CONGRUENCE LATTICES OF P-ALGEBRAS

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ABSTRACT. T. Katriňák proved the following theorem: *Every finite distributive lattice is the congruence lattice of a finite p-algebra.* We provide a short proof, and a generalization, of this result.

1. Introduction

A *p-algebra* P is a pseudocomplemented lattice in which the pseudocomplementation * is regarded as a unary operation; that is, P is an algebra with the binary operations $\vee$ and $\wedge$, the unary operation *, and the nullary operations 0 and 1, such that $\langle P; \vee, \wedge, 0, 1\rangle$ is a bounded lattice with zero 0 and unit 1, and * satisfies $x \leq a^*$ iff $x \wedge a = 0$.

T. Katriňák [8] has proved the following result:

Theorem 1. *Every finite distributive lattice D is isomorphic to the congruence lattice of a finite p-algebra P.*

In this paper we give a new proof of this theorem. We hope that the reader will find the present proof more elementary.

In fact, we prove the following generalization:

Theorem 2. *Let D be an algebraic distributive lattice satisfying the following two conditions:*

1. *The unit element of D is compact.*
2. *Every compact element of D is a finite join of join-irreducible compact elements.*

Then D can be represented as the congruence lattice of a p-algebra P.

To keep the discussion as simple as possible, we shall first prove Theorem 1. In Section 5, we shall then indicate what changes are necessary to prove Theorem 2, and we shall also mention some open problems.

2. Preliminaries

For the notation and basic concepts, we refer the reader to [1].

For a finite distributive lattice D, let $J(D)$ denote the poset of all *join-irreducible elements* of D. For a poset Q, let $H(Q)$ denote the distributive lattice of all *hereditary subsets* of Q (including $\varnothing$). Recall that $H(J(D))$ is isomorphic to D.

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An element d of a p-algebra P is called *dense* if $d^* = 0$; the dense elements form a dual ideal $D(P)$.

The construction of the present paper is based on a theorem of R. P. Dilworth; see [1] and [6].

Theorem 3. *Let D be a finite distributive lattice. Then there exists a finite and sectionally complemented lattice L such that the congruence lattice of L is isomorphic to D.*

A *chopped lattice* S is a lattice L with zero 0 and unit 1 with the unit removed: $S = L - \{1\}$; on S, 0 is a nullary operation, $\wedge$ is an operation, and $\vee$ is a partial operation. Equivalently, a chopped lattice S is a meet-semilattice with zero 0, in which any two elements having an upper bound have a join.

An *ideal* I of S is a subset of S containing 0 with the following two properties, for $x, y \in S$:

$x \in I$ and $y \leq x$ imply that $y \in I$.

If $x, y \in I$ and $x \vee y$ exists, then $x \vee y \in I$.

$\operatorname{Id} S$ denotes the *lattice of ideals* of S.

For $a \in S$, we denote by $(a]$ the *principal ideal* generated by a in S, that is,

$$(a] = \{x \in S \mid x \leq a\}.$$

By identifying $a \in S$ with $(a]$, we regard $\operatorname{Id} S$ an extension of S.

The proof of Theorem 3 (as presented in [6]) was based on the following idea: we represent D as the congruence lattice of a finite chopped lattice S. Then we apply the following result (which is due to H. Lakser and the first author, see [1]):

Lemma . *Let S be a finite chopped lattice. Then for every congruence relation Θ of S, there exists exactly one congruence relation $\bar{\Theta}$ of $\operatorname{Id} S$ such that for $a, b \in S$, $(a] \equiv (b]\ (\bar{\Theta})$ iff $a \equiv b\ (\Theta)$.*

Obviously, by the Lemma, the congruence lattice of the chopped lattice S is isomorphic to the congruence lattice of the lattice $\operatorname{Id} S$.

3. The construction

Let D be the finite distributive lattice we want to represent in Theorem 1. We may assume that D contains at least two elements. Let $m_1, m_2, \dots, m_n$ denote the maximal elements of $J(D)$. For any j with $1 \leq j \leq n$, let

$$M_j = \{x \in J(D) \mid x < m_j\}.$$

M_j may be empty.

To every $M_j \neq \varnothing$, we apply Theorem 3, to obtain a finite sectionally complemented lattice L_j such that the congruence lattice of L_j is isomorphic to $H(M_j)$. Let us denote by o_j and i_j the zero and unit of L_j, respectively. We define the lattice $\widehat{L}_j$ as L_j with a new zero 0_j adjoined, see Figure 1.

If $M_j = \varnothing$, then we define $\widehat{L}_j$ as the two element lattice with elements 0_j and $o_j = i_j$.

Let $L = \Pi(\widehat{L}_j \mid j \leq n)$. Define

$$\mathbf{o} = \langle o_1, o_2, \dots, o_n \rangle,\ \mathbf{0} = \langle 0_1, 0_2, \dots, 0_n \rangle \in L.$$

For every $1 \leq j \leq n$, we define:

$$\mathbf{t}_j = \langle o_1, \dots, o_{j-1}, i_j, o_{j+1}, \dots, o_n \rangle,\ \mathbf{o}_j = \langle 0_1, \dots, 0_{j-1}, o_j, 0_{j+1}, \dots, 0_n \rangle \in L.$$

Obviously, the interval $[\mathbf{o}, \mathbf{t}_j]$ of L is isomorphic to L_j.

Let $p \in M_j$. The lattice L_j is finite and sectionally complemented; therefore, every join-irreducible congruence of L_j can be represented in the form, $\Theta(o_j, x_j)$, where x_j is an atom of L_j. The hereditary set

$$\{h \in J(D) \mid h \leq p\} \subseteq M_j$$

corresponds to a join-irreducible congruence of L_j; this congruence is of the form $\Theta(o_j, p_j)$, where p_j is an atom of L_j. For every p and j with $p \in M_j$, we choose such an atom p_j in L_j, and define the element

$$\mathbf{p}_j = \langle o_1, \ldots, o_{j-1}, p_j, o_{j+1}, \ldots, o_n \rangle \in L.$$

We define a subset of L as follows:

$$S' = \bigcup ((\mathbf{t}_j] \mid 1 \leq j \leq n).$$

It is obvious that S' is a finite chopped lattice. We extend S' as follows.

If for the element $p \in J(D)$, there are at least two different j, $k \leq n$ such that $p \in M_j$ and $p \in M_k$, then we adjoin to S' two new elements $\mathbf{p}_0$ and $\bar{\mathbf{p}}$ so that the elements

$$\mathbf{o},\ \mathbf{p}_0,\ \{\mathbf{p}_j \mid p \in M_j,\ 1 \leq j \leq n\},\ \bar{\mathbf{p}}$$

form a lattice isomorphic to a projective line, with zero $\mathbf{o}$, unit element $\bar{\mathbf{p}}$, and with atoms $\mathbf{p}_0$ and $\{\mathbf{p}_j \mid p \in M_j,\ 1 \leq j \leq n\}$.

The poset S is the poset S' extended by all $\mathbf{p}_0$ and $\bar{\mathbf{p}}$. The poset S is a finite chopped lattice. We define the p-algebra P of Theorem 1 by

$$P = \operatorname{Id} S$$

S is illustrated in Figure 2 with a p that is in M_1 and M_2. On the diagram, we mark the interval $[\mathbf{o}, \mathbf{t}_j]$, with L_j, for $j = 1$, 2, and 3. The dashed lines indicate that only the very top and the very bottom of S have been drawn. Observe that P contains L.

4. Proof of Theorem 1

First, observe that P is a p-algebra. Indeed, L is the direct product of the p@-algebras $\widehat{L}_j$, $1 \leq j \leq n$; hence L is a p-algebra. $D(L)$ is the dual ideal generated by $\mathbf{o}$. The adjoined elements are all in $D(P)$, consequently, P is a p-algebra.

We have to describe the congruence relations of the p-algebra P. By the Lemma, the congruence relations of the *lattice* P are in one-to-one correspondence with the congruences of the chopped lattice S. Therefore, the congruences of the *p-algebra* P are in one-to-one correspondence with the congruences Θ of the chopped lattice S that satisfy the substitution property for pseudo-complementation.

Let $p \in J(D)$. We define a congruence Θ_p on S in four steps.

Step 1. For $r \in J(D)$ and $1 \leq j \leq n$, we define a congruence relation Φ_r^j on $\widehat{L}_j$ as follows:

1. If $r = m_j$, then Φ_r^j is the largest congruence ι on $\widehat{L}_j$.
2. If $r \nleq m_j$, then Φ_r^j is the smallest congruence ω on $\widehat{L}_j$.
3. If $r < m_j$, that is, if $r \in M_j$, then the congruence lattice of L_j is isomorphic to $H(M_j)$. Then Φ_r^j is the congruence which on L_j corresponds to the set

$$\{x \in M_j \mid x \leq r\} \in H(M_j)$$

under this isomorphism; and $\{0_j\}$ forms a singleton class.

Step 2. Let Θ_p^j denote the following congruence on $\widehat{L}_j$:

$$\Theta_p^j = \bigvee(\Phi_r^j \mid r \leq p).$$

Step 3. Now we form the congruence

$$\Pi(\Theta_p^j \mid 1 \leq j \leq n)$$

on

$$L = \Pi(\widehat{L}_j \mid 1 \leq j \leq n).$$

Since S' is contained in L, we obtain by restriction a congruence Θ_p' of S'.

Step 4. Finally, we extend Θ_p' from S' to S to obtain Θ_p. Let the Θ_p classes on S be the same as the Θ_p' classes on S' with the following exception. Recall that $S - S'$ consists of elements of the form $\mathbf{r}_0$ and $\bar{\mathbf{r}}$ where $r \in J(D)$ and there are at least two different $1 \leq j,\ k \leq n$ such that $r \in M_j$ and $r \in M_k$. If $r \leq p$, then there is a congruence class of Θ_p' containing $\mathbf{o}$, $\mathbf{r}_j$, and $\mathbf{r}_k$. In this case, let the Θ_p class containing $\mathbf{o}$ also contain $\mathbf{r}_0$ and $\bar{\mathbf{r}}$. Otherwise, $\{\mathbf{r}_0\}$ and $\{\bar{\mathbf{r}}\}$ are congruence classes under Θ_p.

We have to prove the following statements:

1. Θ_p is a congruence relation of the chopped lattice S.

2. Θ_p is join-irreducible, and every join-irreducible congruence is of this form.

3. The mapping $p \to \Theta_p$ is an order-preserving bijection between $J(D)$ and the poset of all join-irreducible congruences of S.

4. The extension $\bar{\Theta}_p$ of Θ_p to $P = \mathrm{Id}\, S$ is a congruence of the p-algebra P.

All four statements are trivial, or follow from the Lemma.

This concludes the proof of the Theorem 1.

5. Discussion

A chopped lattice S is *locally finite* if every principal ideal of S is finite.

Let D be an algebraic distributive lattice satisfying the two conditions of Theorem 2. Then we can construct a locally finite chopped lattice S with zero whose congruence lattice is isomorphic to D, just as in the finite case, see [1] and [6]. The Lemma then applies. In S, the join-irreducible compact elements are again of the form $\Theta(0, a)$, where 0 is the zero of S and a is an atom.

Because of the second condition on D, the unit element can be represented as the join of join-irreducible compact elements $m_1, m_2, \ldots, m_n$, finite in number, and every compact join-irreducible element is contained in some m_j, $1 \leq j \leq n$. As in the finite case, for any j with $1 \leq j \leq n$, let M_j be the poset $\{x \in J(D) \mid x < m_j\}$. Now observe that Theorem 3 can be applied to the poset M_j (see [1] and [6]) to obtain a locally finite and sectionally complemented lattice whose congruence lattice is isomorphic to $H(M_j)$. The join-irreducible compact congruences are again of the form $\Theta(0, a)$, where 0 is the zero and a is an atom.

Now the proof of Theorem 1 carries through without any change.

The p-algebra P is said to be *decomposable* if for every $x \in P$ there exists a $d \in D(P)$ such that $x = x^{**} \wedge d$.

T. Katriňák [8] proved a stronger version of Theorem 1. He proved that the algebra P can be constructed as a decomposable p-algebra.

It is easy to see that our construction also yields a decomposable p-algebra. Indeed, L is decomposable since each $\widehat{L}_i$ is. Since we expanded only $D(L)$ to obtain P, we conclude that P is decomposable.

A number of problems arise naturally. Since our proof relies on Theorem 3, any generalization or strengthening of this result raises the question whether there is an analogous result for p-algebras. G. Grätzer [2] is a brief survey of this field. Any one of the results mentioned there raises a problem. Let us mention here only two:

Problem 1. Is every finite distributive lattice D isomorphic to the congruence lattice of a *modular* p-algebra P?

Note that the p-algebra is no longer required to be finite. See E. T. Schmidt [7] for the lattice-theoretic result.

Problem 2. Let D be a finite distributive lattice with n join-irreducible elements. Is D isomorphic to the congruence lattice of a p-algebra P of $O(n^2)$ elements?

See G. Grätzer and H. Lakser [3], G. Grätzer, H. Lakser, and E. T. Schmidt [4], and G. Grätzer, I. Rival, and N. Zaguia [5], for the lattice-theoretic result.

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