DO WE NEED COMPLETE-SIMPLE DISTRIBUTIVE LATTICES?

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A complete lattice L is called *complete-simple* if it has no nontrivial complete congruence relations, that is, it has no congruence relation that satisfies the Substitution Property with respect to complete joins and meets. The two-element lattice, $\mathfrak{C}_2$, is trivially a distributive complete-simple lattice. In [3] and [4], we have proved the existence of infinite distributive complete-simple lattices.

We used infinite distributive complete-simple lattices in proving the Representation Theorem in [5]: *Every complete lattice L can be represented as the lattice of complete congruence relations of a complete distributive lattice K.*

Of course, infinite distributive complete-simple lattices are not needed as K; indeed, if K is an infinite distributive complete-simple lattice, then the lattice of complete congruences of K is the same as the lattice of complete congruences of the lattice $\mathfrak{C}_2$.

However, in this note we want to observe that the Representation Theorem does require the existence of infinite distributive complete-simple lattices. Let us define a *complete quotient* of a complete lattice L as the quotient L/Θ modulo a complete congruence Θ. We prove the following two results:

Proposition 1. *Let D be a complete distributive lattice, and let the lattice of complete congruences of D be the three-element chain. Then D has an infinite distributive complete-simple lattice as a complete quotient.*

Proposition 2. *Let D be a complete distributive lattice, and let the lattice of complete congruences of D be the five-element nonmodular lattice, $\mathfrak{N}_5$. Then D has an infinite distributive complete-simple lattice as a complete quotient.*

Proof of Proposition 1. Let Θ be the only nontrivial complete congruence relation of D. Then the quotient lattice, D/Θ, is a complete-simple lattice because Θ is a maximal proper complete congruence relation. If D/Θ is infinite, then it is an infinite distributive complete-simple lattice, and the proposition is proved. By way of contradiction, let us assume that D/Θ is finite; then it is a two-element lattice. Therefore, Θ has two congruence classes. Since Θ is complete, the smaller congruence class has a largest element, a, and the larger congruence class has a smallest element, b. It follows that $a \prec a \vee b$.

It was observed in [1] that the (ordinary) principal congruence relation $\Theta(a, a \vee b)$ is a complete congruence. Obviously, $\Theta(a, a \vee b)$ is the complement of Θ in the lattice of all complete congruence relations of D, contricting that the lattice of all complete congruence relations is the three-element chain. □

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Proof of Proposition 2. Let Θ, Φ, Ψ be the nontrivial complete congruence relation of D, where $\Psi < \Theta$. As in the proof of Proposition 1, by way of contradiction, we can assume that D/Θ is finite, and conclude that there is a covering pair of elements $a \prec a \vee b$ in D such that $\Theta(a, a \vee b)$ is the complement of Θ in the lattice of all complete congruence relations of D. Then $\Phi = \Theta(a, a \vee b)$. It now follows that D/Φ is infinite because if it were finite, so would be D, contradicting that the complete congruence lattice is nondistributive. So D/Φ is the required infinite complete-simple lattice quotient. □

Similar arguments work in many other cases.

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