CONGRUENCE REPRESENTATIONS OF JOIN-HOMOMORPHISMS OF DISTRIBUTIVE LATTICES: A SHORT PROOF

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To the memory of Milan Kolibiar

Abstract. András Huhn proved the following theorem: Let D and E be finite distributive lattices, and let $\psi\colon D \to E$ be a $\{0\}$-preserving join-homomorphism. Then there are finite lattices K and L and there is a lattice homomorphism $\varphi\colon K \to L$ such that $\operatorname{Con} K$ (the congruence lattice of K) represents D, $\operatorname{Con} L$ (the congruence lattice of L) represents E, and the mapping $\operatorname{ext}\varphi\colon \operatorname{Con} K \to \operatorname{Con} L$ (obtained by mapping a congruence of K under φ to L as a binary relation and then forming the minimal extension of this binary relation to a congruence relation of L) represents ψ.

In this note we give a short proof of this theorem. In fact, we prove a much stronger result: for K one can choose any finite lattice whose congruence lattice is isomorphic to D.

1. Introduction

One of the most persistent problems of lattice theory is the representation problem of distributive algebraic lattices as congruence lattice of lattices. A. P. Huhn in [5] attempted to solve this problem by simultaneous representation of finite distributive lattices as congruence lattices of finite lattices.

To state Huhn's result, we need a notation. Let K and L be lattices, and let φ be a homomorphism of K into L. Then φ induces a map $\operatorname{ext}\varphi$ of $\operatorname{Con} K$ into $\operatorname{Con} L$: for a congruence relation Θ of K, let the image Θ under $\operatorname{ext}\varphi$ be the congruence relation of L generated by the set $\Theta\varphi = \{\, \langle a\varphi, b\varphi\rangle \mid a \equiv b \ (\Theta) \,\}$.

The following result was proved by A. P. Huhn in [5] in the special case when ψ is an embedding and was proved for arbitrary ψ in [3] (where you also find for a more complete history of this result):

Theorem 1. *Let D and E be finite distributive lattices, and let*

$$\psi\colon D \to E$$

be a $\{0, \vee\}$-homomorphism. Then there are finite lattices K and L, a lattice homomorphism $\varphi\colon K \to L$, and isomorphisms

$$\alpha\colon D \to \operatorname{Con} K, \qquad \beta\colon E \to \operatorname{Con} L$$

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with

$$\psi\beta = \alpha(\operatorname{ext}\varphi).$$

Furthermore, φ is an embedding iff ψ separates 0*.*

Theorem 1 concludes that the following diagram is commutative:

$$\begin{array}{ccc} D & \xrightarrow{\psi} & E \\ \cong\Big\downarrow\alpha & & \cong\Big\downarrow\beta \\ \operatorname{Con} K & \xrightarrow{\operatorname{ext}\varphi} & \operatorname{Con} L \end{array}$$

In this paper we give a short proof of this theorem. In fact, we prove the following much stronger version:

Theorem 2. *Let K be a finite lattice, let E be a finite distributive lattice, and let $\psi\colon \operatorname{Con} K \to E$ be a $\{0, \vee\}$-homomorphism. Then there is a finite lattice L, a lattice homomorphism $\varphi\colon K \to L$, and an isomorphism $\beta\colon E \to \operatorname{Con} L$ with $\operatorname{ext}\varphi = \psi\beta$. Furthermore, φ is an embedding iff ψ separates* 0*.*

2. Preliminaries

Let M be a finite lattice and let C be a finite set; the elements of C will be called *colors*. A *coloring* μ of M over C is a map

$$\mu\colon \mathfrak{P}(M) \to C$$

of the set of prime intervals $\mathfrak{P}(M)$ of M into C satisfying the condition: if two prime intervals generate the same congruence relation of M, then they have the same color; that is,

$$\mathfrak{p},\ \mathfrak{q} \in \mathfrak{P}(M) \text{ and } \Theta(\mathfrak{p}) = \Theta(\mathfrak{q}) \text{ imply that } \mathfrak{p}\mu = \mathfrak{q}\mu.$$

Since the join-irreducible congruences of M are exactly those that can be generated by prime intervals, equivalently, μ can be regarded as a map of the set $J(\operatorname{Con} M)$ of join-irreducible congruences of M into C:

$$\mu\colon J(\operatorname{Con} M) \to C.$$

In this paper, we need the more general concept. A *multi-coloring* over C is an *isotone map* μ from $\mathfrak{P}(M)$ into $P^+(C)$ (the set of all nonempty subsets of C); *isotone* means that if $\mathfrak{p}$, $\mathfrak{q} \in \mathfrak{P}(M)$ and $\Theta(\mathfrak{p}) \leq \Theta(\mathfrak{q})$, then $\mathfrak{p}\mu \subseteq \mathfrak{q}\mu$. Equivalently, a multi-coloring is an isotone map of the poset $J(\operatorname{Con} M)$ into the poset $P^+(C)$.

We will now show that a multi-colored lattice has a natural extension to a colored lattice.

Lemma. *Let M be a finite lattice with a multi-coloring μ over the set C. Then there exist a lattice M^* with a coloring μ^* over C such that the following conditions holds:*

1. *M^* is the direct product of the lattices M_c, $c \in C$, where M_c is a homomorphic image of M colored by $\{c\}$.*
2. *There is a lattice embedding $a \to a^*$ of M into M^*.*
3. *For every prime interval $\mathfrak{p} = [a, b]$ of M,*

$$\mathfrak{p}\mu = \{\, \mathfrak{q}\mu^* \mid \mathfrak{q} \in \mathfrak{P}(M^*) \text{ and } \mathfrak{q} \subseteq [a^*, b^*]\,\}$$

and the minimal extension of $\Theta(\mathfrak{p})$ *under this embedding into* M^* *is of the form*

$$\prod(\Theta(\mathfrak{p}_c) \mid c \in C),$$

where $\mathfrak{p}_c$ *is a prime interval of* M_c *iff* $c \in \mathfrak{p}\mu$ *and* $\mathfrak{p}_c$ *is a trivial interval otherwise (in which case,* $\Theta(\mathfrak{p}_c) = \omega_{M_c}$*).*

Proof. For $c \in C$, define the binary relation Φ_c on M as follows:

$$u \equiv v \ \ (\Phi_c) \text{ iff } c \notin \mathfrak{p}\mu \text{ for every prime interval } \mathfrak{p} \subseteq [u \wedge v, u \vee v].$$

This relation is obviously reflexive and symmetric. To show transitivity, assume that $u \equiv v \ \ (\Phi_c)$ and $v \equiv w \ \ (\Phi_c)$, and let $\mathfrak{q}$ be a prime interval in $[u \wedge w, u \vee w]$. Then $\mathfrak{q}$ is collapsed by $\Theta(u, v) \vee \Theta(v, w)$, hence there is a prime interval $\mathfrak{p}$ in $[u \wedge v, u \vee v]$ or in $[v \wedge w, v \vee w]$ satisfying $\Theta(\mathfrak{q}) \leq \Theta(\mathfrak{p})$. It follows from the definition of multi-coloring that $\mathfrak{q}\mu \subseteq \mathfrak{p}\mu$; since $c \notin \mathfrak{p}\mu$, it follows that $c \notin \mathfrak{q}\mu$, hence $u \equiv w \ \ (\Phi_c)$. The proof of the Substitution Property is similar.

For $c \in C$, we define the lattice M_c as M/Φ_c. A prime interval $\mathfrak{p}$ of $M^* = \prod(M_c \mid c \in C)$ is uniquely associated with a $c \in C$ and a prime interval of M_c. We define $\mathfrak{p}\mu^* = c$. It is easy to see that μ^* is a coloring of M^* over C, establishing the first condition.

To establish the second condition, for $a \in M$, define a^* so that its M_c-component be $[a]\Phi_c$. The mapping $a \to a^*$ is obviously a lattice homomorphism. We have to prove that it is one-to-one. Let a, $b \in M$ and $a \neq b$; we have to prove that $a^* \neq b^*$. Let $\mathfrak{p}$ be a prime interval in $[a \wedge b, a \vee b]$. Since μ^* is a multi-coloring, there is a $c \in \mathfrak{p}\mu^*$. Obviously, then $a \not\equiv b \pmod{\Phi_c}$, from which the statement follows.

Finally, the third condition is trivial from the definition of M^* and μ^*. ☐

3. Proof of Theorem 2

Let K, E, and ψ be given as in Theorem 2.

Step 1. Since ψ preserves 0 and joins, there is a largest congruence Φ of K such that $\Phi\psi = 0_E$. Let $K_1 = K/\Phi$. The mapping ψ has a natural decomposition, $\psi = \psi_1\psi_2$, where $\psi_1 \colon \operatorname{Con} K \to \operatorname{Con} K_1$ is defined by $\Theta\psi_1 = \Theta \vee \Phi$, and $\psi_2 \colon \operatorname{Con} K_1 \to E$ is the restriction of ψ to $[\Phi) \cong \operatorname{Con} K_1$. Then ψ_2 separates 0 in $\operatorname{Con} K_1$. It is sufficient to prove Theorem 2 for K_1, E, and ψ_2.

Consequently, we need only prove Theorem 2 under the assumption that ψ *separates* 0.

Step 2. We define a map μ of $\mathfrak{P}(K)$ to subsets of $J(E)$:

$$\mathfrak{p}\mu = J(E) \cap (\Theta(\mathfrak{p})\psi].$$

μ is obviously isotone. ψ separates 0, so $\mathfrak{p}\mu \neq \varnothing$. Therefore, μ is a multi-coloring of K over $J(E)$. We apply the Lemma to obtain the lattice

$$K^* = \prod(K_c \mid c \in J(E)).$$

Step 3. Any finite lattice M can be embedded in a finite simple lattice $\overline{M}$ with the same zero and unit. Use such an extension for each K_c to obtain a simple lattice $\overline{K}_c$, then define:

$$L_0 = \prod(\overline{K}_c \mid c \in J(E)),$$

and extend the coloring so that $\overline{K}_c$ is also colored by $\{c\}$. Since L_0 is a direct product of simple lattices, it follows that $J(\mathrm{Con}\, L_0)$ is unordered; the congruence lattice of L_0 is a Boolean lattice with $|J(E)|$ atoms. K is a sublattice of K^* and K^* is a sublattice of L_0, so we obtain an embedding $\varphi\colon K \to L_0$.

Finally, we construct a special ideal of L_0. Let p_c be an arbitrary atom of the direct component $\overline{K}_c$; then the prime interval $[0, p_c]$ of L_0 has color c. The atoms p_c, for $c \in J(E)$, generate an ideal B_0 of L_0 which is a Boolean lattice satisfying the following properties:

(1) any two distinct atoms have different colors;

(2) every color $c \in J(E)$ occurs in B_0.

Step 4. We continue by forming a finite atomistic lattice L_1 with $E \cong \mathrm{Con}\, L_1$ under the isonorphism β_1. For L_1, we take the oldest published construction as in [4], except that we use a uniform "tripling" (first done in [2]) as opposed to "doubling" of non-maximals as in [4]. To recap, using the exposition in [1], we construct a partial lattice P_1 with 0 as follows. For every join-irreducible element p of E, we take three atoms p_1, p_2, and p_3, so that in P_1 they are the three atoms of a sublattice isomorphic to M_3 with zero 0; and if p, $q \in J(E)$, then $p_i \wedge q_j = 0$ $(0 \leq i, j \leq 3)$. If $q \prec p$ in $J(E)$, then we add the element $p(q)$ so that $p_3 \vee q_i = p(q)$ $(0 \leq i \leq 3)$. Let L_1 be the ideal lattice of P_1. The isomorphism $J(E) \cong J(\mathrm{Con}\, L_1)$ is given as follows: for $p \in J(E)$, the congruence $\Theta(0, p)$ of L_1 corresponds to p. Let β_1 denote the corresponding isomorphism $\beta_1 : E \to \mathrm{Con}\, L_1$.

We consider on L_1 the natural coloring over $J(E)$ (a prime interval $\mathfrak{p}$ is colored by $\Theta(\mathfrak{p})\beta_1^{-1} \in J(E)$). Note that L_0 and L_1 are colored over the same set, $J(E)$. Let B_1 be the ideal of L_1 generated by the atoms p_2, for $p \in J(E)$. Then the ideal B_1 is a Boolean lattice satisfying the properties (1) and (2) stated in Step 3.

Step 5. We have the lattice L_0 with the ideal B_0 and L_1 with an ideal B_1. Note that B_0 and B_1 are isomorphic finite Boolean lattices with the same coloring. Take the dual L_2 of L_1; in this lattice B_1 corresponds to a dual ideal B_2. Again, note that B_0 and B_2 are isomorphic finite Boolean lattices with the same coloring. Glue together L_0 and L_2 by a color preserving identification of B_0 and B_2. The resulting lattice is L. The prime intervals of L are colored by $J(E)$, and we have the isomorphism $\beta\colon E \to \mathrm{Con}\, L$. Since L_0 is a sublattice of L, we may view φ as an embedding of K into L.

Step 6. Finally, we have to verify that $\mathrm{ext}\, \varphi = \psi\beta$. It is enough to prove that $\Theta(\mathrm{ext}\, \varphi) = \Theta\psi\beta$ for join-irreducible congruences Θ in K.

So let $\Theta = \Theta(\mathfrak{p})$, where $\mathfrak{p} = [a, b]$ is a prime interval of K. By the Lemma, $\Theta(\mathfrak{p})\, \mathrm{ext}\, \varphi = \Theta(a^*, b^*)$ collapses in K^* the prime intervals of color $\leq \Theta\psi$; the same holds in L_0 and in L.

Computing $\Theta\psi\beta$ we get the same result, hence $\Theta(\mathrm{ext}\, \varphi) = \Theta\psi\beta$, completing the proof.

4. Concluding remarks

The proof in [3] of Theorem 1 gave a slightly stronger result—the lattices K and L can be chosen to be atomistic. In our proof here K can be chosen to be atomistic, but L is not atomistic. However, in his thesis ([7], Lemma 4.18), M. Tischendorf proved that any finite lattice L can be embedded in a finite atomistic lattice L' by an embedding $\varepsilon\colon L \to L'$ with $\mathrm{ext}\, \varepsilon$ an isomorphism. Consequently, extending

L in Theorem 2 by such an L', enables us to choose both K and L atomistic in Theorem 1.

Theorem 2 also yields a substantial simplification of the proof of Huhn's theorem [6] that any algebraic distributive lattice with countably many compact elements is the congruence lattice of a lattice. Let us denote by S the join-semilattice of compact elements of the given algebraic distributive lattice. Huhn observes that S is the direct limit (union) of an increasing countable family $(D_i \mid i < \omega)$ of finite distributive 0-preserving subsemilattices of S. The D_i are, of course, distributive lattices. For each $i < \omega$, let us denote by $\psi_i \colon D_i \to D_{i+1}$ the $\{0, \vee\}$-embedding. Huhn constructs a sequence $(L_i \mid i < \omega)$ of lattices with lattice embeddings $\varphi_i \colon L_i \to L_{i+1}$ such that $\operatorname{ext} \varphi_i \colon \operatorname{Con} L_i \to \operatorname{Con} L_{i+1}$ represents ψ_i. Then, denoting by L the direct limit of the sequence $(L_i \mid i < \omega)$, it follows that $\operatorname{Con} L \cong D$. The construction of the L_i and φ_i is the most complicated part of his paper—it comprises everything but the introduction. However, using our Theorem 2, we can proceed in a straightforward manner. We first represent D_0 by a finite lattice L_0, and, inductively, given L_i we immediately get a finite lattice L_{i+1} and an embedding $\varphi_i \colon L_i \to L_{i+1}$ with $\operatorname{ext} \varphi_i$ representing ψ_i.

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