COMPLETE CONGRUENCE LATTICES OF JOIN-INFINITE DISTRIBUTIVE LATTICES

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In [1] we proved the following Representation Theorem: *Every complete lattice L can be represented as the lattice of complete congruence relations of a complete distributive lattice K.* In this note we consider the same representation problem for special classes of complete distributive lattices.

The two best known infinitary identities are the *Join-Infinite Distributive Identity*:

$$\text{(JID)} \qquad a \wedge \bigvee X = \bigvee (a \wedge x \mid x \in X),$$

and its dual, the *Meet-Infinite Distributive Identity*:

$$\text{(MID)} \qquad a \vee \bigwedge X = \bigwedge (a \vee x \mid x \in X).$$

We shall denote by $(\mathrm{JID}_{\mathfrak{m}})$ the condition that (JID) holds for sets X satisfying $|X| < \mathfrak{m}$, where $\mathfrak{m}$ is a regular cardinal with $\mathfrak{m} > \aleph_0$. We define $(\mathrm{MID}_{\mathfrak{m}})$ dually.

It is easy to see that the lattice K we construct for the Representation Theorem of [1] fails both $(\mathrm{JID}_{\mathfrak{m}})$ and $(\mathrm{MID}_{\mathfrak{m}})$. Now we prove the following result:

Lemma 1. *Let D be a complete distributive lattice satisfying* (JID) *and* (MID). *Let a, $b \in D$, $a \leq b$. Then $\Theta(a, b)$ is a complete congruence.*

Proof. By an early result of the authors (see [2], Theorem II.3.3), for x, $y \in D$, $x \leq y$,

$$x \equiv y \quad (\Theta(a,b)) \quad \text{iff} \quad x \wedge a = y \wedge a \text{ and } y \vee b = y \vee b.$$

It is now obvious that every $\Theta(a,b)$ class has a smallest and largest element by (JID) and (MID). Hence by Lemma 2 of [3], $\Theta(a,b)$ is a complete congruence. ☐

Based on this lemma, we can prove

Theorem 1. *Let L be a complete lattice with more than two elements and with a meet-irreducible zero. Then L cannot be represented as the lattice of complete congruence relations of a complete distributive lattice K satisfying* (JID) *and* (MID).

Proof. Let us assume that L is isomorphic to the lattice of complete congruences of a complete distributive lattice K satisfying (JID) and (MID). Since L has more than two elements, it follows that K has more than two elements; so we can choose

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in K the elements $a < b < c$. By Lemma 1, the congruences $\Theta(a, b)$ and $\Theta(b, c)$ on K are complete. Since K is distributive, it follows that

$$\Theta(a, b) \wedge \Theta(b, c) = \omega.$$

Hence in K the zero is meet-reducible, contradicting that in L the zero is meet-irreducible. This contradiction proves the theorem. ☐

So we can raise the following:

Problem 1. *Characterize the complete congruence lattices of complete distributive lattices satisfying* (JID) *and* (MID).

Or more generally:

Problem 2. *Characterize the lattices of* $\mathfrak{m}$*-complete congruences of* $\mathfrak{m}$*-complete distributive lattices satisfying* ($\mathrm{JID}_{\mathfrak{m}}$) *and* ($\mathrm{MID}_{\mathfrak{m}}$).

Could we improve on the Theorem, then, by requiring only one of (JID) and (MID)? We start with the following:

Lemma 2. *Let D be a complete distributive lattice satisfying* (JID). *Let $a \in D$. Then $\Theta(a, 1)$ is a complete congruence, and the map*

$$\varphi \colon a \to \Theta(a, 1)$$

is a dual embedding of D into the lattice of complete congruences of D.

Proof. This follows immediately from the proof of Lemma 1. Indeed, if $b = 1$, then only (JID) is utilized in the proof. ☐

Theorem 2. *Let L be a finite non-Boolean lattice or a complete chain with more than two elements. Then L cannot be represented as the lattice of complete congruence relations of a complete distributive lattice K satisfying* (JID).

Proof. Indeed, by Lemma 2, K has a dual embedding into L. If L is finite, then so is K. Therefore, the complete congruence lattice of K is the same as the congruence lattice of K, and it is isomorphic to L, which is a finite Boolean lattice, a contradiction. If L is a chain with more than two elements, then by the dual embedding so is K. However, the complete congruence lattice of K is the same as the congruence lattice of K, which is isomorphic to L; this can never be a chain for a complete chain K with more than two elements. ☐

References

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