ISOTONE MAPS AS MAPS OF CONGRUENCES. I. ABSTRACT MAPS

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To L. Fuchs on his 70th birthday

ABSTRACT. Let L be a lattice and let L_1, L_2 be sublattices of L. Let Θ be a congruence relation of L_1. We extend Θ to L by taking the smallest congruence $\overline{\Theta}$ of L containing Θ. Then we restrict $\overline{\Theta}$ to L_2, obtaining the congruence $\overline{\Theta}_{L_2}$ of L_2. Thus we have defined a map $\operatorname{Con} L_1 \to \operatorname{Con} L_2$. Obviously, this is an isotone 0-preserving map of the finite distributive lattice $\operatorname{Con} L_1$ into the finite distributive lattice $\operatorname{Con} L_2$.

The main result of this paper is the converse. Let D_1 and D_2 be finite distributive lattices, and let $\psi\colon D_1 \to D_2$ be an isotone map that preserves 0. Then there is a finite lattice L with sublattices L_1 and L_2 such that $\operatorname{Con} L_1$ represents D_1 and $\operatorname{Con} L_2$ represents D_2, and the map $\operatorname{Con} L_1 \to \operatorname{Con} L_2$ obtained by first extending each congruence relation of L_1 to L by minimal extension and then restricting the resulting congruence relation to L_2 represents ψ.

1. Introduction

It is well-known that, given a lattice L and a convex sublattice K, the map of restriction $\operatorname{Con} L \to \operatorname{Con} K$ is a $\{0,1\}$-preserving lattice homomorphism. In [3], see also [10], the converse is proved: any $\{0,1\}$-preserving homomorphism of finite distributive lattices can be realized as such a restriction and, indeed, as a restriction to an ideal of a finite lattice.

If the sublattice K is not a convex sublattice, then the restriction map $\operatorname{Con} L \to \operatorname{Con} K$ need not preserve join, but it still preserves meet, 0, and 1.

Similarly, we can extend congruences from the sublattice K to L by minimal extension. This map of extension need not preserve meet, but it does preserve join and 0. Furthermore it *separates* 0, that is, nonzero congruences extend to nonzero congruences.

Consequently, if L is a lattice and L_1, L_2 are sublattices of L, then there is a map

$$\operatorname{Con} L_1 \to \operatorname{Con} L_2$$

obtained by first extending each congruence relation of L_1 to L and then restricting the resulting congruence relation to L_2. All we can say about this map is that it is

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isotone and that it preserves 0. The main result of this paper is that this is in fact a characterization of 0-preserving isotone maps between *finite* distributive lattices:

Theorem. *Let D_1 and D_2 be finite distributive lattices, and let*

$$\psi\colon D_1 \to D_2$$

be an isotone map that preserves 0. Then there is a finite lattice L with sublattices L_1 and L_2 and there are isomorphisms

$$\alpha_1 : D_1 \to \operatorname{Con} L_1, \qquad \alpha_2\colon D_2 \to \operatorname{Con} L_2$$

such that the diagram

$$\begin{array}{ccccc} D_1 & & \xrightarrow{\ \psi\ } & & D_2 \\ \cong\Big\downarrow{\scriptstyle \alpha_1} & & & & \cong\Big\downarrow{\scriptstyle \alpha_2} \\ \operatorname{Con} L_1 & \xrightarrow{\ extension\ } & \operatorname{Con} L & \xrightarrow{\ restriction\ } & \operatorname{Con} L_2 \end{array}$$

is commutative.

We actually prove a slight generalization—see Theorem 3 in the following section.

In the title of this paper, "Abstract maps" refers to the fact that in the Theorem, ψ is an isotone map from a finite distributive lattice into another; these finite distributive lattices are "abstract" representations of congruence lattices. We can make this setup "concrete" by starting with two finite lattice K_1 and K_2, an isotone map ψ from $\operatorname{Con} K_1$ into $\operatorname{Con} K_2$ that preserves the zero congruence, and we look for a joint extension L of K_1 and K_2 that represents ψ. This problem is considered in Part II of this paper [4].

2. Statement of the results

We formalize and slightly extend the ideas of Section 1. Let K and L be lattices and let $\varphi\colon K \to L$ be a lattice homomorphism (not necessarily an embedding). We then have the associated *restriction map*

$$\operatorname{rs}\varphi\colon \operatorname{Con} L \to \operatorname{Con} K$$

defined by setting

$$x \equiv y \quad ((\operatorname{rs}\varphi)\Theta) \quad \text{ iff } \quad \varphi x \equiv \varphi y \quad (\Theta)$$

for each $\Theta \in \operatorname{Con} L$, that is,

$$\operatorname{rs}\varphi = (\varphi^2)^{-1}{}_{|\operatorname{Con} L},$$

where $\varphi^2 : K^2 \to L^2$ is the map induced by φ. Now, $\operatorname{rs}\varphi$ preserves $\wedge$ and 1. Observe that $\operatorname{rs}\varphi$ also preserves 0 iff φ is an embedding.

We also have a dual concept, in the technical sense also, as we demonstrate in Section 6. We define the *extension* of φ,

$$\operatorname{xt}\varphi\colon \operatorname{Con} K \to \operatorname{Con} L$$

by setting, for each $\Theta \in \operatorname{Con} K$, $(\operatorname{xt}\varphi)\Theta$ to be the congruence relation of L generated by the subset $\varphi^2(\Theta)$ of L^2:

$$(\operatorname{xt}\varphi)\Theta = \bigvee(\, \Theta_L(\varphi x, \varphi y) \mid x \equiv y \quad (\Theta)\,).$$

Now, $\operatorname{xt}\varphi$ preserves $\vee$ (see Corollary 1 in Section 5 further on) and 0, and φ is an embedding iff $\operatorname{xt}\varphi$ *separates* 0, that is, iff

$$(\operatorname{xt}\varphi)\Theta = 0_{\operatorname{Con} L} \quad \text{implies} \quad \Theta = 0_{\operatorname{Con} K}.$$

We prove here that these situations are typical for finite lattices:

Theorem 1. *Let D and E be finite distributive lattices, and let*

$$\psi\colon E \to D,$$

be a $\{0, \vee\}$-preserving map. Then there are finite lattices K, L, a lattice homomorphism $\varphi\colon K \to L$, and isomorphisms

$$\alpha\colon D \to \operatorname{Con} L, \qquad \beta\colon E \to \operatorname{Con} K$$

with

$$\alpha \circ \psi = (\operatorname{xt}\varphi) \circ \beta,$$

that is, such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\psi} & D \\ \cong\Big\downarrow\beta & & \cong\Big\downarrow\alpha \\ \operatorname{Con} K & \xrightarrow{\operatorname{xt}\varphi} & \operatorname{Con} L \end{array}$$

is commutative. Furthermore, φ is an embedding iff ψ separates 0.

The special case of Theorem 1 when ψ is also an *embedding* was proved by Huhn [6]. This special case also follows immediately from Theorems 5.5 and 5.6 in Tschendorf's thesis [11].

Theorem 2. *Let D and E be finite distributive lattices, and let*

$$\psi\colon D \to E,$$

be a $\{1, \wedge\}$-preserving map. Then there are finite lattices K, L, a lattice homomorphism $\varphi\colon K \to L$, and isomorphisms

$$\alpha\colon D \to \operatorname{Con} L, \qquad \beta\colon E \to \operatorname{Con} K$$

with

$$\beta \circ \psi = (\operatorname{rs}\varphi) \circ \alpha,$$

that is, such that the diagram

$$\begin{array}{ccc} D & \xrightarrow{\psi} & E \\ \cong\Big\downarrow\alpha & & \cong\Big\downarrow\beta \\ \operatorname{Con} L & \xrightarrow{\operatorname{rs}\varphi} & \operatorname{Con} K \end{array}$$

is commutative. Furthermore, φ is an embedding iff ψ preserves 0.

We henceforth refer to a $\{0, \vee\}$-preserving map as a $\{0, \vee\}$-homomorphism and to a $\{1, \wedge\}$-preserving map as a $\{1, \wedge\}$-homomorphism.

Theorem 3. *Let D_1 and D_2 be finite distributive lattices, and let*

$$\psi\colon D_1 \to D_2$$

be an isotone map. Then there are finite lattices L_1, L_2, L, a lattice embedding

$$\varphi_1\colon L_1 \to L,$$

a lattice homomorphism

$$\varphi_2\colon L_2 \to L,$$

and isomorphisms

$$\alpha_1\colon D_1 \to \operatorname{Con} L_1, \qquad \alpha_2\colon D_2 \to \operatorname{Con} L_2$$

such that

$$\alpha_2 \circ \psi = (\operatorname{rs} \varphi_2) \circ (\operatorname{xt} \varphi_1) \circ \alpha_1,$$

that is, such that the diagram

$$\begin{array}{ccccc} D_1 & & \xrightarrow{\psi} & & D_2 \\ \cong \big\downarrow \alpha_1 & & & & \cong \big\downarrow \alpha_2 \\ \operatorname{Con} L_1 & \xrightarrow{\operatorname{xt} \varphi_1} & \operatorname{Con} L & \xrightarrow{\operatorname{rs} \varphi_2} & \operatorname{Con} L_2 \end{array}$$

is commutative. Furthermore, φ_2 is also an embedding iff ψ preserves 0.

The major part of this paper is involved in proving Theorem 1. Theorem 2 follows quite easily from Theorem 1, and the proof of Theorem 3 is a combination of the methods used to prove Theorem 1 and Theorem 2. Section 3 and Section 4 are preparatory to the proof of Theorem 1 in Section 5; our approach is more or less explicit in [8] and [12]. In [7] it is presented explicitly as in this paper, as is the construction in Section 5. This construction is actually closely related to a construction in [8], and is virtually the same as one in [12].

3. Formal inequalities and congruence relations

The congruences of atomistic lattices are especially nice—a congruence relation is determined by those atoms it identifies with 0. The oldest way of constructing lattices whose congruences represent a particular finite distributive lattice (see [5] and [8]) proceeds by constructing an atomistic lattice, controlling the congruences on the atomic level. Such lattices grow very rapidly in size, and very quickly such lattices are too large to investigate their congruences using their diagrams. The method based on formal inequalities is a very efficient method to control congruences, obviating the necessity to draw lattice diagrams.

Let L be a finite atomistic lattice, and let A be its set of atoms. The lattice L can be described completely by listing those atoms below each nonzero element of L. Since each nonzero element of L is a join of atoms, this listing can be presented as a set of expression of the form

$$\text{“}a \leq \bigvee B\text{”}, \tag{1}$$

where $a \in A$ and B is a nonempty subset of A. If $a \in B$, then no information is provided by (1)—we thus exclude this case. Then each subset B of A must contain at least two elements. Of course, L can often be determined by a proper subset of the set of all expressions (1) that hold in L.

We now reverse the above process. Given a finite set A, a *formal inequality* on A is an expression (1) where $a \in A$, where B is a subset of A with at least two elements, and where $a \notin B$. Starting with a finite set A and a set Φ of formal inequalities on A, we construct a finite atomistic lattice. The set Φ of formal inequalities on A gives rise to a *preclosure system* M_Φ on A; for each subset X of A and each $a \in A$, we set $a \in M_\Phi X$ if and only if either $a \in X$ or there is a formal inequality "$a \leq \bigvee B$" in Φ with $B \subseteq X$. A subset $X \subseteq A$ is Φ-*closed* if $M_\Phi X = X$, that is, if for each formal inequality "$a \leq \bigvee B$" in Φ, $B \subseteq X$ implies that $a \in X$.

The set of Φ-closed subsets of A forms a lattice $\mathbf{L}_\Phi(A)$, where the meet operation is set intersection. Often, when it is clear to which set of formal inequalities we are referring, we shall omit the modifier Φ, and we denote the Φ-closure of a subset $X \subseteq A$ by $\overline{X}$.

We set

$$M^0_\Phi(X) = X$$

and, inductively,

$$M^n_\Phi(X) = M_\Phi(M^{n-1}_\Phi(X))$$

for $n > 0$. We then observe:

Lemma 1. *If $a \in A$ and $X \subseteq A$, then*

$$a \in \overline{X} \quad \textit{iff} \quad a \in M^n_\Phi(X) \textit{ for some } n \geq 0.$$

Since the right-hand side of each formal inequality contains at least two elements, the singleton subsets $\{a\}$ of A are closed and so are distinct elements of $\mathbf{L}_\Phi(A)$. Writing a for $\{a\}$, we thus embed A in $\mathbf{L}_\Phi(A)$. We then have the following easy lemma, whose proof is left to the reader:

Lemma 2. *$\mathbf{L}_\Phi(A)$ is a finite atomistic lattice, and A is its set of atoms. For each formal inequality*

$$\text{“}a \leq \bigvee B\text{”}$$

in Φ, the inequality

$$a \leq \bigvee B$$

holds in $\mathbf{L}_\Phi(A)$.

Note that the last formula in the lemma can also be written as

$$a \leq \overline{B},$$

where we really mean $\{a\}$ on the left-hand side, or as

$$a \in \overline{B}.$$

We now turn to the congruence relations of the lattice $\mathbf{L}_\Phi(A)$. In dealing with formal inequalities in general some subtle technicalities arise. If the set B on the right-hand side of each formal inequality (1) has exactly two elements those difficulties disappear. We thus define a *binary formal inequality* on the finite set A as an expression

$$\text{“}a \leq b_1 \vee b_2\text{”}, \tag{2}$$

where a, b_1, b_2 are distinct elements of A. Then, given a set Φ of binary formal inequalities on A, an element $a \in A$, and a subset $X \subseteq A$, it follows that $a \in M_\Phi X$ iff either $a \in X$ or there is a formal inequality

$$\text{“}\, a \leq b_1 \vee b_2 \,\text{”}$$

in Φ with $b_1, b_2 \in X$. In this paper we need only binary formal inequalities.

The question of whether the expressions “$a \leq b_1 \vee b_2$” and “$a \leq b_2 \vee b_1$” are the same—as they are if we think of them as another way of writing “$a \leq \bigvee\{b_1, b_2\}$”—or whether they are different is irrelevant to the definition of $\mathbf{L}_\Phi(A)$. We shall subsequently express our definitions as if they are different—if they are regarded as the same, we shall just end up repeating ourselves.

The basic connection between binary formal inequalities and congruence relations is the following lemma.

Lemma 3. *Let Φ be a set of formal inequalities on the set A, and let*

$$\text{“}\, a \leq b_1 \vee b_2 \,\text{”}$$

be a binary formal inequality in Φ. Then, for each $i = 1, 2$,

$$\Theta(0, a) \leq \Theta(0, b_i)$$

in the congruence lattice of $\mathbf{L}_\Phi(A)$.

Proof. Without loss of generality, let $i = 1$. Then, in $\mathbf{L}_\Phi(A)$,

$$a \leq b_1 \vee b_2$$

and so, since a and b_2 are distinct atoms in $\mathbf{L}_\Phi(A)$,

$$a \wedge b_2 = 0.$$

Now, if $0 \equiv b_1 \pmod{\Theta}$, then

$$b_2 \equiv b_1 \vee b_2 \pmod{\Theta},$$

and so, taking the meet with a,

$$0 \equiv a \pmod{\Theta}.$$

□

Lemma 3 enables us to determine the congruence lattice of $\mathbf{L}_\Phi(A)$ for any set Φ of binary formal inequalities on A. We define a digraph structure $\mathbb{D}_\Phi(A)$, the *dependency graph* of Φ, on A. The nodes of the digraph are the elements of A, and we have an edge

$$\langle a, b\rangle \in \mathbb{D}_\Phi(A)$$

exactly when there is a $c \in A$ and either the formal inequality

$$\text{“}\, a \leq b \vee c \,\text{”} \quad \text{or} \quad \text{“}\, a \leq c \vee b \,\text{”}$$

is in Φ.

A subset X of A is said to be Φ-*arrow-closed* if

$$\langle a, b\rangle \in \mathbb{D}_\Phi(A) \text{ and } b \in X \text{ implies that } a \in X.$$

Since arrow-closure is a unary closure operation, the poset (under set-inclusion) $\mathbb{A}_\Phi(A)$ of arrow-closed subsets of A, in distinction to the poset $\mathbf{L}_\Phi(A)$ of Φ-closed sets of A, is a sublattice of the set of all subsets of A—join is set-union, as well as meet being set-intersection. Again, as for Φ-closure, when the set Φ is evident we shall use the term “arrow-closed”.

Since $\mathbb{A}_\Phi(A)$ is a distributive lattice, its join-irreducible members are of interest. These are the principal arrow-closed sets—an arrow-closed subset C of A is *the principal arrow-closed subset generated by* $a \in A$ if it is the smallest arrow-closed subset of A containing a. To describe it, an element $b \in A$ is an element of C iff either $b = a$ or there is a sequence

$$a = a_0, a_1, \dots, a_{n-1} = b$$

with

$$\langle a_{i+1}, a_i \rangle \in \mathbb{D}_\Phi(A) \quad \text{for all } i.$$

The congruence lattice of $\mathbf{L}_\Phi(A)$ is determined by the following lemma.

Lemma 4. *Let Φ be a set of binary formal inequalities on A. The map that associates with each congruence relation Θ on $\mathbf{L}_\Phi(A)$ the subset*

$$\{\, a \in A \mid a \equiv 0 \pmod{\Theta} \,\}$$

of A is an isomorphism between the congruence lattice of $\mathbf{L}_\Phi(A)$ and the lattice $\mathbb{A}_\Phi(A)$ of arrow-closed subsets of A. The inverse of this isomorphism associates with each arrow-closed subset C of A the congruence

$$\bigvee(\, \Theta(0, a) \mid a \in C \,)$$

of $\mathbf{L}_\Phi(A)$. Furthermore, with the principal arrow-closed subset determined by $a \in A$ is associated the congruence $\Theta(0, a)$.

The concept of dependency graph was formulated by Pudlák and Tůma [9] and this lemma is an easy consequence of Theorem 2.8 of that paper. A formulation of this theorem more in keeping with the spirit of this paper is Lemma 3.4 and Corollary 3.5 of [1]. The details of the relation of Lemma 4 to these results is provided in the Appendix.

4. Representing finite distributive lattices as congruence lattices

In order to prove Theorem 1 we shall have to represent the distributive lattices D and E as congruence lattices of finite lattices and also represent the $\{0, \vee\}$-homomorphism ψ as $\operatorname{xt}\varphi$ for some lattice homomorphism φ. ($\operatorname{xt}\varphi$, the extension of φ, was defined in Section 2.) Both will be accomplished by means of binary formal inequalities.

Let D be a finite distributive lattice. We denote by $J(D)$ its poset of (nonzero) join-irreducible elements. The poset $J(D)$ determines D completely, and to represent D as a congruence lattice of a finite lattice we concentrate on $J(D)$. The advantage therein is that we need not control joins of congruences—we need only control the partial order. A join-irreducible congruence relation on a finite lattice L is one determined by a prime interval, that is, a congruence relation of the form $\Theta(x, y)$ with $x \prec y$ in L. If L is atomistic, then all congruence relations are determined by which elements are identified with 0. Consequently, in a finite atomistic lattice L, the join-irreducible congruence relations are those of the form $\Theta(0, a)$ where a is an atom of L.

We use the concepts of the previous section to represent the finite distributive lattice as a congruence lattice of a finite lattice L. We choose a finite set A, which will be the set of atoms of L. A *coloring* μ of A by $J(D)$ is a map

$$\mu\colon A \to J(D).$$

The congruence relation $\Theta(0,a)$ on L will represent the element μa of D. The correct ordering on the congruence lattice of L will be determined by a set Φ of binary formal inequalities on A. The lattice L will be $\mathbf{L}_\Phi(A)$. By Lemma 3, if

$$“a \leq b_1 \vee b_2” \tag{3}$$

is in Φ, then, in $\operatorname{Con} \mathbf{L}_\Phi(A)$,

$$\Theta(0,a) \leq \Theta(0,b_i)$$

for each $i = 1, 2$. Consequently, we shall require that

$$\mu a \leq \mu b_i \text{ for } i = 1, 2. \tag{4}$$

A formal inequality (3) on A is said to be *adapted to* the coloring $\mu\colon A \to J(D)$ if (4) holds. If the formal inequality is adapted to μ then it cannot ruin the partial order on $J(D)$.

We now present a general lemma that will take care of both the congruences and the homomorphisms.

Lemma 5. *Let D be a finite distributive lattice, let A be a finite set, let $A' \subseteq A$, let*

$$\mu\colon A \to J(D)$$

be a coloring, and let Φ be a set of binary formal inequalities on A such that the following five conditions hold:

(i) *μ is surjective.*
(ii) *Each member of Φ is adapted to μ.*
(iii) *Given $x, y \in J(D)$ with $y \prec x$ in $J(D)$, there are $a, b \in A'$ with $\mu a = x$, $\mu b = y$, and $\langle b, a\rangle \in \mathbb{D}_\Phi(A)$.*
(iv) *Given distinct $a, b \in A'$ with $\mu a = \mu b$, then both*

$$\langle b, a\rangle, \langle a, b\rangle \in \mathbb{D}_\Phi(A).$$

(v) *For each $a \in A - A'$ there are $b, b' \in A'$ with $\mu b = \mu b' = \mu a$ and*

$$\langle a, b\rangle, \langle b', a\rangle \in \mathbb{D}_\Phi(A).$$

Then the map ε from D to the congruence lattice of $\mathbf{L}_\Phi(A)$ determined by

$$\varepsilon\colon x \mapsto \bigvee(\, \Theta(0,a) \mid a \in A, \mu a \leq x\,)$$

is an isomorphism between D and the congruence lattice of $\mathbf{L}_\Phi(A)$. Furthermore, for each $a \in A$,

$$\Theta(0,a) = \varepsilon\mu a.$$

We remark that, in order to represent the congruence lattice, it suffices to set $A = A'$, obviating condition (v). The set $A - A'$ is only needed for homomorphisms, as will be evident in Section 5.

Proof. By the second claim of Lemma 4, we need only show that the map ε' that associates with each $x \in D$ the subset

$$\varepsilon' x = \{\, a \in A \mid \mu a \leq x \,\}$$

is an isomorphism

$$\varepsilon'\colon D \to \mathbb{A}_\Phi(A).$$

By condition (ii), each member of Φ is adapted to μ and so, for each $x \in D$, $\varepsilon' x$ is arrow-closed. Thus ε' indeed maps to $\mathbb{A}_\Phi$. Clearly, ε' is isotone. Thus, to show that ε' is an isomorphism we need only find an isotone inverse δ of ε'.

Define

$$\delta\colon \mathbb{A}_\Phi(A) \to D$$

by setting

$$\delta\colon C \mapsto \bigvee(\, \mu a \mid a \in C\,)$$

for each arrow-closed subset C of A. δ is clearly isotone. Each element of D is determined by the set of join-irreducibles below it; thus, by condition (i), we see that $\delta \circ \varepsilon'$ is the identity map on D.

We need now only show that $\varepsilon' \circ \delta$ is the identity map on $\mathbb{A}_\Phi(A)$, that is, given $C \in \mathbb{A}_\Phi(A)$ and $b \in A$ with

$$\mu b \leq \bigvee(\, \mu a \mid a \in C\,),$$

we must show that $b \in C$. Since μb is join-irreducible, there is an $a \in C$ with $\mu b \leq \mu a$. Since C is arrow-closed, we may assume that $a, b \in A'$ by condition (v). Then, by conditions (iii) and (iv), it follows that $b \in C$. Consequently, $\varepsilon' \delta$ is the identity map on $\mathbb{A}_\Phi(A)$.

Thus, ε' is an isomorphism, and so

$$\varepsilon\colon x \mapsto \bigvee(\, \Theta(0, a) \mid a \in A, \mu a \leq x\,)$$

is an isomorphism as claimed.

Since ε' is an isomorphism and μa is join-irreducible, it follows that

$$\varepsilon' \mu a = \{\, b \in A \mid \mu b \leq \mu a\,\}$$

is a join-irreducible element of $\mathbb{A}_\Phi(A)$ and so is the principal arrow-closed subset determined by some $c \in A$. Now, clearly,

$$a, c \in \varepsilon' \mu a.$$

Thus, $\mu c \leq \mu a$. By condition (ii), $\mu u \leq \mu v$ whenever $\langle u, v\rangle \in \mathbb{D}_\Phi(A)$. Thus, by the description of principal arrow-closed subsets, $\mu a \leq \mu c$ if a and c are distinct. Thus $\mu a = \mu c$ and, by conditions (iv) and (v), $\varepsilon' \mu a$ is the principal arrow-closed subset determined by a. By the last claim of Lemma 4 it follows that

$$\varepsilon \mu a = \Theta(0, a),$$

concluding the proof. □

To relate lattice homomorphisms to formal inequalities, we recall the following lemma essentially due to P. Pudlák ([8], Lemma 4):

Lemma 6. *Let A, B be finite sets and let Φ_A, Φ_B be sets of formal inequalities on A, B, respectively. Let*

$$f\colon A \to B$$

be a partial *map such that*

$$f^{-1}(M_{\Phi_B} X) = M_{\Phi_A}(f^{-1}(X))$$

for each $X \subseteq B$. Then the map

$$\varphi\colon X \mapsto f^{-1}(X) \quad \textit{for each } \Phi_B\textit{-closed } X \subseteq B,$$

is a 0*-preserving lattice homomorphism*

$$\varphi\colon \mathbf{L}_{\Phi_B}(B) \to \mathbf{L}_{\Phi_A}(A).$$

Furthermore, if f is surjective, then φ is an embedding.

Using Lemma 1, the proof is quite easy. Both Φ_A and Φ_B will consist of binary formal inequalities in our applications of Lemma 6.

5. The proof of Theorem 1

Recall that if $\varphi\colon K \to L$ is a lattice homomorphism, then $\operatorname{xt}\varphi\colon \operatorname{Con} K \to \operatorname{Con} L$ is defined by setting

$$(\operatorname{xt}\varphi)\Theta = \Theta_L(\varphi^2(\Theta))$$

for each congruence relation Θ on K. Using Mal'cev's lemma, Theorem 1.10.4 of [2], the proof of the following lemma is easy.

Lemma 7. *Let K and L be lattices, let $\varphi\colon K \to L$ be a lattice homomorphism, and let $X \subseteq K^2$. Then*

$$(\operatorname{xt}\varphi)\Theta_K(X) = \Theta_L(\varphi^2(X)).$$

Corollary 1. *If $\varphi\colon K \to L$ is a lattice homomorphism, then* $\operatorname{xt}\varphi\colon \operatorname{Con} K \to \operatorname{Con} L$ *preserves joins.*

Proof. Let Θ_1 and Θ_2 be congruence relations on K. Since

$$\Theta_1 \vee \Theta_2 = \Theta_K(\Theta_1 \cup \Theta_2)$$

and

$$\varphi^2(\Theta_1 \cup \Theta_2) = \varphi^2(\Theta_1) \cup \varphi^2(\Theta_2),$$

we conclude by Lemma 7 that

$$\begin{aligned}(\operatorname{xt}\varphi)(\Theta_1 \vee \Theta_2) &= \Theta_L(\varphi^2(\Theta_1) \cup \varphi^2(\Theta_2))\\ &= \Theta_L(\varphi^2(\Theta_1)) \vee \Theta_L(\varphi^2(\Theta_2)) = (\operatorname{xt}\varphi)\Theta_1 \vee (\operatorname{xt}\varphi)\Theta_2,\end{aligned}$$

concluding the proof. □

We also remark that $\{0, \vee\}$-homomorphisms of finite lattices are determined by the join-irreducible elements of the lattices:

Lemma 8. *Let D, D' be finite lattices, and let*

$$\psi_1, \psi_2\colon D \to D'$$

be $\{0, \vee\}$-homomorphisms. Then $\psi_1 = \psi_2$ if and only if, for each join-irreducible $x \in D$ and join-irreducible $y \in D'$,

$$y \le \psi_1 x \quad \textit{is equivalent to} \quad y \le \psi_2 x. \tag{5}$$

Proof. If $\psi_1 = \psi_2$ then (5) clearly holds. Conversely, let (5) hold. Since each element of D' is the join of the join-irreducibles beneath it, it follows that

$$\psi_1 x = \psi_2 x \quad \text{for each } x \in J(D).$$

Then, since each element of D is a join of join-irreducibles, we conclude that $\psi_1 = \psi_2$. □

We now proceed to prove Theorem 1.

Let D and E be finite distributive lattices and let

$$\psi\colon E \to D$$

be a $\{0, \vee\}$-homomorphism. We proceed in several steps.

Step 1. We represent E as the congruence lattice of a lattice K.

Let B be a finite set with a coloring

$$\mu_B\colon B \to J(E)$$

and a set Ψ of binary formal inequalities such that the conditions of Lemma 5 hold, whereby

$$\beta\colon E \to \operatorname{Con} \mathbf{L}_\Psi(B),$$

determined by

$$\beta\colon x \mapsto \bigvee(\, \Theta(0, b) \mid b \in B, \mu_B b \leq x \,),$$

is an isomorphism. We set $K = \mathbf{L}_\Psi(B)$.

We present an example of such a B. We set

$$B = B' = \{\, x_i \mid x \in J(E), i \in \{0, 1, 2\} \,\},$$

and set

$$\mu_B x_i = x.$$

Ψ consists of all formal inequalities

$$\text{“} x_i \leq x_j \vee x_k \text{”}, \quad \text{where } x \in J(E) \text{ and } \{i, j, k\} = \{0, 1, 2\} \tag{6}$$

and all formal inequalities

$$\text{“} y_i \leq y_j \vee x_k \text{”} \quad \text{where } y \prec x \text{ in } J(E) \text{ and } \{i, j, k\} = \{0, 1, 2\}. \tag{7}$$

The dependency graph $\mathbb{D}_\Psi(B)$ consists of all edges

$$\langle x_j, x_i \rangle \quad \text{with } i \neq j$$

and

$$\langle y_j, x_i \rangle \quad \text{with } y \prec x \text{ in } J(E) \text{ and } i \neq j.$$

It is easy to see that conditions (i)–(v) (condition (v) vacuously) of Lemma 5 hold.

Step 2. We represent D as the congruence lattice of a finite lattice L. We first choose a finite set A' with a coloring

$$\mu_{A'}\colon A' \to J(D)$$

and a set Φ' of binary formal inequalities such that conditions (ii), (iii), (iv) of Lemma 5, along with the following strengthening of condition (i), hold:

(i') For each $x \in J(D)$ there are at least two distinct $a, a' \in A'$ with

$$\mu_{A'} a = \mu_{A'} a' = x.$$

If we choose A' and Φ' as we chose B and Ψ above, then condition (i') holds, since each $x \in J(D)$ colors three distinct elements of A', the elements x_0, x_1, x_2.

Then A' along with Φ' already represents D. However, in order to get our map φ we must extend A' to a set of atoms A. For each $b \in B$ and each $x \in J(D)$ with

$x \le \psi\mu_B b$ we want an atom in φb of color x. We arrange this by adding to A', for each $b \in B$ and $a \in A'$ with

$$\psi\mu_B b \ge \mu_{A'} a,$$

a new element $\langle b, a\rangle$ of color $\mu_{A'} a$. For fixed $b \in B$, the $\langle b, a\rangle$ will be all the atoms of A lying below φb.

Summarizing,

$$A = A' \cup \{\, \langle b, a\rangle \in B \times A' \mid \psi\mu_B b \ge \mu_{A'} a \,\}$$

with a coloring $\mu_A : A \to J(D)$ defined by setting

$$\mu_A a = \mu_{A'} a$$

for $a \in A'$ and

$$\mu_A \langle b, a\rangle = \mu_A a (= \mu_{A'} a).$$

To ensure that the $\langle b, a\rangle$ yield the correct congruences, we choose for each $\langle b, a\rangle \in A$ an $a' \ne a$ in A' with $\mu_A a' = \mu_A a$ (this is where condition (i') comes in), and add to Φ' the formal inequalities

$$\text{“}\langle b, a\rangle \le a \vee a'\text{”}, \qquad \text{“}a \le \langle b, a\rangle \vee a'\text{”}. \tag{8}$$

These, furthermore, ensure that condition (v) of Lemma 5 hold. Note that the color of each member of the right-hand side of each formal inequality is the same as that of the left-hand side; thus each of these formal inequalities is adapted to μ_A.

We wish to define φ so that

$$\varphi b = \{\, \langle b, a\rangle \mid a \in A' \text{ with } \mu_A a \le \psi\mu_B b \,\}.$$

Consequently, any formal inequality in Ψ relating the various b's must be reflected in a formal inequality on $A - A'$. For each formal inequality

$$\text{“}b \le b_1 \vee b_2\text{”} \tag{9}$$

in Ψ and each $a \in A'$ with $\psi\mu_B b \ge \mu_A a$, we take the binary formal inequality

$$\text{“}\langle b, a\rangle \le \langle b_1, a\rangle \vee \langle b_2, a\rangle\text{”} \tag{10}$$

on A. Since (9) is adapted to μ_B, the right-hand side of (10) makes sense—

$$\mu_B b_i \ge \mu_B b$$

for each $i = 1, 2$, and so

$$\psi\mu_B b_i \ge \psi\mu_B b \ge \mu_A a.$$

Observe also that (10) is trivially adapted to μ_A—all elements there have the same color.

We now let Φ be the set consisting of all the formal inequalities in Φ' along with all the formal inequalities (8) and (10). Set $L = \mathbf{L}_\Phi(A)$. The hypotheses of Lemma 5 hold for Φ. We thus have an isomorphism

$$\alpha \colon D \to \operatorname{Con}(L)$$

given by

$$\alpha x = \bigvee (\, \Theta(0, u) \mid u \in A, \mu u \le x \,).$$

Step 3. We now construct a 0-preserving lattice homomorphism $\varphi \colon K \to L$.

We use Lemma 6. We define a partial map

$$f\colon A \to B$$

with domain $A - A'$ by setting

$$f\langle b, a\rangle = b$$

for each $\langle b, a\rangle \in A - A'$. We then have:

Lemma 9.

$$f^{-1}(M_\Psi X) = M_\Phi(f^{-1}(X))$$

for each $X \subseteq B$.

Proof. Let

$$u \in f^{-1}(M_\Psi X).$$

Since the domain of f is $A - A'$,

$$u = \langle b, a\rangle \in f^{-1}(M_\Psi X)$$

for some $\langle b, a\rangle \in A - A'$. Then

$$b = f\langle b, a\rangle \in M_\Psi X.$$

If $b \in X$, then

$$\langle b, a\rangle \in f^{-1}(X) \subseteq M_\Phi(f^{-1}(X)).$$

Otherwise, there are $b_1, b_2 \in X$ and a formal inequality

$$\text{“}\, b \leq b_1 \vee b_2\, \text{”}$$

in Ψ. Then the formal inequality

$$\text{“}\, \langle b, a\rangle \leq \langle b_1, a\rangle \vee \langle b_2, a\rangle\, \text{”},$$

an instance of (10), is a member of Φ. Since

$$\langle b_1, a\rangle, \langle b_2, a\rangle \in f^{-1}(X),$$

we get

$$u = \langle b, a\rangle \in M_\Phi(f^{-1}(X)),$$

thereby showing that

$$f^{-1}(M_\Psi X) \subseteq M_\Phi(f^{-1}(X)).$$

Conversely, let

$$u \in M_\Phi(f^{-1}(X)).$$

If $u \in f^{-1}(X)$, then, since $X \subseteq M_\Psi X$,

$$u \in f^{-1}(M_\Psi X).$$

Otherwise, the are $u_1, u_2 \in f^{-1}(X)$ with

$$\text{“}\, u \leq u_1 \vee u_2\, \text{”} \tag{11}$$

in Φ. Since the domain of f is $A - A'$, the formal inequality (11) must be an instance of (10). That is, there are $b, b_1, b_2 \in B$ and $a \in A'$ with

$$u = \langle b, a\rangle$$

and

$$u_i = \langle b_i, a \rangle$$

for $i = 1, 2$ such that the formal inequality

$$\text{“} b \leq b_1 \vee b_2 \text{”} \tag{12}$$

is a member of Ψ. Now

$$b_i = f u_i \in X$$

for each $i = 1, 2$. Thus, by (12),

$$b \in M_\Psi X$$

and, consequently,

$$u = \langle b, a \rangle \in f^{-1}(M_\Psi X).$$

We have thus verified that

$$M_\Phi(f^{-1}(X)) \subseteq f^{-1}(M_\Psi X)$$

also, concluding the proof of the lemma. □

By Lemma 6, we have a 0-preserving lattice homomorphism $\varphi\colon K \to L$ given by

$$\varphi\colon X \mapsto f^{-1}(X)$$

for each Ψ-closed subset X of B. So,

$$\varphi\colon b \mapsto \bigvee(\, \langle b, a \rangle \in A \mid \mu_A a \leq \psi \mu_B b \,). \tag{13}$$

Step 4. We show that

$$\alpha \circ \psi = (\mathrm{xt}\, \varphi) \circ \beta.$$

We apply Lemma 8. Let $x \in J(E)$. Any join-irreducible congruence on L is of the form $\Theta(0, u)$ for some $u \in A$. Since

$$\alpha \psi x = \bigvee(\, \Theta(0, v) \mid \mu_A v \leq \psi x \,)$$

and $\Theta(0, u)$ is join-irreducible, we have

$$\begin{aligned} \Theta(0, u) \leq \alpha \psi x \quad &\text{iff} \quad \Theta(0, u) \leq \Theta(0, v) \text{ for some } v \text{ with } \mu_A v \leq \psi x \\ &\text{iff} \quad \mu_A u \leq \psi x, \end{aligned} \tag{14}$$

by the last statement of Lemma 5.

Now, choose any $b \in B$ with $\mu_B b = x$. Then

$$\Theta(0, b) = \beta x$$

and

$$(\mathrm{xt}\, \varphi)(\Theta(0, b)) = \Theta(0, \varphi b) = \bigvee(\, \Theta(0, \langle b, a \rangle) \mid \mu_A a \leq \psi \mu_B b = \psi x \,),$$

the first equality by Lemma 7, since φ preserves 0, and the second equality by (13). Consequently,

$$\begin{aligned} &\Theta(0, u) \leq (\mathrm{xt}\, \varphi) \beta x = (\mathrm{xt}\, \varphi)(\Theta(0, b)) \\ &\qquad \text{iff} \quad \Theta(0, u) \leq \Theta(0, \langle b, a \rangle) \text{ for some } a \in A' \text{ with } \mu_A a \leq \psi \mu_B b = \psi x \\ &\qquad \text{iff} \quad \mu_A u \leq \psi x. \end{aligned} \tag{15}$$

By (14) and (15),

$$\Theta(0, u) \leq \alpha\psi x \quad \text{iff} \quad \Theta(0, u) \leq (\operatorname{xt} \varphi)\beta x.$$

Thus, by Lemma 8,

$$\alpha \circ \psi = (\operatorname{xt} \varphi) \circ \beta.$$

Step 5. Since α and β are isomorphisms, ψ separates 0 iff $\operatorname{xt} \varphi$ separates 0, and so the final claim of Theorem 1 holds—see the comment just before the statement of the theorem in Section 2.

We have thus concluded the proof of Theorem 1.

Observe how the construction neatly separates the determination of the congruences of L and the determination of the homomorphism φ—the lattice structure of the congruences is determined in A' by the formal inequalities Φ', while φ is determined in $A - A'$ by the formal inequalities (10). The formal inequalities (8) serve only to identify congruences determined in $A - A'$ with those in A'.

6. The proof of Theorem 2

Theorem 2 follows quite easily from Theorem 1 by a rather straight-forward duality theory between $\{0, \vee\}$-homomorphisms and $\{1, \wedge\}$-homomorphisms of finite lattices.

Let D and D' be finite lattices, and let

$$\varphi \colon D' \to D$$

be a $\{0, \vee\}$-homomorphism. We define the M-*dual of* φ ("M" for meet)

$$\varphi_M \colon D \to D'$$

by setting

$$\varphi_M x = \bigvee(\, y \in D' \mid \varphi y \leq x \,)$$

for each $x \in D$. Then φ_M is completely determined by the condition

$$y \leq \varphi_M x \quad \text{iff} \quad \varphi y \leq x \tag{16}$$

for all $x \in D$, $y \in D'$. Indeed, if $y \leq \varphi_M x$, then

$$\varphi y \leq \varphi \bigvee(\, z \in D' \mid \varphi z \leq x \,) = \bigvee(\, \varphi z \in D' \mid \varphi z \leq x \,) \leq x.$$

Conversely, if $\varphi y \leq x$, then, clearly, $y \leq \varphi_M x$.

From (16) we have, for all $y \in D'$,

$$y \leq \varphi_M 1,$$

that is,

$$\varphi_M 1 = 1.$$

Also, given x_1, $x_2 \in D$, then, for each $y \in D'$,

$$\begin{aligned} y \leq \varphi_M(x_1 \wedge x_2) \quad &\text{iff} \quad \varphi y \leq x_1 \wedge x_2 \\ &\text{iff} \quad \varphi y \leq x_1 \text{ and } \varphi y \leq x_2 \\ &\text{iff} \quad y \leq \varphi_M x_1 \text{ and } y \leq \varphi_M x_2 \\ &\text{iff} \quad y \leq \varphi_M x_1 \wedge \varphi_M x_2. \end{aligned}$$

Thus,

$$\varphi_M(x_1 \wedge x_2) = \varphi_M x_1 \wedge \varphi_M x_2.$$

Consequently, $\varphi_M : D \to D'$ is a $\{1, \wedge\}$-homomorphism.

By considering the dual ordering of D and D', we get for each $\{1, \wedge\}$-homomorphism $\varphi\colon D' \to D$ a $\{0, \vee\}$-homomorphism

$$\varphi_J : D \to D',$$

the J-*dual of* φ ("J" for join) characterized by

$$\varphi_J x \leq y \quad \text{iff} \quad x \leq \varphi y \tag{17}$$

for all $x \in D$, $y \in D'$.

Since the J-dual and M-dual are dual concepts (in the sense of partial order), each of their properties yields a dual property by exchanging the operators J and M and reversing the partial order of the lattices involved.

We observe that the operators J and M are inverses of each other:

Lemma 10. *If $\varphi\colon D' \to D$ is a $\{1, \wedge\}$-homomorphism of finite lattices, then*

$$(\varphi_J)_M = \varphi.$$

Proof. By (16), the characterization of M-dual, this is exactly what (17) states. ☐

We, of course, have the dual result $(\varphi_M)_J = \varphi$ for any $\{0, \vee\}$-homomorphism φ.

The operators M and (dually) J are (contravariantly) functorial:

Lemma 11. *If D', D, and D'' are finite lattices and $\varphi\colon D' \to D$, $\psi\colon D \to D''$ are $\{0, \vee\}$-homomorphisms, then*

$$(\psi \circ \varphi)_M = \varphi_M \circ \psi_M.$$

Proof. Let $x \in D''$. For each $y \in D'$,

$$\begin{aligned} y \leq (\psi \circ \varphi)_M x \quad &\text{iff} \quad \psi\varphi y \leq x \\ &\text{iff} \quad \varphi y \leq \psi_M x \\ &\text{iff} \quad y \leq \varphi_M \psi_M x, \end{aligned}$$

proving the claim. ☐

Next, we consider an isomorphism $\varphi\colon D' \to D$. Then φ is both a $\{0, \vee\}$- and a $\{1, \wedge\}$-homomorphism. We have:

Lemma 12. *If $\varphi\colon D' \to D$ is an isomorphism, then*

$$\varphi_M = \varphi_J = \varphi^{-1}.$$

Proof. Let $x \in D$. Then, for each $y \in D'$,

$$y \leq \varphi_M x \quad \text{iff} \quad \varphi y \leq x \quad \text{iff} \quad y \leq \varphi^{-1} x,$$

and so $\varphi_M x = \varphi^{-1} x$.

Dually, $\varphi_J = \varphi^{-1}$. ☐

Observe, finally:

Lemma 13. *If $\varphi\colon D' \to D$ is a $\{0, \vee\}$-homomorphism of finite lattices, then φ separates 0 iff φ_M preserves 0.*

Proof.

$$\begin{aligned} \varphi \text{ separates } 0 \quad &\text{iff} \quad \varphi x \leq 0 \text{ implies } x = 0 \\ &\text{iff} \quad x \leq \varphi_M 0 \text{ implies } x = 0 \\ &\text{iff} \quad \varphi_M 0 = 0. \end{aligned}$$

☐

We now relate this duality theory to maps of congruences.

Lemma 14. *Let K and L be finite lattices, and let $\varphi\colon K \to L$ be a lattice homomorphism. Then*

$$\operatorname{rs}\varphi = (\operatorname{xt}\varphi)_M$$

and

$$\operatorname{xt}\varphi = (\operatorname{rs}\varphi)_J.$$

Proof. Let Θ be any congruence relation on L. Then, for each congruence relation Θ' on K,

$$\Theta' \leq (\operatorname{rs}\varphi)(\Theta) \quad \text{iff} \quad \varphi^2(\Theta') \leq \Theta$$

by the definition of $\operatorname{rs}\varphi$, and

$$\varphi^2(\Theta') \leq \Theta \quad \text{iff} \quad (\operatorname{xt}\varphi)(\Theta') \leq \Theta$$

by the definition of $\operatorname{xt}\varphi$. Then, by (16) and (17),

$$\operatorname{rs}\varphi = (\operatorname{xt}\varphi)_M$$

and

$$\operatorname{xt}\varphi = (\operatorname{rs}\varphi)_J,$$

concluding the proof. ☐

Theorem 2 now follows quite easily from Theorem 1. If $\psi\colon D \to E$ is a $\{1, \wedge\}$-homomorphism, then $\psi_J\colon E \to D$ is a $\{0, \vee\}$-homomorphism. By Theorem 1, there are finite lattices K and L, a lattice homomorphism $\varphi\colon K \to L$, and isomorphisms $\alpha\colon D \to \operatorname{Con} L$, $\beta\colon E \to \operatorname{Con} K$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\psi_J} & D \\ \cong\big\downarrow\beta & & \cong\big\downarrow\alpha \\ \operatorname{Con} K & \xrightarrow{\operatorname{xt}\varphi} & \operatorname{Con} L \end{array}$$

is commutative. Taking the M-dual we get the commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{(\psi_J)_M} & E \\ \big\uparrow\alpha_M & & \big\uparrow\beta_M \\ \operatorname{Con} L & \xrightarrow{(\operatorname{xt}\varphi)_M} & \operatorname{Con} K \end{array}$$

by Lemma 11. So, by Lemma 10, Lemma 12, and Lemma 14, we have the commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{\psi} & E \\ \cong\big\uparrow\alpha^{-1} & & \cong\big\uparrow\beta^{-1} \\ \operatorname{Con} L & \xrightarrow{\operatorname{rs}\varphi} & \operatorname{Con} K \end{array}$$

which is equivalent to the commutative diagram of Theorem 2.

An appeal to Lemma 13 completes the proof of Theorem 2.

7. The proof of Theorem 3

The proof of Theorem 3 follows by combining the methods of Section 5 and Section 6.

First, we show that any isotone map can be written as the composition of a $\{0, \vee\}$-homomorphism and a $\{1, \wedge\}$-homomorphism.

Lemma 15. *Let D_1 and D_2 be finite lattices, and let $\psi\colon D_1 \to D_2$ be an isotone map. Then there are a finite distributive lattice D, a $\{0, \vee\}$-homomorphism*

$$\psi_1\colon D_1 \to D$$

that separates 0, and a $\{1, \wedge\}$-homomorphism

$$\psi_2\colon D \to D_2$$

with

$$\psi = \psi_2 \circ \psi_1.$$

Proof. Recall that an *order-filter* in a poset P is a subset $X \subseteq P$ such that whenever $x \in P$ and $y \geq x$, then $y \in P$. If $x \in P$ then $[x)$ denotes the order filter

$$[x) = \{\, y \in P \mid y \geq x \,\}.$$

We then let D be the set of order-filters of D_1, including the empty order-filter $\varnothing$. We partially order D by the opposite of set-containment:

$$X \leq Y \quad \text{iff} \quad Y \subseteq X.$$

Then D is a finite distributive lattice, with

$$X \vee Y = X \cap Y \text{ and } X \wedge Y = X \cup Y.$$

The 0 of D is the order-filter D_1 itself, and the 1 of D is the empty order-filter. We define

$$\psi_1\colon D_1 \to D$$

by setting

$$\psi_1 x = [x).$$

Then

$$\psi_1 x = 0_D \quad \text{iff} \quad [x) = D_1 \quad \text{iff} \quad x = 0$$

and

$$\psi_1(x) \vee \psi_1(y) = [x) \cap [y) = [x \vee y) = \psi_1(x \vee y),$$

that is, ψ_1 is a $\{0, \vee\}$-homomorphism that separates 0.

We next define

$$\psi_2\colon D \to D_2$$

by setting

$$\psi_2 X = \bigwedge \psi(X),$$

—where $\psi(X)$ denotes the image of the subset X under ψ—for each order-filter X of D_1. Then

$$\psi_2 1_D = \bigwedge \psi(\varnothing) = \bigwedge \varnothing = 1$$

and

$$\psi_2(X \wedge Y) = \bigwedge(\psi(X \cup Y)) = \bigwedge(\psi(X) \cup \psi(Y)) = \bigwedge \psi(X) \wedge \bigwedge \psi(Y)$$
$$= \psi_2 X \wedge \psi_2 Y.$$

Thus ψ_2 is a $\{1, \wedge\}$-homomorphism.

Finally, for each $x \in D_1$,

$$\psi_2 \psi_1 x = \bigwedge \psi[x) = \psi x,$$

since ψ is isotone and x is the smallest element of $[x)$. Thus,

$$\psi = \psi_2 \circ \psi_1,$$

concluding the proof. □

We now proceed to prove Theorem 3. We are given an isotone map

$$\psi \colon D_1 \to D_2.$$

So, by Lemma 15, we have a finite distributive lattice D, a $\{0, \vee\}$-homomorphism $\psi_1 \colon D_1 \to D$ that separates 0, and a $\{1, \wedge\}$-homomorphism $\psi_2 \colon D \to D_2$ with $\psi_2 \circ \psi_1 = \psi$. We then have the $\{0, \vee\}$-homomorphism $(\psi_2)_J \colon D_2 \to D$. We shall simultaneously represent ψ_1 and $(\psi_2)_J$ as extension maps. We proceed exactly as in Section 5.

For uniformity of notation, we set

$$\overline{\psi}_1 = \psi_1$$

and

$$\overline{\psi}_2 = (\psi_2)_J.$$

For each $i = 1$, 2, we choose a finite set B_i, a coloring

$$\mu_{B_i} \colon B_i \to J(D_i)$$

and a set Φ_i of binary formal inequalities on B_i satisfying the conditions of Lemma 5.

Again, exactly as in Section 5, we choose a finite set A', a coloring

$$\mu_{A'} \colon A' \to J(D),$$

and a set of binary formal inequalities Φ' such that the following conditions hold: (ii), (iii), and (iv) of Lemma 5 and condition (i'),

For each $x \in J(D)$ there are at least two distinct $a, a' \in A'$ with

$$\mu_{A'} a = \mu_{A'} a' = x,$$

in Step 2 in Section 5.

We next extend A' to A and Φ' to Φ. Our procedure differs from that in Section 5 only in that we now take into account both B_1 and B_2. Set

$$A = A' \cup \{\, \langle b, a \rangle \in B_1 \times A' \mid \overline{\psi}_1 \mu_{B_1} b \geq \mu_{A'} a \,\}$$
$$\cup \{\, \langle b, a \rangle \in B_2 \times A' \mid \overline{\psi}_2 \mu_{B_2} b \geq \mu_{A'} a \,\}$$

and set

$$\mu_A a = \mu_{A'} a$$

for $a \in A'$ and

$$\mu_A \langle b, a \rangle = \mu_A a.$$

For each $\langle b, a\rangle \in A$ we choose an $a' \neq a$ with $\mu_A a' = \mu_A a$ and add to Φ' the formal inequalities

$$\text{“}\langle b, a\rangle \leq a \vee a'\text{”}, \qquad \text{“}a \leq \langle b, a\rangle \vee a'\text{”}. \tag{18}$$

For each $i = 1$, 2, each formal inequality

$$\text{“}b \leq b_1 \vee b_2\text{”}$$

in Ψ_i, and each $a \in A'$ with $\overline{\psi}_i \mu_{B_i} b \geq \mu_A a$, we further add to Φ' the formal inequality

$$\text{“}\langle b, a\rangle \leq \langle b_1, a\rangle \vee \langle b_2, a\rangle\text{”}, \tag{19}$$

thereby getting the set Φ of binary formal inequalities on A. Exactly as in Section 5, the conditions of Lemma 5 hold.

For each $i = 1$, 2, we set

$$L_i = \mathbf{L}_{\Phi_i}(B_i)$$

and we set

$$L = \mathbf{L}_{\Phi}(A).$$

We then have isomorphisms

$$\alpha_i \colon D_i \to \operatorname{Con} L_i$$

and

$$\alpha \colon D \to \operatorname{Con} L.$$

For each $i = 1$, 2, we define a partial map

$$f_i \colon A \to B_i$$

with domain $\{\, \langle b, a\rangle \in A \mid b \in B_i \,\}$ by setting

$$f_i\langle b, a\rangle = b \quad \text{if } b \in B_i.$$

These determine 0-preserving lattice homomorphisms

$$\varphi_i \colon L_i \to L$$

such that the diagrams

$$\begin{array}{ccc} D_i & \xrightarrow{\overline{\psi}_i} & D \\ \cong\Big\downarrow \alpha_i & & \cong\Big\downarrow \alpha \\ \operatorname{Con} L_i & \xrightarrow{\operatorname{xt} \varphi_i} & \operatorname{Con} L \end{array}$$

are commutative. That is, the diagrams

$$\begin{array}{ccc} D_1 & \xrightarrow{\psi_1} & D \\ \cong\Big\downarrow \alpha_1 & & \cong\Big\downarrow \alpha \\ \operatorname{Con} L_1 & \xrightarrow{\operatorname{xt} \varphi_1} & \operatorname{Con} L \end{array} \tag{20}$$

and

$$\begin{array}{ccc} D_2 & \xrightarrow{(\psi_2)_J} & D \\ \cong\Big\downarrow \alpha_2 & & \cong\Big\downarrow \alpha \\ \operatorname{Con} L_2 & \xrightarrow{\operatorname{xt}\varphi_2} & \operatorname{Con} L \end{array} \tag{21}$$

are commutative.

Since ψ_1 separates 0, it follows that φ_1 is an embedding.

Taking the M-dual of (21), exactly as in Section 6, we get the commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{\psi_2} & D_2 \\ \cong\Big\downarrow \alpha & & \cong\Big\downarrow \alpha_2 \\ \operatorname{Con} L & \xrightarrow{\operatorname{rs}\varphi_2} & \operatorname{Con} L_2 \end{array} \tag{22}$$

Combining (20) and (22) we get the commutative diagram of Theorem 3.

Finally, since ψ_1 separates 0, ψ preserves 0 iff ψ_2 preserves 0 iff φ_2 is an embedding, concluding the proof of Theorem 3.

8. Concluding remarks

The construction in Lemma 15 can be dualized, expressing an isotone map as a $\{1, \wedge\}$-homomorphism followed by a $\{0, \vee\}$-homomorphism. Specifically, we can let D be the lattice of order-ideals of D_1 and define $\psi_1 \colon D_1 \to D$ by setting

$$\psi_1 \colon x \mapsto (x] = \{\, y \in D_1 \mid y \leq x \,\}.$$

The map ψ_1 is a $\{1, \wedge\}$-homomorphism. We then define a $\{0, \vee\}$-homomorphism $\psi_2 \colon D \to D_2$ by setting

$$\psi_2 \colon X \mapsto \bigvee \psi(X).$$

Then $\psi = \psi_2 \circ \psi_1$.

We then choose a finite set A and a set of binary formal inequalities Φ on A such that the congruence lattice of $L = \mathbf{L}_\Phi(A)$ represents D. As in Section 5, we construct lattices L_1, L_2 and lattice homomorphisms $\varphi_1 \colon L \to L_1$, $\varphi_2 \colon L \to L_2$ such that $\operatorname{xt}\varphi_1$ represents $(\psi_1)_J$ and $\operatorname{xt}\varphi_2$ represents ψ_2. Then $(\operatorname{xt}\varphi_2) \circ (\operatorname{rs}\varphi_1)$ represents ψ.

However, because of the special role played by the 0 congruence relation, the conditions for φ_1 and φ_2 to be embeddings are more complicated than in Theorem 3. Note, first, that, in the construction outlined, φ_1 will not be an embedding since

$$\psi_1 0 = (0] \neq \varnothing = 0_D.$$

If ψ does not preserve 0, then ψ_2 will separate 0:

$$\psi_2 X = \bigvee \psi(X) \geq \psi 0 > 0 = \psi_2 \varnothing.$$

Thus, if ψ does not preserve 0, then φ_1 will not be an embedding and φ_2 will be an embedding. It is also clear that if ψ does not preserve 0, then, no matter how we choose D and the $\{1, \wedge\}$-homomorphism $\psi_1 \colon D_1 \to D$, ψ_1 will not preserve 0 since ψ_2 must preserve 0.

On the other hand, if ψ preserves 0, then we can change our choice of D—now D will consist of the *nonempty* order-ideals of D_1, whereby $0_D = (0]$ and so ψ_1

preserves 0, and thus φ_1 is an embedding. Then, since ψ preserves 0, $\psi_2\colon D \to D_2$, given by

$$\psi_2\colon X \mapsto \bigvee \psi(X),$$

will preserve 0 and will furthermore separate 0 exactly when ψ separates 0.

We thus have the following theorem:

Theorem 4. *Let D_1 and D_2 be finite distributive lattices, and let*

$$\psi\colon D_1 \to D_2$$

be an isotone map. Then there are finite lattices L_1, L_2, L, lattice homomorphisms

$$\varphi_1\colon L \to L_1, \qquad \varphi_2\colon L \to L_2,$$

and isomorphisms

$$\alpha_1\colon D_1 \to \operatorname{Con} L_1, \qquad \alpha_2\colon D_2 \to \operatorname{Con} L_2$$

such that

$$\alpha_2 \circ \psi = (\operatorname{xt} \varphi_2) \circ (\operatorname{rs} \varphi_1) \circ \alpha_1,$$

that is, such that the diagram

$$\begin{array}{ccccc} D_1 & & \xrightarrow{\psi} & & D_2 \\ \cong \big\downarrow \alpha_1 & & & & \cong \big\downarrow \alpha_2 \\ \operatorname{Con} L_1 & \xrightarrow{\operatorname{rs} \varphi_1} & \operatorname{Con} L & \xrightarrow{\operatorname{xt} \varphi_2} & \operatorname{Con} L_2 \end{array}$$

is commutative.

If ψ does not preserve 0, then φ_1 cannot be an embedding, but φ_2 can be chosen to be an embedding. If ψ preserves 0, then φ_1 can be chosen to be an embedding. If ψ also separates 0, then both φ_1 and φ_2 can be chosen to be embeddings.

9. Appendix

In [9] and [1] the congruence lattice of a finite lattice L is characterized in terms of a dependency graph on the set $J(L)$ of join-irreducible elements of L. Here we are only concerned with finite atomistic lattices L. In this case, the set $J(L)$ is the same as the set $A(L)$ of atoms of L. We formulate the results in [9] and [1] for such lattices.

We consider inequalities in L of the form

$$a \leq \bigvee B$$

where $a \in A(L)$ and $B \subseteq A(L)$. Such an inequality is said to be *minimal* if

$$a \notin B$$

and, for all proper subsets B' of B,

$$a \nleq \bigvee B'.$$

We define a digraph structure $\mathbb{D}_L$, the *dependency graph* of L, on $A(L)$. The nodes of the digraph are the elements of $A(L)$, and we have an edge

$$\langle a, b \rangle \in \mathbb{D}_L$$

exactly when there is a $B \subseteq A(L)$ with $b \in B$ such that

$$a \leq \bigvee B$$

is is aminimal inequality in L.

A subset X of $A(L)$ is said to be *arrow-closed* if

$$\langle a, b \rangle \in \mathbb{D}_L \text{ and } b \in X \text{ implies that } a \in X.$$

We denote by $\mathbb{A}_L$ the set of arrow-closed subsets of $A(L)$.

Then Theorem 2.8 of [9] and Lemma 3.4 and Corollary 3.5 of [1] can be formulated for atomistic lattices as follows.

Lemma 16. *Let L be a finite atomistic lattice. The map that associates with each congruence relation Θ on L the subset*

$$\{\, a \in A(L) \mid a \equiv 0 \quad (\Theta) \,\}$$

of $A(L)$ is an isomorphism between the congruence lattice of L and the lattice $\mathbb{A}_L$ of arrow-closed subsets of $A(L)$. The inverse of this isomorphism associates with each arrow-closed subset C of $A(L)$ the congruence

$$\bigvee (\, \Theta(0, a) \mid a \in C \,)$$

of L. Furthermore, with the principal arrow-closed subset determined by $a \in A(L)$ is associated the congruence $\Theta(0, a)$.

Lemma 4 then follows immediately from the following observation.

Lemma 17. *Let A be a finite set and let Φ be a set of binary formal inequalities defined on A. Then, setting $L = \mathbf{L}_\Phi(A)$,*

$$\mathbb{A}_L = \mathbb{A}_\Phi(A).$$

Proof. We need only show that a subset X of A is arrow-closed in the sense of this section iff it is Φ-arrow-closed.

If X is arrow-closed, then X is Φ-arrow-closed since the formal inequalities in Φ hold in L and are clearly minimal.

On the other hand, let X be Φ-arrow-closed. We show that X is arrow-closed. Let $b \in X$ and let $\langle a, b \rangle \in \mathbb{D}_L$. Then some minimal inequality

$$a \leq \bigvee B$$

with $b \in B \subseteq A$ holds in L. Then, by Lemma 1 and in virtue of the fact that this inequality is minimal, there is a sequence

$$b = a_0, a_1, \ldots, a_{n-1} = a$$

with

$$\langle a_{i+1}, a_i \rangle \in \mathbb{D}_\Phi(A) \quad \text{for all } i.$$

It then follows that $a \in X$, showing that X is arrow-closed and concluding the proof. $\square$

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