COVER-PRESERVING EMBEDDING OF MODULAR LATTICES

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Abstract. In this note we prove: If a subdirect product of finitely many finite projective geometries has the cover-preserving embedding property, then so does each factor.

In what follows all the lattices will be finite modular ones. A finite lattice K has the cover-preseving embedding property, abbreviated as CPEP with respect a variety V of lattices if whenever K can be embedded into a finite lattice L in V, then K has a cover-preserving embedding into L, that is an embedding f with the property that if a covers b in K then $f(a)$ covers $f(b)$ in L. In a paper of E. Fried, G. Grätzer and H. Lakser, [1] it was proved that a finite projective geometry has the cover-preserving embedding property with respect to the variety M of all modular lattices if and only if one of the following three conditions hold: (i) the length of P is 1; (ii) the length of P is 2 and P is isomorphic to $\mathbf{M}_3$; (iii) the length of P is greater then 2 and either P is non-arguesian or P is arguesian and for some prime p each interval of P of length 2 contains $p+1$ atoms (i.e. P is a projective geometry over a prime field). Later in E. T. Schmidt, [2] the following theorem was formulated:

Theorem 1. *If a finite modular lattice L has the CPEP with respect to M then L is the subdirect product of projective geometries of type* (i)-(iii).

Really in [2] the following was proved: If a finite modular lattice L has the CPEP with respect to M then L is the subdirect product of projective geometries. The proof that the subdirect components are just the projective geometries (i)-(iii) was missing. This statment seems in the first moment quite trivial, but it is far not so.

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In this short note we prove that the factors in Theorem 1 are projective geometries of the given type, indeed. First of all we formulate an interesting property of the finite complemented modular lattices. It is easy to see: if $\phi : M \mapsto P$ is a surjection of a finite modular lattice M onto a complemented modular lattice P, then M contains an interval P' such that the restriction of ϕ to P' is an isomorphism between P' and P. If P is a projective geometry then we can prove somewhat more:

Lemma 1. *Let M be any finite modular lattice, P a finite projective geometry with the bounds 0 and 1 and $\phi : M \mapsto P$ a surjective homomorphism. Then, we have:*

1. *There exists a (unique) $a \in M$ with $\phi(a) = 0$ such that $\phi(x) = 0$ implies $x \leq a$.*
2. *There exists a (unique) $b \in M$ with $\phi(b) = 1$ and $b \geq a$ such that $\phi(y) = 1$ and $y \geq a$ together imply $y \geq b$.*
3. *$\phi \restriction [a, b]$ is an isomorphism.*

The interval $[a, b]$ will be called the "natural coimage" *of ϕ.*

Proof. 1. The set $\{x; \phi(x) = 0\}$ is an ideal which is principal by finitness. If it is generated by a, then a has the desired property.

2. The set $\{y; \phi(y) = 1, y \geq a\}$ is a filter which is principal by finitness. If it is generated by b, then b has the desired property.
3. The restriction $\psi = \phi \restriction [a, b]$ is obviously onto. Suppose $x = \psi(u) = \psi(v)$. We may suppose $u \leq v$, as well. Let y be a complement of x. By surjectivity, we have a $w \in [a, b]$ such that $\psi(w) = y$. Then

$$\psi(v \wedge w) = 0 \quad \text{and} \quad \psi(u \vee w) = 1$$

(1) and (2) imply $v \wedge w = a$ and $u \vee w = b$, respectively. Hence, by modularity, $u = v$. □

Corollary 1. *Let $\phi : M \mapsto P$ as in* Lemma 1 *and let $\chi : M \mapsto K$ be another homomorphism to the lattice K. Then the restriction of χ maps the natural coimage $[a, b]$ of ϕ either to a single element of K or this restriction is one-to-one on $[a, b]$.*

Lemma 2. *Let L be a subdirect product of the finite projective geometries P_i together with the natural projections $\phi_i : L \mapsto P_i$ and with the natural coimages $[a_i, b_i]$, $(i \in \{1, 2, ..., n\})$. Suppose, this is a shortest decomposition. Then, ϕ_i maps $[a_j, b_j]$ to a single element, for $i \neq j$.*

Proof. Suppose, say, that ϕ_2 does not map $[a_1, b_1]$ to a single element. Then by Corollary 1, $\phi_2 \restriction [a_1, b_1]$ is one-to-one.

We are going to show, that in this case we may omit P_1 from the subdirect product. In other words, for $x \in L$ the mapping

$$\psi_1 : x \mapsto (\phi_2(x), ..., \phi_n(x))$$

is one-to-one.

Let $x \neq y$ be elements of L. If $\phi_1(x) = \phi_1(y)$, then for some i we have $\phi_i(x) \neq \phi_i(y)$, i.e., $\psi_1(x) \neq \psi_1(y)$. Otherwise,

$$\phi_1((a_1 \vee x) \wedge b_1) = \phi_1(x) \neq \phi_1(y) = \phi_1((a_1 \vee y) \wedge b_1),$$

hence, by our condition, $\psi_1((a_1 \vee x) \wedge b_1) \neq \psi_1((a_1 \vee y) \wedge b_1)$. Therefore, we must have $\psi_1(x) \neq \psi_1(y)$, as well. □

Corollary 2. *Let L be a subdirect product of the projective geometries P_i together with the natural projections $\phi_i : L \mapsto P_i$ and with the natural coimages, $[a_i, b_i]$, $(i \in \{1, 2, ..., n\})$. Suppose, this is a shortest decomposition. Let, further, $\psi : L \mapsto K$ a homomorphism which sends $[a_i, b_i]$ onto K for some i. Then ψ sends all $[a_j, b_j]$ to a single element for each $j \neq i$.*

Proof. We have by, Lemma 2,

$$\phi_i((a_i \vee a_j) \wedge b_i) = \phi_i(a_j) = \phi_i(b_j) = \phi_i((a_i \vee b_j) \wedge b_i),$$

hence, $(a_i \vee a_j) \wedge b_i = (a_i \vee b_j) \wedge b_i$, since ϕ_i is one-to-one on $[a_i, b_i]$, yielding

$$\psi(a_j) = (\psi(a_i) \vee \psi(a_j)) \wedge \psi(b_i) = \psi((a_i \vee a_j) \wedge b_i)$$
$$= \psi((a_i \vee b_j)) \wedge b_i) = (\psi(a_i) \vee \psi(b_j)) \wedge \psi(b_i) = \psi(b_j).$$

This finishes the proof of the Corollary. □

Lemma 3. *Let L be an irreducible subdirect product of the finite projective geometries $P_1, ..., P_t$. If one of the factors fails the CPEP, then so does L.*

Proof. We arrange the factors so that the first s fails CPEP and the other $t - s$ satisfies it. Let A_i denote the number of atoms in P_i. We choose P_1 so that it has the highest dimension among the first s component and, that $A_1 \geq A_2$ for $i \leq s$ if dim(P_i)=dim(P_1). We arrange the first s factors so that $P_1, ..., P_r$ are isomorphic to P_1 and $P_{r+1}, ..., P_s$ are non-isomorphic to P_1. By our assumption there exist a lattice Q_1 such that P_1 has an embedding into it but P_1 has no cover-preserving embedding into it.

Now, we define $Q_i = Q_1$ for $i \leq r$ and $Q_i = P_i$ for $i > r$, and consider the direct product

$$\widehat{L} = Q_1 \times Q_2 \times ... \times Q_r \times Q_{r+1} \times ... \times Q_t.$$

L has an obvious embedding into $\widehat{L}$. We prove that L has no cover-preserving embedding into $\widehat{L}$. Assume,by way of contradiction, that $g : L \mapsto \widehat{L}$ is a cover-preserving embedding. Then, the restriction of g to each natural coimage $[a_i, b_j]$ is a cover-preserving embedding, as well. Let g_j denote the embedding g followed by the j-th projection of $\widehat{L}$. By Corollary 1., the restriction of g_j is either a cover-preserving embedding of $[a_i, b_i]$ into Q_j or it sends this interval to a single element. Since g is an embedding and $\widehat{L}$ is written as a direct product, forevery i must exit a j such that g_j yields a cover-preserving embedding of $[a_i, b_i]$ into Q_j. However, we are going to prove that this is is impossible for $i = 1$. We have to distinguish some cases.

Case 1. $j \leq r$. By our choice, $[a_1, b_1] \cong P_1$ has no cover-preserving embedding into Q_1.

Case 2. $r < j \leq s$ and $\dim(P_j) < \dim(P_1)$. Then, $Q_j = P_j$, hence P_1 cannot be a sublattice of P_j.

Case 3. $r < j \leq s$. Let $\dim(P_k) \geq \dim(P_1)$. Then, by our choice, we must have $\dim(P_j) = \dim(P_1)$. However, in this case we have $A_j \leq A_1$. If $A_j < A_i$, then P_1 has no embedding into P_j, whereas $A_j = A_1$ yealds $P_j \cong P_1$, i.e., $j \leq r$, which was discussed in Case 1.

Case 4.$j > s$. Since $Q_j = P_j$ inthis case, there exist isomorhisms h_j:$Q_j \longrightarrow [a_j, b_j]$ for $(j = s + 1, ..., t)$. Let, further, k_j denote the restriction $g_j \restriction [a_j, b_j]$. By $j.s$ (i.e., by $Q_j = P_j$) and by Corollary 1., if $\mathrm{Im}(k_j)$ has more than one element, then g_j maps all the other $[a_i, b_i]$ to a single element. In other words:

($\bigstar$): If g_j maps $[a_i, b_i]$ isomorphically into Q_j, then k_j is trivial (i.e., maps to a single element).

(In what follows, we shall use the notation g_j for the restriction $g_j \restriction [a_n, b_n]$, as well, provided that the image of this interval has more then one element.)

Now, we have the cover-preserving embedding $[a_1, b_1] \longrightarrow P_{j_1}$. Then, by ($\bigstar$) k_{j_1} is trivial. Hence, we must have a g_{j_2} embedding $[a_{j_1}, b_{j_2}]$ into Q_{j_2}. If $j_2 > s$, then we can continue our procedure. Corollary 1. assure that this chain cannot close, that is there exists an n, such that $j_n \leq s$. Now, consider the diagram:

$$[a_1, b_1] \xrightarrow{g_{j_1}} Q_{j_1} \xrightarrow{h_{j_1}} [a_{j_1}, b_{j_1}] \xrightarrow{g_{j_2}} Q_{j_2} \xrightarrow{h_{j_2}} ... \xrightarrow{h_{j_{n-1}}} [a_{j_{n-1}}, b_{j_{n-1}}] \xrightarrow{g_{j_n}} Q_{j_n}.$$

Here, the first, third, etc. mappings are cover-preserving embeddings whereas the second, fourth, etc are isomorphisms. Hence, their product yields a cover-preserving embedding of $[a_1, b_1]$ into Q_{j_n} for some $j_n \leq s$ contradicting one of the first three cases.

$\square$

Remarks.1.Some results of this paper remain valid for modular lattices of finite length.

2.The Theorem gives only a necessary condition for modular lattices to have (CPEP). It seems to be very complicated to characterize the finite distributive lattices satisfying (CPEP).

References

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