CONGRUENCE LATTICES OF FUNCTION LATTICES

G. GRÄTZER AND E. T. SCHMIDT

Abstract. The *function lattice* L^P is the lattice of all isotone maps from a poset P into a lattice L.

D. Duffus, B. Jónsson, and I. Rival proved in 1978 that for a *finite poset* P, the congruence lattice of L^P is a direct power of the congruence lattice of L; the exponent is $|P|$.

This result fails for infinite P. However, utilizing a generalization of the L^P construction, the $L[D]$ construction (the extension of L by D, where D is a bounded distributive lattice), the second author proved in 1979 that $\operatorname{Con} L[D]$ is isomorphic to $(\operatorname{Con} L)[\operatorname{Con} D]$ for a *finite lattice* L.

In this paper we prove that the isomorphism $\operatorname{Con} L[D] \cong (\operatorname{Con} L)[\operatorname{Con} D]$ holds for a lattice L and a bounded distributive lattice D iff $\operatorname{Con} L$ or D is finite.

1. Introduction

For a lattice L and a poset P, let L^P denote the set of all order-preserving S:intro
maps of P to L partially ordered by $f \leq g$ if and only if $f(x) \leq g(x)$, for all $x \in P$. Then L^P is a lattice; it is called a *function lattice*. This lattice plays a major role in lattice theory.

D. Duffus, B. Jónsson, and I. Rival [2] (see also [1]) obtained the following result:

Theorem 1. *Let L be a lattice, and let P be a finite poset. If P has n elements,* T:FIN
then the following isomorphism holds:

$$\operatorname{Con} L^P \cong (\operatorname{Con} L)^n.$$

It is evident that Theorem 1 does not remain valid for an infinite poset P. However in [8], using Priestley's representation of distributive lattices, the second author generalized a special case of this result (namely, the case of a *finite lattice* L), as follows.

Let D be a bounded distributive lattice, and let X denote the poset of all ultrafilters of D with the usual topology. For a lattice (or in general, for a join-semilattice) L, let $L[D]$ denote the lattice (resp, join-semilattice) of all continuous isotone maps of the space X into the discrete space L; we call $L[D]$ a *generalized function (semi-) lattice*. The constant maps form a sublattice of $L[D]$; we identify L with this sublattice. It is easy to see that for a finite distributive lattice D, the lattice $L[D]$ is a function lattice, namely, $L[D]$ is isomorphic to L^P where $P = J(D)$ is the poset of join-irreducible elements of D.

Date: April 2, 2000.

1991 *Mathematics Subject Classification.* Primary 06B10; Secondary 08A05.

Key words and phrases. Congruence lattice, function lattice.

The research of the first author was supported by the NSERC of Canada.

The research of the second author was supported by the Hungarian National Foundation for Scientific Research, under Grant No. 1903.

The result from E. T. Schmidt [8] is as follows:

Theorem 2. *Let L be a finite lattice, and let D be a bounded distributive lattice. Then* T:INF

$$\operatorname{Con} L[D] \cong (\operatorname{Con} L)[\operatorname{Con} D].$$

Our main result in this paper is the following:

Theorem 3. *Let L be a lattice, and let D be a bounded distributive lattice. Then the isomorphism* T:newMain

$$\operatorname{Con} L[D] \cong (\operatorname{Con} L)[\operatorname{Con} D]. \tag{Con}$$

holds if and only if either $\operatorname{Con} L$ *is finite or* D *is finite.*

Corollary . *Let L be a lattice. If the isomorphism* (Con) *holds for an arbitrary bounded distributive lattice D, then* $\operatorname{Con} L$ *is finite.*

And conversely, if $\operatorname{Con} L$ *is finite, then the isomorphism* (Con) *holds for an arbitrary bounded distributive lattice D.*

The statement in Theorem 3 that the isomorphism (Con) holds if D is finite is just a restatement of Theorem 1. Indeed, if D is finite, then $L[D]$ is isomorphic to L^P, where $P = J(D)$; and $\operatorname{Con} D$ is the Boolean lattice 2^n, so $(\operatorname{Con} L)^n$ is the same as $(\operatorname{Con} L)[\operatorname{Con} D]$.

The other positive statement in Theorem 3, namely, that the isomorphism holds if $\operatorname{Con} L$ is finite, generalizes Theorem 2. Schmidt's proof of Theorem 2 is based on the observation that $L[D]$ is an extension of L in which every prime interval of L contains a copy of D. So it is somewhat surprising that we can generalize Theorem 2 to non-discrete lattices.

Theorem 3 shows that these two positive results are best possible. If $\operatorname{Con} L$ is infinite *and* D is also infinite, then the isomorphism (Con) *always fails*.

In Section 2, we shall investigate congruence lattices of function lattices with finite exponent. We give a very short proof of Theorem 1, and we also prove some results needed in the proof of the main result. In Section 3, we shall show (Theorem 4) that by working with $\operatorname{Comp} L$, the join-semilattice of compact congruences rather than with $\operatorname{Con} L$, we can generalize Theorem 2 from finite lattices to arbitrary lattices.

The main result is presented in Section 4. The proof of this result is based on the results of the previous sections and on Theorem 5 which investigates when the free distributive product of two algebraic distributive lattices is not complete.

2. The case of finite exponent

We start by giving a very brief proof of Theorem 1. S:fin

Proof of Theorem 1. Let L be a lattice, and let P be a finite poset. L^P is a subdirect product of n copies of L; therefore, L^P has congruences $\Phi_1, \dots, \Phi_n$ such that the intersection of these congruences is ω and the quotient lattices are isomorphic to L. By congruence distributivity, $\operatorname{Con} L^P$ can be embedded into $(\operatorname{Con} L)^n$ (more generally, if S is a subdirect product of the lattices $L_1, \dots, L_n$, then $\operatorname{Con} S$ is a sublattice of the direct product of $\operatorname{Con} L_1, \dots, \operatorname{Con} L_n$). Therefore, to prove Theorem 1, it is sufficient to verify that distinct congruences of L^n have distinct restrictions.

So let $\Theta \neq \Phi$ be two distinct congruences of L^n; we represent them in the form $\Theta = \Pi(\Theta_p \mid p \in P)$ and $\Phi = \Pi(\Phi_p \mid p \in P)$, where Θ_p, Φ_p, $p \in P$, are congruences of L. Since $\Theta \neq \Phi$, there exists an $r \in P$ such that $\Theta_r \neq \Phi_r$. So we can choose $a < b \in L$ such that $a \equiv b \ (\Theta_r)$ but $a \equiv b \ (\Phi_r)$ fails (or symmetrically). Define $\mathbf{a}$, $\mathbf{b} \in L^n$ as follows:

$$\mathbf{a}(p) = \begin{cases} a, & \text{if } p \leq r; \\ b, & \text{otherwise;} \end{cases} \qquad \mathbf{b}(p) = \begin{cases} a, & \text{if } p < r; \\ b, & \text{otherwise.} \end{cases}$$

Then $\mathbf{a}$, $\mathbf{b} \in L^P$; moreover, $\mathbf{a} \equiv \mathbf{b} \ (\Theta)$ holds and $\mathbf{a} \equiv \mathbf{b} \ (\Phi)$ fails. This completes the proof of Theorem 1. □

The lattice L^P has the following interesting property. For $u \in L$ and $r \in P$, define the element $\mathbf{u}_r \in L^P$ as follows:

$$\mathbf{u}_r(p) = \begin{cases} 1, & \text{if } r < p; \\ u, & \text{if } r = p; \\ 0, & \text{otherwise.} \end{cases}$$

For every $r \in P$, the sublattice $L_r = [0_r, 1_r]$ is a convex sublattice of L^n. It is easy to see that L_r is a congruence class of the congruence relation Θ_r of L^n, yielding another proof of Theorem 1.

Let Comp A denote the join-semilattice with zero of compact congruences of the algebra A. With this notation, we can state an easy consequence of Theorem 1:

Corollary . *Let L be a lattice, and let P be a finite partially ordered set. If P has n elements, then the following isomorphism holds:*

$$\operatorname{Comp} L^P \cong (\operatorname{Comp} L)^n.$$

Proof. Indeed, in the proof of Theorem 1 we verify that if Θ is a congruence relation of L^P, and we take the congruence $\bar{\Theta}$ of L^n generated by Θ, then $\bar{\Theta}$ restricted to L^P yields Θ. Now the Corollary follows since if Θ is compact, then so is $\bar{\Theta}$. □

Another property for finite exponents, stated in terms of finite distributive extensions, is the following:

Lemma 1. *Let D_1 be a $\{0,1\}$-sublattice of a finite distributive lattice D_2. Then* L:F
$L[D_2]$ is an extension of $L[D_1]$, and the Congruence Extension Property holds.

Proof. Let P_i denote the dual of the poset of join-irreducible elements $J(D_i)$ of D_i, $i = 1$, 2. The duality between finite distributive lattices and finite posets provides an isotone map $\varphi \colon P_2 \to P_1$ (namely, $\varphi(x)$ is the smallest element in D_1 containing x, for $x \in J(D_2)$).

We define an embedding of L^{P_1} into L^{P_2}. Let f be an element of L^{P_1}. Then f is an isotone map: $P_1 \to L$; consequently, $\bar{f} = f \circ \varphi$ is an isotone map from P_2 into L, so $\bar{f} \in L^{P_2}$.

L^{P_1} is a subdirect power of L; consequently, every congruence relation Θ of L^{P_1} is a restriction of a congruence $\Pi(\Theta_p \mid p \in P_1)$ of $L^{|P_1|}$ to L^{P_1}.

Now we consider the congruence $\bar{\Theta} = \Pi(\Theta_{\varphi(q)} \mid q \in P_2)$ of $L^{|P_2|}$. It is easy to see that the restriction of $\bar{\Theta}$ to L^{P_2} is an extension of the congruence Θ of L^{P_1}. □

Finally, we state an interesting property of function lattices. This result was first proved for finite exponents as a crucial step in the proof of our main result. We state here for arbitrary exponents.

Lemma 2. *Let K be a lattice, and let P be a poset. Then* L:power

$$\operatorname{Id} K^P \cong (\operatorname{Id} K)^P.$$

Proof. For $F \in \operatorname{Id} K^P$, we define a map φ^F of P to $\operatorname{Id} K$:

$$\varphi^F \colon p \to p\varphi^F = \{pf \mid f \in F\}.$$

$p\varphi^F$ is an ideal of K. Indeed, if $x,\ y \in p\varphi^F$, then $x = pf$ for some $f \in F$ and $y = pg$ for some $g \in F$; therefore, $x \vee y = pf \vee pg = p(f \vee g)$ and $f \vee g \in F$, since F is an ideal of K^P. If $x \leq y \in p\varphi^F$, then $y = pf$ for some $f \in F$. Define $g \in K^P$ by $qg = qf \wedge x$, for $q \in p$. Then $g \leq f \in F$, so $f \in F$. Therefore, $x \in p\varphi^F$, proving that it is an ideal.

φ^F is obviously an isotone map.

Now we can set up the map α from $\operatorname{Id} K^P$ to $(\operatorname{Id} K)^P$:

$$\alpha \colon F \to \varphi^F \in (\operatorname{Id} K)^P, \text{ for } F \in \operatorname{Id} K^P.$$

α is obviously an isotone map from $\operatorname{Id} K^P$ to $(\operatorname{Id} K)^P$. It is also one-to-one since $F\alpha$ determines F:

$$f \in F \quad \text{iff} \quad pf \in p\varphi^F, \text{ for all } p \in P.$$

We see that α is onto by describing its inverse, β:

$$\beta \colon \Phi \to F_\Phi = \{f \in K^P \mid pf \in p\Phi, \text{ for all } p \in P\}.$$

So α is an isomorphism between $\operatorname{Id} K^P$ and $(\operatorname{Id} K)^P$. ☐

3. Compact congruences

The isomorphism in Theorem 2 does not hold in general. But we can prove a version of it in the general case by switching from congruence lattices to join-semilattices of compact congruences. S:comp

Theorem 4. *Let L be a lattice, and let D be a bounded distributive lattice. Then* T:New

$$\operatorname{Comp} L[D] \cong (\operatorname{Comp} L)[\operatorname{Comp} D]. \tag{Comp}$$

Let B_D denote the Boolean lattice generated by D. Then $\operatorname{Comp} D$ is isomorphic to B_D. So the isomorphism of the theorem can be restated as follows:

$$\operatorname{Comp} L[D] \cong (\operatorname{Comp} L)[B_D].$$

Let K be a join-semilattice with zero; let $\operatorname{Id} K$ denote the ideal lattice of K. Theorem 4 has the following consequence:

Corollary 1 . *Let L be a lattice, and let D be a bounded distributive lattice. Then the following isomorphism holds:*

$$\operatorname{Con} L[D] \cong \operatorname{Id}((\operatorname{Comp} L)[\operatorname{Comp} D]).$$

Corollary 2 . *Let L be a lattice, and let D be a bounded distributive lattice. Then the following isomorphism holds:*

$$\operatorname{Con} L[D] \cong \operatorname{Id}((\operatorname{Comp} L) * (\operatorname{Comp} D)).$$

In the last corollary, $*$ denotes the bounded free product of bounded distributive lattices.

We need three lemmas to prove Theorem 4 and its corollaries. The first is due to R. W. Quackenbush [7]:

Lemma 3. *Let D_1 and D_2 be bounded distributive lattices. Then* L:Q

$$D_1[D_2] \cong D_1 * D_2.$$

In particular,

$$D_1[D_2] \cong D_2[D_1].$$

Consider the *direct limit system* of algebras

$$\Re = \{A_i, \varphi_{i,j} \mid i,j \in I, i \leq j\};$$

see, e.g., [3]. Thus I is an updirected poset, A_i, $i \in I$ are algebras; $\varphi_{i,j}$ is a homomorphism of A_i into A_j, for $i \leq j$. These are subject to the usual conditions.

An element $\mathbf{v}$ of the *direct limit* $\varinjlim A_i$ is a vector such that $\mathbf{v}(i) \in A_i$, and for some $h \in I$, $\mathbf{v}(i)$ is defined for all $h \leq i$; the components satisfy the usual condition: $\mathbf{v}(i)\varphi_{i,j} = \mathbf{v}(j)$ for $i \leq j$.

For $i \leq j$ $(i,j \in I)$ and a congruence Φ_i of A_i, form the congruence Φ_j of A_j generated by the $\varphi_{i,j}$ image of Φ_i in A_j. Obviously, if Φ_i is compact in A_i, then Φ_j is compact in A_j. So we obtain a direct limit system $\operatorname{Comp}\Re$.

The following lemma is a generalization of Lemma 2.4 of M. Tischendorf [9].

Lemma 4. L:T

$$\varinjlim \operatorname{Comp} A_i \cong \operatorname{Comp} \lim\nolimits_{\to} A_i.$$

Proof. If Φ is a compact congruence of $\varinjlim \operatorname{Comp} A_i$, then

$$\Phi = \Theta(\mathbf{a}^1, \mathbf{b}^1) \vee \ldots \vee \Theta(\mathbf{a}^n, \mathbf{b}^n).$$

Choose any $k \in I$ for which all these vectors are defined. Obviously,

$$\Phi(k) = \Theta(\mathbf{a}^1(k), \mathbf{b}^1(k)) \vee \ldots \vee \Theta(\mathbf{a}^n(k), \mathbf{b}^n(k))$$

is a compact congruence of A_k. Mapping Φ to $\langle \Phi(i) \rangle$ gives the isomorphism of this lemma. $\square$

Let $\Re_D$ be the family of all finite $\{0,1\}$-sublattices of a bounded distributive lattice D, with set inclusion as the partial ordering and embeddings as maps. Obviously, $\varinjlim \Re_D \cong D$.

For a bounded distributive lattice E, we can form B_E, the Boolean lattice generated by E. With the obvious definition, the Boolean lattices B_E, $E \in \Re_D$, form a direct limit system, with limit isomorphic to B_D.

Let L be a lattice. For $E \in \Re_D$, we can form $L[E]$. The lattices $L[E]$ with the induced set inclusions and embeddings form a direct limit system $\Re_{D,L}$. Obviously, the direct limit of this system is $L[D]$.

Now we are ready to prove Theorem 4.

Proof of Theorem 4. Let L and D be given as in the theorem. We form the direct limit system $\Re_{D,L}$. By the Corollary to Theorem 1, for any $E \in \Re_D$,

$$\operatorname{Comp} L[E] \cong (\operatorname{Comp} L)[B_E]$$

(where in the right-side of this equation we use the definition of a join-semilattice extended by a bounded distributive lattice).

By Lemma 4, taking the limit of the left-side of the last equation, we obtain

$$\varinjlim \operatorname{Comp} L[E] \cong \operatorname{Comp} \lim\nolimits_{\to} L[E] \cong \operatorname{Comp} L[D].$$

Taking the limit of the right-side of this equation, we get

$$\lim_{\rightarrow} \operatorname{Comp} L[B_E] \cong (\operatorname{Comp} L)[B_D],$$

obtaining the isomorphism (Comp). □

Taking the ideal lattice of both sides of the isomorphism (Comp), we get Corollary 1. By applying Lemma 3 to Corollary 1, we obtain Corollary 2.

4. The main result

To prove our main theorem, we need the following result: S:main

Theorem 5. *Let E be a complete distributive lattice with an infinite ascending chain. Let A be a complete distributive algebraic lattice with an infinite complete-join independent antichain. Then $E[A]$ is not complete.* T:brandnew

Proof. Let $d_1 < d_2 < \cdots < d_n < \dots$ be an infinite ascending chain in E. Set $\delta = \bigvee(d_i \mid i < \omega)$. Let $\{a_i \mid 0 < i < \omega\}$ be an infinite complete-join independent antichain in A. Set $\alpha = \bigvee(a_i \mid i < \omega)$.

Let $L = E[A]$. By Lemma 3, L is isomorphic to $E * A$, the free bounded-distributive product. We shall prove that the set

$$X = \{d_1 \wedge a_1, d_2 \wedge a_2, \dots, d_n \wedge a_n, \dots\}$$

has no join in $E * A$. First, we verify some easy claims.

Claim 1. Let $d \in E$ and $a \in A$. Then $d \vee a$ is an upper bound of X iff $d_i \leq d$ or $a_i \leq a$, for every $i < \omega$.

Proof. By the structure theorem of free bounded-distributive products ([4]),

$$d_i \wedge a_i \leq d \vee a \quad \text{iff} \quad d_i \leq d \text{ or } a_i \leq a.$$

□

For $d \in E$, we define the elements $\delta(d)$ and $\alpha(d)$ of E as follows:
a) If $d_i \leq d$ for all $i < \omega$, then $\delta(d) = \delta$ and $\alpha(d) = 0$.
b) If $d_i \leq d$ and $d_{i+1} \nleq d$, then $\delta(d) = d_i$ and $\alpha(d) = \bigvee(a_j \mid i < j < \omega)$.
c) If $d_i \nleq d$ for all $i < \omega$, then $\delta(d) = 0$ and $\alpha(d) = \alpha$.

Claim 2. $d \vee a$ is an upper bound of X iff $\delta(d) \leq d$ and $\alpha(d) \leq a$.

Proof. This is obvious by the definition of $\delta(d)$ and $\alpha(d)$. □

Claim 3. An element $d \vee a$ is minimal in the set

$$\{d \vee a \mid d \vee a \text{ is an upper bound of } X\}$$

iff $\delta(d) = d$ and $\alpha(d) = a$.

Proof. This is obvious since if $a \vee d$ is an upper bound of X, then so is $\delta(d) \vee \alpha(d)$. □

Let $\varepsilon_0 = \alpha$, $\varepsilon_\omega = \delta$, and for every $0 < i < \omega$, let $\varepsilon_i = \delta(d_i) \vee \alpha(d_i)$. With this notation, we restate the previous claim:

Claim 4. If $\delta < 1$, then $\{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n, \dots, \varepsilon_\omega\}$, otherwise, $\{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n, \dots\}$ is the set of all minimal upper bounds of X of the form $d \vee a$.

Claim 5. Any upper bound u of X can be represented in the form $u = u_1 \wedge \ldots \wedge u_n$, where $1 \leq n < \omega$, and every u_j contains some ε_{i_j}, $0 \leq i_j \leq \omega$; $i_j = \omega$ is permitted if $\delta < 1$.

Proof. Every element u of $E * A$ can be represented as a finite meet of elements of the form $d \vee a$, where $d \in E$ and $a \in A$. If u is an upper bound of X, then so are its components $d \vee a$, so this claim follows from the previous claim. $\square$

Now we are ready to prove the theorem. Let us assume that $u = \bigvee X$ exists. Then by the last claim, u has a representation of the form $u = \varepsilon_{i_1} \wedge \varepsilon_{i_2} \wedge \ldots \wedge \varepsilon_{i_n}$, where $0 \leq i_1 < i_2 < \cdots < i_n \leq \omega$; $i_n = \omega$ is permitted if $\delta < 1$.

We shall distinguish four cases.

Case 1. $u = \alpha$. Then $u \wedge \varepsilon_1$ is an upper bound for X. Since $a_1 < u$ but $a_1 \nleq u \wedge \varepsilon_1$, it follows that $u \wedge \varepsilon_1 < u$, contradicting $u = \bigvee X$.

Case 2. $u = \delta < 1$. Then $u \wedge \varepsilon_1$ is an upper bound for X. Since $d_2 < u$ but $d_2 \nleq u \wedge \varepsilon_1$, it follows that $u \wedge \varepsilon_1 < u$, contradicting $u = \bigvee X$.

Case 3. $u = \varepsilon_{i_1} \wedge \varepsilon_{i_2} \wedge \ldots \wedge \varepsilon_{i_n}$, where $0 \leq i_1 < i_2 < \cdots < i_n < \omega$. Choose a $p > i_n$, $p < \omega$. Then $u \wedge \varepsilon_p$ is an upper bound for X. Since $a_1 \wedge a_2 \wedge \ldots \wedge a_p < u$ but $a_1 \wedge a_2 \wedge \ldots \wedge a_p \nleq u \wedge \varepsilon_p$, it follows that $u \wedge \varepsilon_p < u$, contradicting $u = \bigvee X$.

Case 4. $u = \varepsilon_{i_1} \wedge \varepsilon_{i_2} \wedge \ldots \wedge \varepsilon_{i_{n-1}} \wedge \delta$, where $0 \leq i_1 < i_2 < \cdots < i_{n-1} < \omega$. Choose a $p > i_{n-1}$, $p < \omega$. Then $u \wedge \varepsilon_p$ is an upper bound for X. Since $a_1 \wedge a_2 \wedge \ldots \wedge a_p \wedge \delta < u$ but $a_1 \wedge a_2 \wedge \ldots \wedge a_p \wedge \delta \nleq u \wedge \varepsilon_p$, it follows that $u \wedge \varepsilon_p < u$, contradicting $u = \bigvee X$.

This completes the proof of the theorem. $\square$

Corollary . *Let E be a distributive algebraic lattice. Then $E[A]$ is complete (or algebraic) for every distributive algebraic lattice A iff E is finite.*

Proof. Let E be finite. We shall prove that $E[A]$ is an algebraic lattice. By Lemma 3, $E[A]$ is isomorphic to $A[E]$; so we shall prove instead that $A[E]$ is an algebraic lattice. Let $P = J(E)$. Then $A[E]$ is isomorphic to A^P, and A^P is a complete sublattice of A^n, where $n = |P|$. Since A^n is an algebraic lattice, it follows that $A[E]$ is algebraic.

Conversely, if E is an infinite distributive lattice, then E has an infinite ascending or descending chain. Without loss of generality we can assume that E has an infinite ascending chain. Let A be a complete distributive algebraic lattice with an infinite complete-join independent antichain, for instance, the ideal lattice of the power set of an infinite set. Then by Theorem 5, $E[A]$ is not complete. $\square$

Now we are ready to prove the main result:

Proof of Theorem 3. Let L be a lattice, and let D be a bounded distributive lattice.

Let D be finite. Then the isomorphism (Con) follows from Theorem 1.

Let $\operatorname{Con} L$ be finite; again we have to verify the isomorphism (Con). Let $P = J(\operatorname{Con} L)$. Now compute (observe that both $\operatorname{Comp} L$ and $\operatorname{Comp} D$ are lattices):

$$\begin{aligned} \operatorname{Con} L[D] &\cong \operatorname{Id}((\operatorname{Comp} L)[\operatorname{Comp} D]) && \text{by Corollary 1 to Theorem 4} \\ &\cong \operatorname{Id}((\operatorname{Comp} D)[\operatorname{Comp} L]) && \text{by Lemma 3} \\ &\cong \operatorname{Id}(\operatorname{Comp} D)^P. \end{aligned}$$

On the other hand,

$$\begin{aligned}(\operatorname{Con} L)[\operatorname{Con} D] &\cong (\operatorname{Con} D)[\operatorname{Con} L] \quad \text{by Lemma 3}\\ &\cong (\operatorname{Con} D)^P \cong (\operatorname{Id}\operatorname{Comp} D)^P.\end{aligned}$$

So for $\operatorname{Con} L$ finite, the isomorphism (Con) is equivalent to the isomorphism

$$\operatorname{Id}(\operatorname{Comp} D)^P \cong (\operatorname{Id}\operatorname{Comp} D)^P,$$

which follows from Lemma 2.

Conversely, let us assume that both D and $\operatorname{Con} L$ are infinite. Then D is distributive, so it contains an infinite ascending or descending chain, $x_1, x_2, \dots$.

We set $A = \operatorname{Con} D$ and $E = \operatorname{Con} L$. Then E is an infinite distributive lattice, so it contains an infinite ascending or descending chain.

Case 1. E contains an infinite ascending chain. A is a complete distributive algebraic lattice with an infinite complete-join independent antichain, namely, $\{\Theta(x_i, x_{i+1}) \mid i = 1, 2, \dots\}$. So Theorem 5 applies, and $E[A]$ is not complete. Thus the required isomorphism (Con) cannot hold since the left side is complete, while the right side is not.

Case 2. E contains an infinite descending chain. In this case, we choose in A an infinite complete-meet independent antichain:

$$\{\bigvee(\Theta(x_j, x_{j+1}) \mid j \neq i) \mid i = 1, 2, \dots\},$$

and apply the dual of Theorem 5. □

References

[1] B. A. Davey, D. Duffus, R. W. Quackenbush, and I. Rival, *Exponents of finite simple lattices*, J. London Math. Soc. **17** (1978), pp. 203–211.

[2] D. Duffus, B. Jónsson, and I. Rival, *Stucture results for function lattices*, Canad. J. Math. **33** (1978), pp. 392–400.

[3] G. Grätzer, *Universal Algebra*, Second Edition, Springer-Verlag, New York–Heidelberg–Berlin, 1979.

[4] G. Grätzer and H. Lakser, *Chain conditions in the distributive free product of lattices*, Trans. Amer. Math. Soc. **144** (1969), pp. 301–312.

[5] G. Grätzer, H. Lakser, and R. W. Quackenbush, *The structure of tensor products of semilattices with zero*, Trans. Amer. Math. Soc. **267** (1981), pp. 503–515.

[6] A. Mitschke und R. Wille, *Freie modulare Verbände $FM(DM_3)$*, Proc. Univ. Houston Lattice Theory Conf., Houston, 1973, pp. 383–396.

[7] R. W. Quackenbush, *Free products of bounded distributive lattices*, Algebra Universalis **2** (1972), pp. 393–394.

[8] E. T. Schmidt, *Remark on generalized function lattices*, Acta Math. Hungar. **34** (1979), pp. 337–339.

[9] M. Tischendorf, *The representation problem for algebraic distributive lattices*, Ph. D. Thesis, Darmstadt, 1992.

Department of Mathematics, University of Manitoba, Winnipeg, Man. R3T 2N2, Canada

E-mail address: `George_Gratzer@umanitoba.ca`

Department of Mathematics, Transport Engineering Faculty, Technical University of Budapest, Műegyetem rkp. 9, 1111 Budapest, Hungary

E-mail address: `h1175sch@ella.hu`