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# HOMOMORPHISMS OF DISTRIBUTIVE LATTICES AS RESTRICTION OF CONGRUENCES: THE PLANAR CASE

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Given a lattice $L$ and a sublattice $L'$, then the map of $\operatorname{Con} L$ to $\operatorname{Con} L'$ determined by restriction is a meet-homomorphism preserving 0 and 1. If $L'$ is a convex sublattice, then this map is a lattice homomorphism. G. Grätzer and H. Lakser [1] proved that any $\{0,1\}$-preserving homomorphism of finite distributive lattices can be realized by restricting the congruence lattice of some finite planar lattice $L$ to the congruence lattice of an ideal $L'$ of $L$. In this note we give a short proof of this result.

THEOREM. *Let $D$ and $D'$ be finite distributive lattices and let $\Psi\colon D \to D'$ be a $\{0,1\}$-preserving lattice homomorphism. Then there exist a finite planar lattice $L$, an ideal $L'$ of $L$ and lattice isomorphisms*

$$\rho : D \to \operatorname{Con} L, \quad \rho' : D' \to \operatorname{Con} L'$$

*such that $\Psi\rho'$ is the composition of $\rho$ with the restriction of* $\operatorname{Con} L$ *to* $\operatorname{Con} L'$. *Moreover, the lattices $L$ and $L'$ have no nontrivial automorphisms* (see Figure 1).

$$\begin{tikzcd}
D' \arrow[dr] \arrow[ddr, dashed, "\rho'"'] & & D \arrow[ll, "\text{lattice hom.}"', "\Psi"] \arrow[dr] \arrow[ddr, dashed, "\rho"'] & \\
& L' \arrow[rr, hook, two heads, "\text{embedding}"] \arrow[d] & & L \arrow[d] \\
& \operatorname{Con} L' & & \operatorname{Con} L \arrow[ll, "\text{restriction}"']
\end{tikzcd}$$

*Fig. 1*

PROOF. Let $\Psi : D \to D'$ be the given $\{0,1\}$-preserving homomorphism. By the duality between finite distributive lattices and finite posets $\Psi$ determines an isotone map $\varphi : \mathcal{J}(D') \to \mathcal{J}(D)$. Conversely, $\Psi$ is determined by $\varphi$.

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Let $T$ be the set $\mathcal{J}(D) \cup \mathcal{J}(D')$. We can extend $\varphi$ to $T$ by setting $x\varphi = x$ for $x \in \mathcal{J}(D)$. Denote $p_1, p_2, \dots, p_m$ resp. $p_{m+1}, \dots, p_n$ the elements of $\mathcal{J}(D')$ resp. $\mathcal{J}(D)$. $\varphi$ can be characterized by a quasi-ordering $\preceq$ on $T$:

$$(*) \qquad p_i \preceq p_j \text{ if and only if } \begin{cases} p_i \leqq p_j \text{ in } \mathcal{J}(D'), & i, j \leqq m \text{ and} \\ p_i\varphi \leqq p_j\varphi \text{ in } \mathcal{J}(D) & \text{otherwise.} \end{cases}$$

It is easy to check that $\preceq$ is a quasi-ordering. Let $\Theta$ be the equivalence relation of $T$ induced by this relation, i.e., $p_i \Theta p_j$ iff $p_i \preceq p_j$ and $p_j \preceq p_i$. Then $T/\Theta$ is a poset. By $(*)$ if $0 < i \leqq m$ then $p_i \preceq p_i\varphi$ and $p_i\varphi \preceq p_i$, i.e., $p_i \Theta p_i\varphi$. This implies $T/\Theta \cong \mathcal{J}(D)$.

We define two types of lattices $A_{ij}$ and $B_{ij}$, $0 < j < i \leqq n$ by the diagrams illustrated in Figure 2. Let $\underline{n} = \{0 < 1 < \cdots < n\}$ be an $n+1$-element chain. $A_{ij}$ is the direct product $\underline{n} \times \underline{2}$ augmented with the elements $c_i, c_j, c_{j-1}, \dots, c_0$. $B_{ij}$ is $\underline{n} \times \underline{2}$ augmented with the elements $d_j, d_i, d_{i+1}, \dots, d_n$.

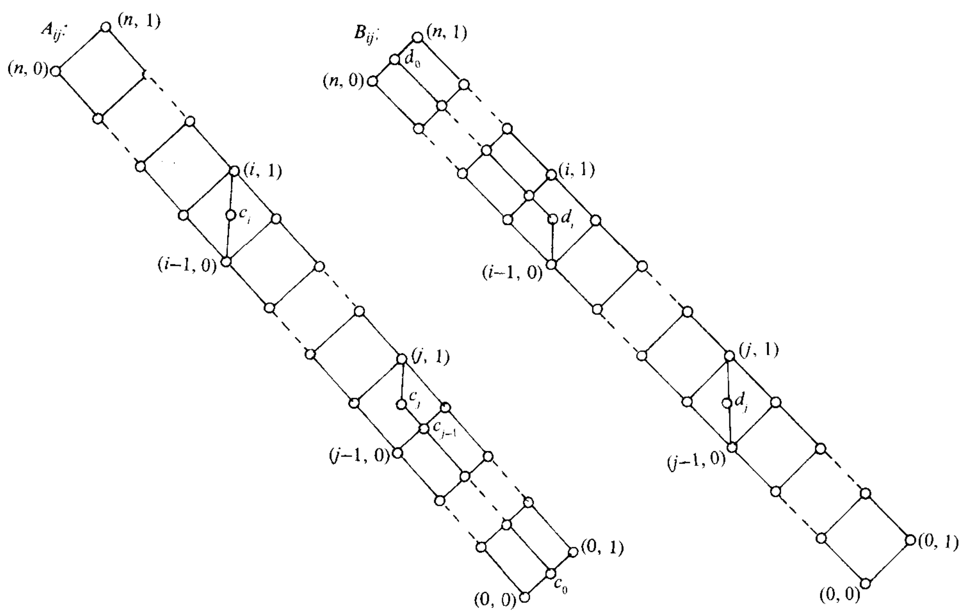


*Fig. 2*

Let $\mathcal{J}_{ij}$ be the ideal of $A_{ij}$ (resp. $B_{ij}$) generated by $(n, 0)$ and let $F_{ij}$ be the filter of $A_{ij}$ (resp. $B_{ij}$) generated by $(0, 1)$. Then $\mathcal{J}_{ij} \cong F_{ij} \cong \underline{n}$. An easy computation shows that the following hold:

(i) Every congruence relation of $A_{ij}$ (resp. $B_{ij}$) is determined by its restriction to $\mathcal{J}_{ij}$ and similarly to $F_{ij}$.

(ii) If $l \neq k$ then $(l-1, 0) \equiv (l, 0)$ forces $(k-1, 0) \equiv (k, 0)$ in $A_{ij}$ iff $l = i$ and $k = j$.

(iii) For $l \neq k$ $(l-1,0) \equiv (l,0)$ forces $(k-1,0) \equiv (k,0)$ in $B_{ij}$ iff $l=j$, $k=i$.

We define the lattice $L$. Consider the bijection $\sigma : [i-1, i] \to p_i$ between the prime intervals of $\underline{n}$ and the elements of $T$ ($\sigma$ is called a coloring of $\underline{n}$).

For $1 \leqq j < i \leqq n$ define

$$R_{ij} \cong \begin{cases} A_{ij} & \text{if } p_j \prec p_i \text{ in } T, \\ B_{ij} & \text{if } p_i \prec p_j \text{ in } T. \end{cases}$$

Order the pairwise disjoint $R_{ij}$-s say $R_{i_0j_0}, R_{i_1j_1}, \ldots, R_{i_sj_s}, \ldots$ such that $(i_0, j_0)$, $(i_1, j_1)$, $\ldots, (i_s, j_s)$ are exactly the pairs which satisfy $1 \leqq i_k$, $j_k \leqq m$ and either $p_{i_k} < p_{j_k}$ or $p_{j_k} < p_{i_k}$ in $\mathcal{J}(D')$. Now we apply the Hall–Dilworth gluing: the filter $F_{i_0j_0}$ of $R_{i_0j_0}$ is isomorphic to the ideal $\mathcal{J}_{i_1j_1}$ of $R_{i_1j_1}$. Identify $F_{i_0j_0}$ and $\mathcal{J}_{i_1j_1}$ via the isomorphism, we obtain the lattice $R_{i_0j_0} \cup \cup R_{i_1j_1}$ which contains $F_{i_1j_1}$ as a filter. Then take $R_{i_2j_2}$ and its ideal $\mathcal{J}_{i_2j_2}$. We apply again the gluing construction, by identifying $F_{i_1j_1}$ and $\mathcal{J}_{i_2j_2}$. We continue this procedure, the resulting lattice is $L$. Then $\mathcal{J} = \mathcal{J}_{i_0j_0}$ is an ideal of $L$. $R_{i_0j_0}$ is isomorphic to one of the $A_{ij}$-s or $B_{ij}$-s, let $k^*$ be the element of $\mathcal{J}_{i_0j_0} \subseteq R_{i_0j_0}$ which corresponds to the element $(k, 0)$ of $A_{ij}$ (or $B_{ij}$).

The properties (i), (ii) and (iii) imply that every congruence relation of $L$ is determined by its restriction to $\mathcal{J}$ and $(j-1)^* \equiv j^*$ forces $(i-1)^* \equiv i^*$ $(i \neq j)$ in $L$ iff $p_i \preceq p_j$ in $T$. Consequently, $\mathcal{J}(\operatorname{Con} L) \cong T/\Theta$ which proves $\operatorname{Con} L \cong D$.

Finally, we define the ideal $L'$ of $L$. If $D'$ is a Boolean lattice then $L' = \{0^* < 1^* < \cdots < s^*\}$.

$R_{i_sj_s}$ is isomorphic to one of the $A_{ij}$-s or $B_{ij}$-s. Denote $t \in R_{i_sj_s} \subseteq L$ the element which corresponds to $(0,1) \in A_{ij}$ (or $B_{ij}$) by this isomorphism, $m^* \in \in \mathcal{J}_{i_0j_0}$ ($m$ is the cardinality of $\mathcal{J}(D')$) and consider the ideal $L'$ generated by the element $m^* \vee t$ (see Figure 3).

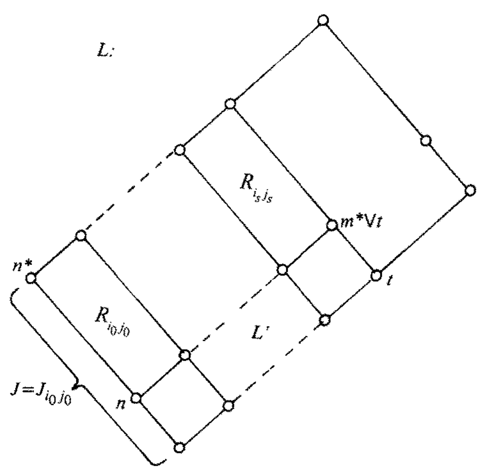


*Fig. 3*

By the given ordering of the "rows" $R_{ij}$ we obtain that in $\mathcal{J} \cap L' = \mathcal{J}_{i_0 j_0} \cap L'$, $(j-1)^* \equiv j^*$ forces $(i-1)^* \equiv i^*$ $(i \neq j)$ iff $p_i < p_j$ in $\mathcal{J}(D')$, i.e., $\mathcal{J}(\operatorname{Con} L') \cong \mathcal{J}(D')$. This is equivalent to $\operatorname{Con} L' \cong D'$. It is clear that the restriction of $\operatorname{Con} L$ to $\operatorname{Con} L'$ is just the given $\{0,1\}$-preserving homomorphism $\Psi$.

If $\alpha$ is an arbitrary automorphism of $L$ (and similarly of $L'$) then its restriction to a "row" $R_{i_k j_k}$ is an automorphism of $R_{i_k j_k}$. Therefore we have only two special cases if $\alpha$ is a nontrivial homomorphism of $R_{i_k j_k}$. In these cases we modify the construction slightly.

(1) If $R_{i_0 j_0} \cong A_{ij}$ and $i = n$. Then the interval $[(n-1,0),(n,1)]$ is isomorphic to $M_3$. We replace this block by the lattice illustrated in Figure 4.

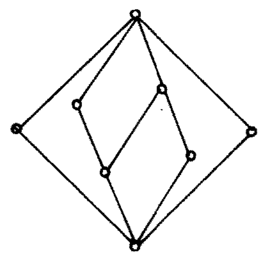

*Fig. 4*

If $[(n-1,0),(n,1)]$ is isomorphic to the lattice illustrated by Figure 5a, then replace this lattice defined by Figure 5b.

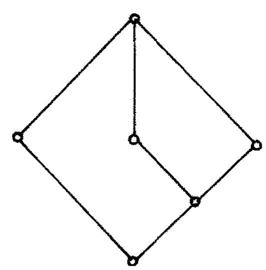

*Fig. 5a*

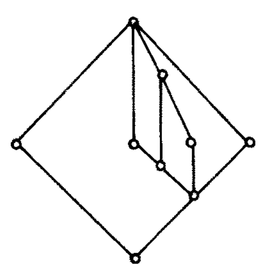

*Fig. 5b*

(2) We use the same modificated construction if $R_{i_k j_k}$ is the least row and $R_{i_k j_k} \cong A_{ij}$ or $B_{ij}$ where $j = 1$ then the first block is again $M_3$ or the lattice illustrated by the dual of the lattice defined by Figure 5a.

## REFERENCE

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