A LATTICE CONSTRUCTION AND CONGRUENCE-PRESERVING EXTENSIONS

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Abstract. A *chopped lattice* is a partial lattice we obtain from a bounded lattice by removing the unit element.

Under a very natural condition, (FG), the finitely generated ideals of a chopped lattice M form a lattice which is a *congruence-preserving extension* of M; that is, every congruence of M has exactly one extension to this lattice.

In this paper, we investigate how we can obtain from a pair of lattices A and B by amalgamation a chopped lattice. We establish a set of six sufficient conditions.

We then investigate when the chopped lattice so obtained will satisfy Condition (FG). A typical result is the following: if $C = A \cap B$ is a principal ideal of both A and B and A is modular, then Condition (FG) holds.

We apply this to prove that if L is a lattice with a nontrivial distributive interval, then L has a proper congruence-preserving extension.

1. Introduction

To find a simple proof of the congruence lattice characterization theorem of finite lattices, H. Lakser and the first author (see [1]) introduced a special type of finite partial lattices: a meet-semilattice in which any two elements with a common upper bound have a join. If M is such a finite partial lattice, then the ideal lattice of M is a congruence-preserving extension of M; that is, every congruence of M has exactly one extension to the ideal lattice.

In [2], we introduced the name *chopped lattice* for such partial lattices, no longer necessarily finite. Of course, if M is no longer finite, we cannot expect the ideal lattice to be a congruence-preserving extension. It is natural to consider, instead, finitely generated ideals; unfortunately, they do not, in general, form a lattice. In Section 2 we introduce Condition (FG) under which the finitely generated ideals form a lattice.

Given two lattices A and B, sharing the sublattice $C = A \cap B$, we obtain the lattice $M(A, B)$ by amalgamation. If C is a principal ideal of both A and B, then $M(A, B)$ is a chopped lattice.

In Section 3, we introduce (see Definition 3) a set of sufficient conditions under which $M(A, B)$ is a chopped lattice. If A and B satisfy the conditions of Definition 3, we shall call A, B a *chopped pair*. Theorem 1 states that if A, B is a chopped pair, then $M(A, B)$ is a chopped lattice. The concept of a chopped pair does not

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seem strong enough to compute with it. In Section 4, we introduce two stronger versions: *sharp* and *full* chopped pairs.

In Section 5 we investigate finitely generated ideals in $M(A, B)$ for a chopped pair A, B. For a sharp chopped pair A and B, if $C = A \cap B$ satisfies the Ascending Chain Condition, then we obtain Condition (FG) (which guarantees that the finitely generated ideals form a lattice) for $M(A, B)$.

In Section 6 we investigate modular lattices. If A, B is a sharp chopped pair and both A and B are modular, then $M(A, B)$ satisfies Condition (FG) (Theorem 3). If A, B is a full chopped pair, then it is enough to assume that one of them is modular to obtain the same conclusion (Theorem 4).

In Section 7 we deal with the problem whether every lattice has a proper congruence-preserving extension. We apply Theorem 4 to prove that if there exist a nontrivial distributive interval in a lattice, then it has a proper congruence-preserving extension.

A modular example of a congruence-preserving extension is outlined in Section 6.

1.1. **Notation.** We refer the reader to [1] for the basic concepts and notation.

In a lattice L, $[x, y]_L$ denotes the interval in L, and $(a]_L$ the principal ideal generated by a. If there is no confusion, the subscript is dropped.

If L is a sublattice of K, then we call K an *extension* of L. If L has a zero, and it is also the zero of K, then K is $\{0\}$-*extension* of L.

2. Chopped lattices

A *chopped lattice* M is a lattice L with zero, 0, and unit, 1, with the unit removed: $M = L - \{1\}$; on M, 0 is a nullary operation, $\wedge$ is an operation, and $\vee$ is a partial operation. Equivalently, a chopped lattice M is a meet-semilattice with zero, 0, in which any two elements having an upper bound have a join. M will be regarded as a partial algebra $\langle M; \wedge, \vee, 0\rangle$.

We shall use the concept of *extension* for chopped lattices; observe that, by definition, an extension of a chopped lattice is a $\{0\}$-extension.

An *ideal* I of M is a subset of M containing 0 with the following two properties for x, $y \in M$:

$x \in I$ and $y \leq x$ imply that $x \in I$.

If x, $y \in I$ and $x \vee y$ exists, then $x \vee y \in I$.

For $H \subseteq M$, there is a smallest ideal $(H]$ of M containing H. If an ideal I can be generated in the form $(H]$ for some finite set H, then the ideal I is called *finitely generated*. In particular, for $a \in M$, we let $(a] = (\{a\}]$ be the *principal ideal* generated by a in M, that is,

$$(a] = \{x \mid x \in M \text{ and } x \leq a\}.$$

$\operatorname{Id} M$ denotes the *lattice of ideals* of M. Obviously, $\operatorname{Id} M$ is a lattice. $\operatorname{Id}_{\mathrm{fg}} M$, the *finitely generated ideals* of M, form a join-sublattice of $\operatorname{Id} M$.

By identifying $a \in M$ with $(a]$, we regard $\operatorname{Id} M$ an extension of M.

Definition 1. *A chopped lattice M satisfies* Condition (FG) *if every finitely generated ideal is a finite union of principal ideals.*

If M satisfies Condition (FG), then $\operatorname{Id}_{\mathrm{fg}} M$ is a sublattice of $\operatorname{Id} M$. Indeed, if

$$I = (a_1] \cup \ldots \cup (a_n],$$
$$J = (b_1] \cup \ldots \cup (b_m],$$

then

$$I \cap J = \bigcup ((a_i \wedge b_j] \mid 1 \leq i \leq n,\ 1 \leq j \leq m).$$

Lemma II.3.19 in [1] states the following:

Lemma 1. *Let M be a finite chopped lattice. Then* $\mathrm{Id}\, M$ *is a congruence-preserving extension of M.*

The proof of this lemma implicitly contains the following two lemmas.

Lemma 2. *Let M be a chopped lattice. Then every congruence relation of M has an extension to* $\mathrm{Id}\, M$.

Proof. Let Θ be a congruence of M; define a relation $\bar{\Theta}$ on $\mathrm{Id}\, M$ as follows:

$$I \equiv J \quad (\bar{\Theta})$$

if for every $i \in I$ there exists a $j \in J$ such that $i \equiv j \ (\Theta)$, and symmetrically. The proof is the same as in [1]. ☐

Lemma 3. *Let M be a chopped lattice, and let $S \supseteq M$ be a sublattice of* $\mathrm{Id}\, M$. *Let us assume that in S every ideal $I \in S$ is a finite union of principal ideals. Then every congruence relation of M has a unique extension to S.*

Proof. First observe that if $a \in M$ and $I \in S$, then $(a] \cap I$ is principal. Indeed,

$$I = (a_1] \cup \ldots \cup (a_n],$$

and so $(a] \cap I$ is generated by $\{a \wedge a_1, \ldots, a \wedge a_n\}$. Since this set has an upper bound (namely a), it has a join b (since M is a chopped lattice), and b obviously generates $(a] \cap I$.

Let Φ be an extension of Θ from M to S. Let I, $J \in S$, $I \equiv J \ (\Phi)$, and $a \in I$. Then $I \wedge (a] \equiv J \wedge (a] \ (\Phi)$. By the statement in the previous paragraph, there is a $b \in J$ such that $(a] \wedge J = (b]$; obviously, $a \equiv b \ (\Theta)$. We conclude that $I \equiv J \ (\bar{\Theta})$. So $\Phi \subseteq \bar{\Theta}$.

Conversely, let I, $J \in S$ with $I \equiv J \ (\bar{\Theta})$. By the assumption on S, we can represent these ideals as

$$I = (a_1] \cup \ldots \cup (a_n],$$
$$J = (b_1] \cup \ldots \cup (b_m].$$

By the definition of $\bar{\Theta}$, for every a_i there is a c_i in J with $a_i \equiv c_i \ (\Theta)$. Symmetrically, for every b_j there is a d_j in I with $d_j \equiv b_j \ (\Theta)$. Since Φ is an extension of Θ, these congruences hold for Φ. The join of these $n + m$ congruences yields $I \equiv J$ (Φ), proving that $\bar{\Theta} \subseteq \Phi$. Thus $\bar{\Theta} = \Phi$, and so every congruence of M has a unique extension to S. ☐

Therefore, the following is true:

Lemma 4. *Let M be a chopped lattice satisfying Condition* (FG). *Then* $\mathrm{Id}_{\mathrm{fg}}\, M$ *is a congruence-preserving extension of M.*

In fact, a congruence-preserving $\{0\}$-extension.

3. Chopped pairs

Let A and B be lattice, let $C = A \cap B \neq \varnothing$. Then we can form the amalgamation $M = M(A, B)$ of A and B over C. It is well-known that on M we can define a partial ordering:

Definition 2. *The partial ordering $\leq_M$ is defined on M as follows:*

1. *For x, $y \in A$, let $x \leq_M y$ iff $x \leq_A y$.*
2. *For x, $y \in B$, let $x \leq_M y$ iff $x \leq_B y$.*
3. *For $x \in A$ and $y \in B$, let $x \leq_M y$ iff there exists a $c \in C$ such that $x \leq_A c$ and $c \leq_B y$; and symmetrically, for $x \in B$ and $y \in A$.*

The subscripts of $\leq$ will be dropped whenever there is no danger of confusion.

We shall use the following notation: $M(A, B) = A \cup B$ is the *poset* obtained by amalgamating A and B over C. In A we form the ideal I_A generated by C; we set $C_A = I_A - C$; symmetrically, we define I_B and C_B. Note that the ideal C_M generated by C in M is the disjoint union of C, C_A, and C_B.

Sometimes, the poset $M(A, B)$ is a chopped lattice. The next definition formulates some natural conditions under which this is the case.

Definition 3. *A pair of lattices A and B is called a* chopped pair *iff the following conditions are satisfied:*

1. *The lattices A and B have a common zero 0.*
2. *Let C denote the lattice $A \cap B$. Then C has a largest element i.*
3. *For $x \in C_M$, there is a smallest $\overline{x} \in C$ satisfying $x \leq \overline{x}$.*
4. *For $x \in M(A, B)$, there is a largest $\underline{x} \in C$ satisfying $\underline{x} \leq x$.*
5. *For $x \in C_A$ and $y \in C_B$, the two elements: $x \vee \overline{y}$ (formed in A) and $\overline{x} \vee y$ (formed in B) are comparable (in $M(A, B)$).*
6. *For $x \in A - B$ and $y \in B - A$, the two elements: $x \wedge \underline{y}$ (formed in A) and $\underline{x} \wedge y$ (formed in B) are comparable (in $M(A, B)$).*

Theorem 1. *Let A, B be a chopped pair. Then $M(A, B)$ is a chopped lattice.*

Proof. There are two claims to verify.

Claim 1. $M(A, B)$ is a meet-semilattice.

Let x, $y \in M(A, B)$. We have to find $u = \inf_{M(A,B)}\{x, y\}$. We shall distinguish several cases.

Case 1.1. x, $y \in A$. Let $u = x \wedge y$ be formed in A. Obviously in $M(A, B)$, $u \leq x$ and $u \leq y$. Now let $v \in M(A, B)$ be a common lower bound of x and y. There are two subcases to consider.

Case 1.1a. $v \in A$. By Definition 2.1, v is a common lower bound of x and y in A, hence, $v \leq u$.

Case 1.1b. $v \in B$. By Definition 2.3, there are elements c_x and c_y in C such that $v \leq_B c_x \leq_A x$ and $v \leq_B c_y \leq_A y$. Then $c_x \wedge c_y \in C$, and $v \leq_B c_x \wedge c_y \leq_A u$. So indeed, $u = \inf_{M(A,B)}\{x, y\}$.

Case 1.2. x, $y \in B$. Proceed as in Case 1.1.

Case 1.3. $x \in A$, $y \in B$. In view of the previous cases, we can assume that $x \in A - B$ and $y \in B - A$. Since by Definition 2.3, any common lower bound must be in C_M, we can replace x by $x \wedge i$ and y by $y \wedge i$. So again referring to the previous cases, we can assume that $x \in C_A$ and $y \in C_B$. Now take a common lower bound v of x and y.

Now we claim that of the common lower bounds $v \in A$, there is a largest one, $x \wedge \underline{y}$. Indeed, $x \wedge \underline{y}$ is a lower bound. If $t \in A$ is also a lower bound, then $t \leq y$ in $M(A,B)$, hence by Definition 2.3, there is a $c \in C$ satisfying $t \leq_A c \leq_B y$. Obviously, $c \leq \underline{y}$, and so $t \leq_A x \wedge \underline{y}$, as claimed.

Now we claim that of the common lower bounds $v \in B$, there is a largest one, $\underline{x} \wedge y$. To prove this, proceed as in the previous paragraph.

Finally, by Definition 3.6, $x \wedge \underline{y}$ and $\underline{x} \wedge y$ are comparable, hence $\inf_{M(A,B)}\{x,y\}$ exists and it equals $\sup\{x \wedge \underline{y}, \underline{x} \wedge y\}$.

Case 1.4. $x \in B$, $y \in A$. Proceed as in Case 1.3.

This completes the proof of Claim 1.

Claim 2. In $M(A,B)$, any two elements, x and y, having a common upper bound, v, have a join.

Let x, $y \in M(A,B)$, and let v be an upper bound of x and y. We have to find $u = \sup_{M(A,B)}\{x,y\}$. We shall distinguish several cases.

Case 2.1. x, $y \in A$. Form $u = x \vee y$ in A. We have to show that if t is any upper bound of x and y in $M(A,B)$, then $u \leq t$.

Case 2.1a. $t \in A$. This case is obvious.

Case 2.1b. $t \in B$. By Definition 2.3, there are c_x, $c_y \in C$ so that $x \leq_A c_x \leq_B t$ and $y \leq_A c_y \leq_B t$. Therefore, $u = x \vee y \leq_A c_x \vee c_y \leq_B t$; so again, by Definition 2.3, $u \leq_{M(A,B)} t$, completing Case 2.1.

Case 2.2. x, $y \in B$. Proceed as in Case 2.1.

Case 2.3. $x \in A$ and $y \in B$. In view of Cases 2.1–2.2, we can assume that $x \in A - B$ and $y \in B - A$. Without loss of generality, we can assume that $t \in A$. It follows that $y \in C_B$. Again, we distinguish two subcases.

Case 2.3a. $x \in C_A$. If $t \in A$ is an upper bound of x and y, then $x \vee \overline{y} \leq t$. Similarly, if $t \in B$ is an upper bound of x and y, then $\overline{x} \vee y \leq t$. By Definition 3.5, the elements $x \vee \overline{y}$ and $\overline{x} \vee y$ are comparable, hence,

$$\sup\{x,y\} = \inf\{x \vee \overline{y}, \overline{x} \vee y\}$$

Case 2.3b. $x \notin C_A$. In this case, no upper bound of x is in B, hence, $\sup\{x,y\} = x \vee \overline{y}$ formed in A.

Case 2.4. $x \in B$ and $y \in A$. Proceed as in Case 2.3.

This completes the proof of Claim 2 and of the lemma. □

4. Some examples and special cases

It is easy to give examples that last two strange conditions of Definition 3 do not follow from the others. Here is one: let $A = B$ be the direct product of the two element chain $\{0,1\}$ with the three element chain $\{0,a,1\}$. The elements are of the form $\langle x,y\rangle$, where $x \in \{0,1\}$ and $y \in \{0,a,1\}$. We make A and B disjoint (we shall denote $\langle x,y\rangle \in A$ by $\langle x,y\rangle_A$, and the same for B), then we identify elements as follows:

$\langle 0,0\rangle_A$ with $\langle 0,0\rangle_B$;
$\langle 1,0\rangle_A$ with $\langle 0,1\rangle_B$;
$\langle 0,1\rangle_A$ with $\langle 1,0\rangle_B$;
$\langle 1,1\rangle_A$ with $\langle 1,1\rangle_B$.

So $C = \{\langle 0,0\rangle_A, \langle 1,0\rangle_A, \langle 0,1\rangle_A, \langle 1,1\rangle_A\}$ is a four-element Boolean lattice. It is easy to see that Definitions 3.1–3.4 hold, but both Definitions 3.5 and 3.6 fail. Indeed, let $x = \langle a,0\rangle_A \in C_A$ and $y = \langle a,0\rangle_B \in C_B$. Then $\overline{x} = \langle 1,0\rangle_A$ and

$\overline{y} = \langle 1, 0\rangle_B = \langle 0, 1\rangle_B$. Hence,

$$x \vee \overline{y} = \langle a, 1\rangle_A \text{ and } \overline{x} \vee y = \langle a, 1\rangle_B,$$

and these two elements are not comparable.

If A, B is a chopped pairs, then we know that in $M(A, B)$ any pair of elements with a common upper bound has a join. To perform computations we need more; we must have a formula for the join we can work with.

Definition 4. *A chopped pair of lattices A and B, is called* sharp *iff*

$$x \vee \overline{y} = \overline{x} \vee y,$$

for $x \in C_A$ and $y \in C_B$, and

$$x \wedge \underline{y} = \underline{x} \wedge y,$$

for $x \in A - B$ and $y \in B - A$.

There are many equivalent forms of these conditions; for instance, the first is equivalent to

$$x \vee \overline{y} \in C,$$

for $x \in C_A$ and $y \in C_B$; or to

$$x \vee y = \overline{x} \vee \overline{y}.$$

Observe that if A and B form a sharp chopped pair, then in $M(A, B)$, we have $x \wedge y \in C$, for $x \in C_A$ and $y \in C_B$; and $x \vee y \in C$, for $x \in C_A$ and $y \in C_B$.

Two important examples of chopped pairs follow in which C is largest and smallest possible:

Example 1. $C = (i]$ is a principal ideal of both A and B.

We considered this special case for finite lattices in a previous paper [2]. In this case, $C_A = C_B = \varnothing$; for every $x \in M(A, B)$, $\underline{x} = x \wedge i$; and for every $x \in C = C_M$, $\overline{x} = x$. The conditions of Definition 3 and Definition 4 are trivially satisfied—in fact,

$$x \vee \overline{y} = \overline{x} \vee y = x \vee y \text{ and } x \wedge \underline{y} = \underline{x} \wedge y = x \wedge y \wedge i.$$

Example 2. $C = \{0, i\}$.

In this case, again, the conditions of Definition 3 are trivially satisfied—in fact,

$$x \vee \overline{y} = \overline{x} \vee y = i \text{ and } x \wedge \underline{y} = \underline{x} \wedge y = 0.$$

In these two examples, the conditions of Definition 3 and Definition 4 hold in a much stronger form.

We name the first example:

Definition 5. *A chopped pair of lattices, A and B, is called* full *if $C = (i]_A = (i]_B$.*

5. Finitely generated ideals

In this section, we shall investigate conditions under which $M(A, B)$ satisfies Condition (FG). The following two lemmas are easy to verify, but they are crucial to our investigations. First some definitions.

Definition 6. *Let A, B be a chopped pair, $C = A \cap B$. Let $a \in A - C$ and $b \in B - C$. We define the elements:*

$$
\begin{aligned}
a_0 &= a, && \\
b_0 &= b, && \\
b_1 &= b_0 \vee \overline{a_0 \wedge i} && \textit{(formed in } B\textit{)},\\
a_1 &= a_0 \vee \overline{b_1 \wedge i} && \textit{(formed in } A\textit{)},\\
b_2 &= b_1 \vee \overline{a_1 \wedge i}\ (= b \vee \overline{a_1 \wedge i}) && \textit{(formed in } B\textit{)},\\
a_2 &= a_1 \vee \overline{b_1 \wedge i}\ (= a \vee \overline{b_1 \wedge i}) && \textit{(formed in } A\textit{)},\\
&\dots && \\
b_{n+1} &= b_n \vee \overline{a_n \wedge i}\ (= b \vee \overline{a_n \wedge i}) && \textit{(formed in } B\textit{)},\\
a_{n+1} &= a_n \vee \overline{b_{n+1} \wedge i}\ (= a \vee \overline{b_{n+1} \wedge i}) && \textit{(formed in } A\textit{)},\\
&\dots &&
\end{aligned}
$$

See Figure 1—the white filled elements are in A (and maybe in C); the shaded elements are in B (and maybe in C), and the black filled elements are in C.

Lemma 5. *Let A and B be a sharp chopped pair. Then in $M(A, B)$, the following inequalities hold:*

$$a = a_0 \leq a_1 \leq a_2 \leq \dots \ \textit{(in } A\textit{)}, \tag{1}$$

$$b = b_0 \leq b_1 \leq b_2 \leq \dots \ \textit{(in } B\textit{)}, \tag{2}$$

and

$$\overline{a_0 \wedge i} \leq \overline{b_1 \wedge i} \leq \overline{a_1 \wedge i} \leq \overline{b_2 \wedge i} \leq \overline{a_2 \wedge i} \leq \dots \ \textit{(in } C\textit{)}. \tag{3}$$

If, for some n, $a_n = a_{n+1}$, then (1) terminates at n, and (2) terminates at $n + 1$; and symmetrically, for (2). If (3) does not terminate, neither do (1) and (2).

So either all three sequences terminate or none terminate.

Proof. Let $a_n = a_{n+1}$; then $\overline{a_n \wedge i} = \overline{a_{n+1} \wedge i}$. Therefore,

$$b_{n+2} = b \vee \overline{a_{n+1} \wedge i} = b \vee \overline{a_n \wedge i} = b_{n+1};$$

and so $\overline{b_{n+1} \wedge i} = \overline{b_{n+2} \wedge i}$. By the definition of a_{n+1} and a_{n+2}, it follows that $a_{n+1} = a_{n+2}$. Hence, $\overline{a_{n+1} \wedge i} = \overline{a_{n+2} \wedge i}$, so $b_{n+2} = b_{n+3}$. It is now clear that

$$a_n = a_{n+1} = a_{n+2} = \dots,$$

and

$$b_{n+1} = b_{n+2} = \dots$$

Finally,

$$\overline{a_n \wedge i} \leq \overline{b_{n+1} \wedge i} \leq \overline{a_{n+1} \wedge i} \leq \overline{b_{n+2} \wedge i},$$

$\overline{a_n \wedge i} = \overline{a_{n+1} \wedge i}$ and $\overline{b_{n+1} \wedge i} = \overline{b_{n+2} \wedge i}$; therefore,

$$\overline{a_n \wedge i} = \overline{b_{n+1} \wedge i} = \overline{a_{n+1} \wedge i} = \overline{b_{n+2} \wedge i}, \dots,$$

so sequence (3) also terminates. Conversely, if sequence (3) terminates, then sequences (1) and (2) terminate by the definitions of a_{n+1} and b_{n+1} in Definition 6. ∎

Lemma 6. *Let A and B be a sharp chopped pair; let $a \in A - C$, $b \in B - C$. The ideal $(a, b]$ of $M(A, B)$ generated by $\{a, b\}$ can be described as follows:*

$$(a, b] = \bigcup((a_n]_A \mid n < \omega) \cup \bigcup((b_n]_B \mid n < \omega).$$

This is not a finitely generated ideal if, and only if, none of the sequences of Lemma 5 terminate. If $(a, b]$ is a finitely generated ideal, then $(a, b] = (a_n] \cup (b_n]$ for some $n < \omega$.

Proof. Let $R = \bigcup((a_n]_A \mid n < \omega) \cup \bigcup((b_n]_B \mid n < \omega)$. If we know that R is an ideal of $M(A, B)$, then it is straightforward to verify that R is the ideal of $M(A, B)$ generated by $\{a, b\}$, and the rest follows from Lemma 5.

So we verify that R is an ideal of $M(A, B)$.

Firstly, let $x \in R$ and $y \le x$ in $M(A, B)$. Without loss of generality we can assume that $x \le a_n$ for some n and $y \le x$. If $y \in A$, then $y \le a_n$; therefore $y \le a_n$ in A, and so $y \in R$. If $y \in B$, then $\overline{y} \le a_n$, and so $\overline{y} \le a_n \wedge i \le b_n$. This implies that $y \le b_n$ in B, therefore $y \in R$; completing the proof of $y \in R$.

Secondly, let x, $y \in R$, and let x and y have a common upper bound z in $M(A, B)$. Without loss of generality we can assume that $z \in A$. We want to show that $x \vee y \in R$. We shall distinguish several cases.

Case 1. x, $y \in A$.

Case 1.1. $x \le a_n$ and $y \le a_m$ for some n and m. In this case, as in all the subsequent cases, we can assume without loss of generality that $n = m$. Then $x \vee y \le a_n$, so $x \vee y \in R$.

Case 1.2. $x \le a_n$ and $y \le b_n$. Since $y \in A$ and $b_n \in B$, the condition $y \le b_n$ implies that $y \le i$. Hence, $y \le b_n \wedge i \le a_n$, and so $x \vee y \le a_n$, yielding $x \vee y \in R$.

Case 1.3. $x \le b_n$ and $y \le a_n$. Proceed as in Case 1.2.

Case 1.4. $x \le b_n$ and $y \le b_n$. As in Case 1.2, we can verify that $x \le a_n$ and $y \le a_n$, so Case 1.1 completes this case.

Case 2. $x \in A$, $y \in B$. Observe that $y \le i$ since $y \le z$, $y \in B$ and $z \in A$.

Case 2.1. $x \le a_n$ and $y \le a_n$. So $x \vee y = x \vee \overline{y} \le a_n$, hence $x \vee y \in R$.

Case 2.2. $x \le a_n$ and $y \le b_n$. Since $y \le i$, it follows that $y \le b_n \wedge i$, so $y \le a_n$; hence $x \vee y \le a_n$, yielding $x \vee y \in R$.

Case 2.3. $x \le b_n$ and $y \le a_n$. Proceed as in Case 2.2.

Case 2.4. $x \le b_n$ and $y \le b_n$. Then as in Case 2.2, $x \le a_n$ and $y \le a_n$, so we can proceed as in Case 1.

Case 3. $x \in B$, $y \in A$. This is symmetric to Case 2.

Case 4. x, $y \in B$.

Case 4.1. $x \le a_n$ and $y \le a_n$. Using the argument of Case 2.2, we obtain that $x \le b_{n+1}$ and $y \le b_{n+1}$, which is symmetric to Case 1.1. Hence $x \vee y \in R$.

Case 4.2. $x \le a_n$ and $y \le b_n$. Again, $x \in B$ and $x \le a_n$ imply that $x \le b_{n+1}$, which is symmetric to Case 1.1.

Case 4.3. $x \le b_n$ and $y \le a_n$. Proceed as in Case 4.2.

Case 4.4. $x \le b_n$ and $y \le b_n$. This is symmetric to Case 1.1. ☐

Observe that this lemma fully describes all finitely generated ideals, since a finitely generated ideal of $M(A, B)$ is obviously one- or two-generated.

Now we prove:

Theorem 2. *Let A and B be form a sharp chopped pair, and let $C = A \cap B$. Let us assume that C satisfies the Ascending Chain Condition. Then $M(A, B)$ satisfies* *condition* (FG)*, and* $\mathrm{Id_{fg}}\, M(A,B)$ *is a congruence-preserving extension of* $M(A,B)$ *(in fact, a congruence-preserving* $\{0\}$*-extension).*

Proof. If C satisfies the Ascending Chain Condition, then sequence (3) of Lemma 5 must terminate. By Lemma 5, the sequences (1) and (2) terminate, and so the statement of the Theorem follows from Lemma 6.

Finally, the statement concerning congruence-preserving extension follows from Lemma 4. □

For full chopped pairs, Definition 6, Lemma 5, and Lemma 6 take on a much simpler form:

Definition 7. *Let* A*,* B *be a full chopped pair,* $C = A \cap B$*. Let* $a \in A - C$ *and* $b \in B - C$*. Then we define the elements:*

$$
\begin{aligned}
a_0 &= a,\\
b_0 &= b,\\
b_1 &= b_0 \vee (a_0 \wedge i),\\
a_1 &= a_0 \vee (b_1 \wedge i),\\
b_2 &= b_1 \vee (a_1 \wedge i)\ (= b \vee (a_1 \wedge i)),\\
a_2 &= a_1 \vee (b_1 \wedge i)\ (= a \vee (b_1 \wedge i)),\\
&\dots\\
b_{n+1} &= b_n \vee (a_n \wedge i)\ (= b \vee (a_n \wedge i)),\\
a_{n+1} &= a_n \vee (b_{n+1} \wedge i)\ (= a \vee (b_{n+1} \wedge i)),\\
&\dots
\end{aligned}
$$

See Figure 2—the white filled elements are in A (and maybe in C); the shaded elements are in B (and maybe in C), and the black filled elements are in C.

Lemma 7. *Let* A *and* B *be a full chopped pair. Then in* $M(A,B)$*, the following inequalities hold:*

$$a = a_0 \leq a_1 \leq a_2 \leq \dots \ \textit{(in } A\textit{)}, \tag{4}$$

$$b = b_0 \leq b_1 \leq b_2 \leq \dots \ \textit{(in } B\textit{)}, \tag{5}$$

and

$$a_0 \wedge i \leq b_1 \wedge i \leq a_1 \wedge i \leq b_2 \wedge i \leq a_2 \wedge i \leq \dots \ \textit{(in } C\textit{)}. \tag{6}$$

If, for some n*,* $a_n = a_{n+1}$*, then* (4) *terminates at* n*, and* (5) *terminates at* $n+1$*; and symmetrically, for* (5)*. If* (6) *does not terminate, neither do* (4) *and* (5)*.*

The proof of this lemma is a simplified version of the proof of Lemma 5. Lemma 6 remains valid for full chopped pairs; in this case, the sequences a_n and b_n will be the ones defined in Definition 7.

6. Modular lattices

By inspecting Figure 1, we can see that if A and B are modular, then a lot of elements must collapse. In fact, we have the following result:

Theorem 3. *Let A and B form a sharp chopped pair. Let us assume that both A and B are modular. Then $M(A,B)$ satisfies condition* (FG), *and $\mathrm{Id_{fg}}\, M(A,B)$ is a congruence-preserving extension of $M(A,B)$ (in fact, a congruence-preserving $\{0\}$-extension).*

Proof. Let A and B be modular. The equations (see Figure 1)

$$a_0 \wedge \overline{b_1 \wedge i} = a_0 \wedge (a_1 \wedge i) = a_0 \wedge i,$$

$$a_0 \vee \overline{b_1 \wedge i} = a_0 \vee (a_1 \wedge i) = a_1$$

hold in $M(A,B)$. By the modularity of A, the two equations imply that $\overline{b_1 \wedge i} = a_1 \wedge i$. So

$$\overline{a_1 \wedge i} = \overline{\overline{b_1 \wedge i}} = \overline{b_1 \wedge i}.$$

By the modularity of B, a similar argument yields that $\overline{b_2 \wedge i} = \overline{a_1 \wedge i}$, and so on. So the sequence (3) has only one or two members; it terminates. By Lemma 5, the sequences (1) and (2) terminate. So the statement of the Theorem follows from Lemma 6.

Finally, the statement concerning congruence-preserving extension follows from Lemma 4. ☐

We can prove a stronger statement for full chopped pairs.

Lemma 8. *Let A, B be a full chopped pair. If A is a modular lattice, then*

$$(a,b] = (a_1] \cup (b_1].$$

Proof. As in Theorem 3, the modularity of A implies that $b_1 \wedge i = a_1 \wedge i$. Hence $b_2 = b_1 \vee (a_1 \wedge i) = b_1 \vee (b_1 \wedge i) = b_1$, and $a_2 = a_1 \vee (b_2 \wedge i) = a_1 \vee (b_1 \wedge i) = a_1 \vee (a_1 \wedge i) = a_1$. So the statement of the Lemma follows from Lemma 6. ☐

So now we can conclude a stronger form of Theorem 3 for full chopped pairs:

Theorem 4. *Let A, B be a full chopped pair. If A is a modular lattice, then $M(A,B)$ satisfies condition* (FG).

7. CONGRUENCE-PRESERVING EXTENSIONS

In [2] we raised the following question:

Problem . *Is it true that every lattice with more than one element has a proper congruence-preserving extension?*

We proved in [2] that in the finite case this is true. This result is generalized by the following theorem:

Theorem 5. *Let L be a lattice with zero, 0. If there exists an element $\alpha > 0$ in L such that the interval $[0,\alpha]$ is distributive, then L has a proper congruence-preserving extension K.*

Proof. To prove this result, we need a construction due to the second author. Let M_3 denote the five-element modular nondistributive lattice on the set $\{0,a,b,c,1\}$, and let D be a bounded distributive lattice. Let

$$M_3[D] = \{\langle x,y,z\rangle \in D^3 \mid x \wedge y = x \wedge z = y \wedge z\}.$$

Then $M_3[D]$ is a modular lattice; it contains M_3 as a $\{0,1\}$-sublattice (on the set $\{\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle, \langle 1,1,1\rangle\}$), and each prime interval of this M_3

contains (in $M_3[D]$) a copy of D; for instance, the interval $[\langle 0,0,0\rangle, \langle 1,0,0\rangle]$ can be described as $\{\langle d,0,0\rangle \mid d \in D\}$. If we identify D with $\{\langle d,0,0\rangle \mid d \in D\}$, we find that the lattice $M_3[D]$ is a congruence-preserving $\{0\}$-extension of D.

Now let $D = [0,\alpha]$, and let $A = M_3[D]$. Then A has a spanning M_3; let $i = \langle a,0,0\rangle$. Let $B = L$, and define $i = \alpha$ in B. Then $A \cap B = (i]$, and A, B form a full chopped pair in which A is modular. So we can form the chopped lattice $M(A,B)$. Obviously, $M(A,B)$ is a proper congruence-preserving $\{0\}$-extension of L. By Theorem 4, (FG) holds for $M(A,B)$. Therefore, by Lemma 4, $\mathrm{Id}_{\mathrm{fg}}\, M(A,B)$ is a congruence-preserving $\{0\}$-extension of $M(A,B)$. We conclude that $\mathrm{Id}_{\mathrm{fg}}\, M(A,B)$ is a proper congruence-preserving $\{0\}$-extension of L. ☐

The following result is a generalization of Theorem 5.

Theorem 6. *Let L be a lattice. If there exist a nontrivial distributive interval in L, then L has a proper congruence-preserving extension K.*

Proof. Let $[\alpha,\beta]$ be a nontrivial distributive interval in L. Let us form the lattice $B = [\alpha)$ in L. Obviously, B satisfies the conditions of Theorem 5; therefore, B has a congruence-preserving $\{0\}$-extension K_1. Clearly, B is an ideal of K_1 and a dual ideal of L; hence we can glue L and K_1 over B; let K be the resulting lattice.

Let Θ be a congruence relation on L. Let Θ_B be the restriction of Θ to B. Since K_1 is a congruence-preserving extension of B, there is a unique extension Φ of Θ_B to K_1. It is easy to see that $\overline{\Theta} = \Theta \cup \Phi$ is the unique extension of Θ to K. Hence K is a congruence-preserving extension of L. Obviously, it is a proper extension. ☐

8. A modular example

It is easy to give examples of classes of lattices that have proper congruence-preserving extensions that have nothing to do with distributivity. For instance, every simple lattice with more than one element has a proper simple extension; this is obviously a proper congruence-preserving extension.

In this section we outline a modular example with no proper distributive sublattice.

Let C be a continuous geometry with zero, 0, and unit, 1. Then C has the following properties:

1. For $a < b$, the interval $[a,b]$ is isomorphic to C.
2. C is a simple lattice.

Let I be a nonprincipal ideal of C and F a nonprincipal dual ideal of C satisfying $I \cap F = \varnothing$. Let L be the sublattice $I \cup F$. The congruence lattice of L is the three element chain.

We choose in C a spanning $M_3 = \{0 < a,\ b,\ c < 1\}$. The interval $[0,a]$ is isomorphic to C. Therefore, we find in $[0,a]$ a copy I_a of I and a copy F_a of F. The projectivities in the spanning M_3 define the ideals and dual ideals, I_b, I_c, F_b, F_c in the intervals $[0,b]$ and $[0,c]$. Similarly, we obtain the ideal I_a^u and dual ideal F_a^u in $[a,1]$, I_b^u and F_b^u in $[b,1]$, I_c^u and F_c^u in $[c,1]$.

Let I be the ideal of C generated by the three "small" ideals, I_a, I_b, I_c. Similarly, the three dual ideals F_a^u, F_b^u,F_c^u generate a dual ideal F. We consider the sublattice

$$K = I \cup F \cup F_a \cup F_b \cup F_c \cup I_a^u \cup I_b^u \cup I_c^u.$$

It is easy to see that K is a sublattice of C, and it is a congruence-preserving extension of the sublattice $L \subseteq [0,a]$.

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