This paper gives a new proof of a theorem of G. Birkhoff: *Every group $\mathfrak{G}$ can be represented as the automorphism group of a distributive lattice D; if $\mathfrak{G}$ is finite, D can be chosen to be finite.* The new proof is short, and it is easily visualized.

1. Introduction

In [1], G. Birkhoff proved the following result:

Theorem. *Every group $\mathfrak{G}$ can be represented as the automorphism group of a distributive lattice D; if $\mathfrak{G}$ is finite, then D can be chosen to be finite.*

See also the references for alternative proofs. In this note, we present a new proof of this result which may be the simplest and most direct of all the proofs.

2. The proof

Let $\mathfrak{G}$ be the given group defined on the set $G = \{ g_\gamma \mid \gamma < \alpha \}$ with $g_0 = 1$; we assume that $|G| > 1$. We view ordinals as well-ordered chains, so

$$\gamma \leq \delta < \alpha \text{ and } \gamma \cong \delta \quad \text{imply that} \quad \gamma = \delta. \tag{1}$$

For every $x,\ y \in G$ with $y \neq 1$ (equivalently, with $x \neq yx$), we construct the poset $P(x, y)$ of Figure 1, defined on the set $\{x, yx, \langle x, y\rangle\} \cup (\psi \times \langle x, y\rangle)$, where $y = g_\psi$, $\psi < \alpha$; we partially order this set by

$$x < \langle 0, \langle x, y\rangle\rangle < \langle 1, \langle x, y\rangle\rangle < \cdots < \langle \gamma, \langle x, y\rangle\rangle < \dots, \quad \gamma < \psi; \tag{2}$$

$$yx < \langle x, y\rangle < \langle 0, \langle x, y\rangle\rangle. \tag{3}$$

The two minimal elements of $P(x, y)$ are x and yx, both in G.

Let $P = \bigcup(P(x, y) \mid x,\ y \in G,\ y \neq 1)$ be partially ordered by $a < b$ in P iff $a < b$ in some $P(x, y)$. It is sufficient to prove that $\operatorname{Aut} P \cong \mathfrak{G}$. Indeed, $\operatorname{Aut} P \cong \operatorname{Aut} L$, where L is the distributive lattice completely freely generated by P; moreover, if G is finite, then P and L are both finite.

To prove that $\operatorname{Aut} P \cong \mathfrak{G}$, let Σ be an automorphism of P. Since G is the set of minimal elements of P, so Σ permutes G. Let $a = 1\Sigma$ and let $b \in G$. We want to show that $b\Sigma = ba$.

If $b = 1$, this holds by the definition of a. So let $b \neq 1$. Let $b = g_\beta$ with $\beta < \alpha$. Then the poset $P(1, b)$, with minimal elements 1 and b, is defined (since $1 \neq b = b1$).

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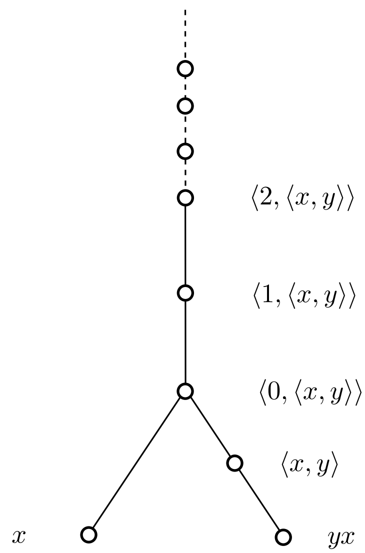

FIGURE 1. The poset $P(x, y)$

Since $b \neq 1$, also $b\Sigma \neq a$ $(= 1\Sigma)$ and so $b\Sigma = ua$, for some $u \in G$ with $u \neq 1$. Therefore, $P(a, u)$ with minimal elements $a = 1\Sigma$ and $ua = b\Sigma$, is defined.

Thus Σ takes the minimal elements of $P(1, b)$ into the minimal elements of $P(a, u)$, hence it must take all of $P(1, b)$ to $P(a, u)$, so $P(1, b) \cong P(a, u)$. Thus the top chain of $P(a, u)$ is the same as the top chain of $P(1, b)$, that is, β, and so $u = b$, proving that $b\Sigma = ba$.

For every $a \in G$, define Σ_a by $b\Sigma_a = ba$. Then we have just proved that every automorphism of P restricted to G is of this form; the converse is trivial. This completes the proof of the claim and of the Theorem.

References

[1] G. Birkhoff, *On the groups of automorphisms* (Spanish), Revista Unió Mat. Argentina **11** (1946), 155–157.

[2] R. Frucht, *Herstellung von Graphen mit vorgegebener abstrakter Gruppe*, Compos. Math. **6** (1938), 239–250.

[3] ———, *Lattices with a given group of automorphisms*, Canad. J. Math. **2** (1950), 417–419.

[4] G. Grätzer, *General Lattice Theory*, Pure and Applied Mathematics **75**, Academic Press, Inc. (Harcourt Brace Jovanovich, Publishers), New York-London; Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften, Mathematische Reihe, Band 52. Birkhäuser Verlag, Basel-Stuttgart; Akademie Verlag, Berlin, 1978. xiii+381 pp.

[5] G. Grätzer and H. Lakser, *Homomorphisms of distributive lattices as restrictions of congruences. II. Planarity and automorphisms*, Canadian J. Math. **46 (1)** (1994), 3–54.

[6] G. Grätzer, H. Lakser, and E. T. Schmidt, *On a result of Birkhoff*, Period. Math. Hungar. **30** (1995), 183-188.

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