COMPLETE CONGRUENCE LATTICES OF COMPLETE DISTRIBUTIVE LATTICES

G. GRÄTZER AND E. T. SCHMIDT

Abstract. In 1983, R. Wille raised the question: *Is every* complete lattice L *isomorphic to the lattice of complete congruence relations of a suitable* complete lattice K*?* In 1988, this was answered in the affirmative by the first author. A number of papers have been published on this problem by R. Freese, P. Johnson, H. Lakser, S.-K. Teo, and the authors.

In the present paper we prove that K can always be choosen as a *complete distributive lattice.* In fact, we prove the following more general result:

Theorem . *Let* $\mathfrak{m}$ *be a regular cardinal* $> \aleph_0$*. Every* $\mathfrak{m}$-algebraic lattice L *can be represented as the lattice of* $\mathfrak{m}$*-complete congruence relations of an* $\mathfrak{m}$-complete distributive lattice K*.*

1. Introduction

In 1945, G. Birkhoff (see [1] and [2]) formulated the celebrated characterization problem of congruence lattices of (finitary and infinitary) algebras. The finitary problem was solved in 1961 by the authors. The infinitary problem was solved by G. Grätzer and W. A. Lampe; see [6] (Chapter 2, Appendices 1 and 7) for a detailed discussion of this problem.

In 1983, R. Wille raised the following closely connected question (see, e.g., K. Reuter and R. Wille [16]):

Problem 1. *Is every* complete lattice L *isomorphic to the lattice of complete congruence relations of a suitable* complete lattice K*?*

S.-K. Teo [17] solved this problem in the finite case. In 1988, the first author announced an affirmative answer to Wille's question in [7]. G. Grätzer and H. Lakser [9] constructed a *planar complete lattice* K.

Let K be an $\mathfrak{m}$*-complete* lattice; it was noted in [10] that the lattice L of all $\mathfrak{m}$-complete congruence relations of K is $\mathfrak{m}$*-algebraic*. Grätzer and Lakser proved a partial converse:

Let $\mathfrak{m}$ *be a regular cardinal* $> \aleph_0$*, and let* L *be an* $\mathfrak{m}$-algebraic lattice *with an* $\mathfrak{m}$*-compact unit element. Then* L *is isomorphic to the lattice of* $\mathfrak{m}$*-complete congruences of an* $\mathfrak{m}$-complete lattice K*.*

A much sharper form of the original result was proved in the paper R. Freese, G. Grätzer, and E. T. Schmidt [3]:

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Every complete lattice L *is isomorphic to the lattice of complete congruence relations of a* complete modular lattice K.

The $\mathfrak{m}$-algebraic direction and the modular direction were combined by the present authors in [13]:

Let $\mathfrak{m}$ *be a regular cardinal* $> \aleph_0$. *Every* $\mathfrak{m}$-algebraic lattice L *is isomorphic to the lattice of* $\mathfrak{m}$*-complete congruence relations of a suitable* $\mathfrak{m}$-complete modular lattice K.

G. Grätzer, P. Johnson, and E. T. Schmidt [8] presents the same construction with a simplified proof.

In this paper we prove the following sharper result:

Theorem . *Let* $\mathfrak{m}$ *be a regular cardinal* $> \aleph_0$. *Every* $\mathfrak{m}$-algebraic lattice L *can be represented as the lattice of* $\mathfrak{m}$*-complete congruence relations of an* $\mathfrak{m}$-complete distributive lattice K.

Since the proof is somewhat long and technical, we start out in Section 2 to give an outline of the proof. The basic notation is introduced in Section 3. In Section 4 we review $\mathfrak{m}$-complete and $\mathfrak{m}$-algebraic lattices. $\mathfrak{m}$-complete congruences are briefly surveyed in Section 5. Some preliminary constructions (one from [12]) and results are presented in Section 6. The construction of K really starts in Section 7 with the introduction of the tower construction; we also describe complete congruences of towers. In constructing K, we use two special towers described in Section 8. The towers are merged in Section 9, providing the analogue of the "second basic construction" of the outline in Section 2. Finally, in Section 10 we construct K, and in Section 11 we prove that K satisfies the conditions of the Theorem. A number of related and problems are listed in Section 12.

2. Outline of proof

We outline the proof of the following version of the Theorem:

Theorem' . *Every* complete lattice L *can be represented as the lattice of complete congruence relations of a* complete distributive lattice K.

Earlier proofs of results of this type proceeded as follows. We construct a complete lattice K with the following properties:

1. K is *weakly atomic*, that is, every proper interval $[a, b]$ ($a, b \in K$, and $a < b$) contains an element p such that $a \prec p \leq c$.
2. There is a map φ of the prime intervals of K into $L - \{0\}$ such that two prime intervals generate the same complete congruence iff they have the same image under φ.

   The φ image of a prime interval is called the *color* of the prime interval.
3. Every complete congruence $> \omega$ is generated by a prime interval.

   For $a \in L - \{0\}$, let Θ^a denote the complete congruence generated by any (all) prime interval $\mathfrak{p}$ with $\mathfrak{p}\varphi = a$.
4. For $X \subseteq L - \{0\}$,
$$\bigvee(\Theta^a \mid a \in X) = \Theta^{\bigvee X}.$$

Property (1) is necessary to ensure Property (3). For Property (2) to hold, we take two prime intervals in general position and add an $\mathfrak{M}_3$ as shown in Figure 1; this is the *first basic construction*. Then to ensure Property (4) we do the *second basic construction* illustrated in Figure 2 for $X = \{a, b\}$. In Figure 2, the color of many prime intervals are shown. Note that the complete congruence generated by the prime intervals of color a joined by the complete congruence generated by the prime intervals of color b equals the complete congruence generated by the prime intervals of color $a \vee b$.

Now the construction of K for Theorem' (in the nondistributive case) is as follows:

Step 1: Construct a complete lattice K_1 in which for every $a \in L - \{0\}$ there is a prime interval $\mathfrak{p}_a$, such the (complete) congruences generated by the prime intervals $\{\mathfrak{p}_a \mid a \in L - \{0\}\}$ are "independent."

Step 2: Apply the first basic construction to obtain K_2 with Property (2).

Step 3: Apply the second basic construction to K_2 to obtain K_3 in which Property (4) holds.

If we attempt to carry out such a construction for complete distributive lattices, we find that there are two obstacles in the way.

- The first basic construction cannot be done for distributive lattices. The congruences generated by two prime intervals in general position are always independent.
- It was observed in [3] that the congruence relation generated by a prime interval in a complete distributive lattice is always a complete congruence. Therefore, the complete distributive lattice K we construct cannot, in general, contain prime intervals.

Instead of prime intervals, we shall use infinite complete-simple complete distributive lattices. Such lattices were constructed in [11] and [12]. It can be shown that the representation of the three-element chain must use such a lattice.

The identification of congruences—Property (2)—will be done using the $D^{\langle 2 \rangle}$ construction and the Π^* product, see Definition 3 and Lemma 7. Every complete congruence will be generated by an interval which is an infinite complete-simple complete distributive lattice.

To find a substitute for the second basic construction, we introduce the tower construction in Sections 7 and 8. The merging of the towers in Section 9 will have properties analogous to the properties of the second basic construction.

We modify (2) as follows. There is a map φ of some intervals of K into $L - \{0\}$; each of these intervals is an infinite complete- simple complete distributive lattice. Two of these intervals generate the same complete congruence iff they have the same image under φ.

Finally, in Section 10, we construct a complete distributive lattice with sufficiently many infinite complete-simple intervals in a general position and we add enough Π^* products and merged towers to be able to control the complete congruences.

3. Notation

For the notation and basic concepts, we refer the reader to [5].

$\mathfrak{C}_2$ denotes the two-element chain. The symbol $\cong$ denotes isomorphism.

A *bounded* lattice is synonymous with *a lattice with* 0 *and* 1. In such a context, 0 is the least (zero) element of the lattice, and 1 is the largest (unit) element.

For a lattice L, let $L + 1$ denote the lattice obtained from L by adjoining a new unit element. Conversely, if the lattice L has a join-irreducible unit element, then L^- denotes the lattice obtained from L by removing its unit element.

Let L_i, $i \in I$ be lattices. Then $\Pi(L_i \mid i \in I)$ denotes their (complete) *direct product*; for $\mathbf{v} \in \Pi(L_i \mid i \in I)$, let $\mathbf{v}(i)$ denote the *i-th component* of $\mathbf{v}$; by definition, $\mathbf{v}(i) \in L_i$. For $\mathbf{v} \in \Pi(L_i \mid i \in I)$, let $\operatorname{supp}\mathbf{v}$ denote the *support of* $\mathbf{v}$, that is, $\operatorname{supp}\mathbf{v} = \{i \in I \mid \mathbf{v}(i) > 0\}$.

$\mathfrak{c}$ denotes the power of the continuum.

Clause 2 of Lemma 3 will be referred to as Lemma 3.2.

4. $\mathfrak{m}$-complete and $\mathfrak{m}$-algebraic lattices

A cardinal $\mathfrak{m}$ is *regular* if whenever J is a set with $|J| < \mathfrak{m}$ and $(I_j \mid j \in J)$ is a family of sets satisfying $|I_j| < \mathfrak{m}$, for all $j \in J$, then $|\bigcup(I_j \mid j \in J)| < \mathfrak{m}$. For instance, all cardinals of the form $\aleph_{\alpha+1}$ are regular. In this paper, $\mathfrak{m}$ stands for a fixed *uncountable regular cardinal*. A set of cardinality less than $\mathfrak{m}$ is said to be *small*.

A lattice M is $\mathfrak{m}$*-complete* if $\bigvee X$ and $\bigwedge X$ exist in M for every small nonempty subset X of M.

Let L_i, $i \in I$, be lattices. Then $\Pi_{\mathfrak{m}}(L_i \mid i \in I)$ denotes their $\mathfrak{m}$*-weak direct product*, that is, the sublattice of $\Pi(L_i \mid i \in I)$ consisting of all $\mathbf{v} \in \Pi(L_i \mid i \in I)$ with small support. If all L_i, $i \in I$, are $\mathfrak{m}$-complete lattices, then so is $\Pi_{\mathfrak{m}}(L_i \mid i \in I)$.

We refer the reader to [5] and [6] for elementary facts about algebraic lattices; natural extensions to $\mathfrak{m}$-algebraic lattices, which first appeared in [4], we shall now restate.

An element c of a lattice M is $\mathfrak{m}$*-compact* if $c \leq \bigvee X$ implies that $c \leq \bigvee X_1$ for some small subset X_1 of X. A lattice M is $\mathfrak{m}$*-algebraic* if it is complete and every element is a complete join of $\mathfrak{m}$-compact elements.

Henceforth, L will denote the $\mathfrak{m}$-algebraic lattice in the Theorem; K will be the lattice we construct to prove the Theorem. If $|L| \leq 2$, then we can take $K = L$. We shall henceforth assume that $|L| > 2$.

Let C denote the set of *nonzero* $\mathfrak{m}$*-compact* elements of L. Since $|L| > 2$, it follows that $|C| \geq 2$. The set C inherits a partial order from L and it is closed under small nonempty joins in L.

An $\mathfrak{m}$*-complete ideal* (or $\mathfrak{m}$*-ideal*) of C is a nonempty subset I of C with the property that, for each small nonempty subset X of C,

$$\bigvee X \in I \text{ iff } X \subseteq I.$$

Equivalently, I is closed under small nonempty joins and $c \in I$, $c' \in C$, and $c' \leq c$ imply that $c' \in I$. Let $\operatorname{Id}_{\mathfrak{m}} C$ denote the set, ordered by inclusion, of all $\mathfrak{m}$-ideals of C together with the "empty ideal" $\varnothing$. It is a complete lattice, since $\operatorname{Id}_{\mathfrak{m}} C$ is closed under arbitrary intersection.

The importance of C in the proof of the Theorem stems in part from the following easy result (see [4]).

Lemma 1. *The map*

$$x \mapsto \{c \in C \mid c \leq x\}$$

is an isomorphism between L and $\operatorname{Id}_{\mathfrak{m}} C$.

5. Congruences

It is well known that it is sufficient to describe congruences for comparable elements only.

If Θ is a congruence relation on a lattice M, and $a \in M$, then $[a]\Theta$ will denote the set

$$[a]\Theta = \{x \mid x \in M \text{ and } a \equiv x \quad (\Theta)\}.$$

We shall call a *isolated* under Θ if $[a]\Theta = \{a\}$.

A congruence relation Θ on a lattice M is a *complete congruence relation* if the Infinite Substitution Property holds; that is, if $x_i \equiv y_i$ (Θ) for all $i \in I$, and $\bigvee(x_i \mid i \in I)$ and $\bigvee(y_i \mid i \in I)$ exit, then

$$\bigvee(x_i \mid i \in I) \equiv \bigvee(y_i \mid i \in I) \quad (\Theta);$$

and if $\bigwedge(x_i \mid i \in I)$ and $\bigwedge(y_i \mid i \in I)$ exit, then

$$\bigwedge(x_i \mid i \in I) \equiv \bigwedge(y_i \mid i \in I) \quad (\Theta).$$

The lattice $\mathrm{Con_c}\, M$ of all complete congruence relations of a (complete) lattice M is a complete lattice.

A congruence relation Θ on a lattice M is an $\mathfrak{m}$-*complete congruence* (or $\mathfrak{m}$-*congruence*) if the Infinite Substitution Property holds for fewer than $\mathfrak{m}$ elements. With one exception (Lemma 7), we shall only consider $\mathfrak{m}$-congruences on $\mathfrak{m}$-complete lattices.

If M is a lattice and a, $b \in M$, then $\Theta(a, b)$ (resp., $\Theta_c(a, b)$, $\Theta_{\mathfrak{m}}(a, b)$) denotes the *principal congruence relation* (resp., *principal complete congruence relation, principal* $\mathfrak{m}$-*congruence relation*) generated by a and b, that is, the smallest congruence relation (resp., complete congruence relation, $\mathfrak{m}$-congruence relation) under which a and b are congruent.

Observe that a congruence relation Θ on a lattice M is a complete congruence relation iff it is an $\mathfrak{m}$-congruence for all infinite regular cardinals $\mathfrak{m}$.

The lattice $\mathrm{Con}_{\mathfrak{m}}\, M$ of all $\mathfrak{m}$-congruence relations of an $\mathfrak{m}$-complete lattice M is an $\mathfrak{m}$-algebraic lattice (see [4]); its lattice operations are denoted by $\wedge$, $\vee_{\mathfrak{m}}$, and the non-binary variants by $\bigwedge$, $\bigvee_{\mathfrak{m}}$. Note that in $\mathrm{Con}_{\mathfrak{m}}\, M$ the operations $\wedge$ and $\bigwedge$ agree with the set intersection.

The smallest and largest congruence will be denoted by ω and ι, respectively; they are $\mathfrak{m}$-congruences. If L is a complete lattice and it has no other complete congruence, then we shall call L *complete-simple*. Similarly, if L is an $\mathfrak{m}$-complete lattice and it has no other $\mathfrak{m}$-congruence, then we shall call L $\mathfrak{m}$-*simple*.

Let M_i, $i \in I$, be $\mathfrak{m}$-complete lattices. With every $i \in I$ we associate the congruence Ψ_i on $\Pi_{\mathfrak{m}}\,(M_i \mid i \in I)$, called a *factor congruence*, defined as follows: $\mathbf{v} \equiv \mathbf{w}$ (Ψ_i) iff $\mathbf{v}(j) = \mathbf{w}(j)$ for all $j \in I$, $j \neq i$. The factor congruences are $\mathfrak{m}$-congruences and their family determines the $\mathfrak{m}$-weak product.

The following lemma is from [3]:

Lemma 2. *Let M be a complete lattice, and let Θ be a congruence relation on M. Then Θ is a complete congruence if and only if every Θ class is an interval.*

We now generalize this to $\mathfrak{m}$-congruences:

Lemma 3. *Let M be an $\mathfrak{m}$-complete lattice, and let Θ be a congruence relation on M. Then Θ is an $\mathfrak{m}$-congruence if and only if every Θ class is $\mathfrak{m}$-complete.*

Proof. It is obvious that if Θ is an $\mathfrak{m}$-congruence, then every Θ class is $\mathfrak{m}$-complete. Conversely, let every Θ class be $\mathfrak{m}$-complete. Let $x_i \equiv y_i$ (Θ) for all i in some nonempty small set I. Without loss of generality we can assume that $x_i \leq y_i$, for all $i \in I$. Let $a = \bigvee(x_i \mid i \in I)$. Then $a \equiv a \vee y_i$ (Θ), for all $i \in I$. Hence $a \equiv \bigvee(a \vee y_i \mid i \in I)$ (Θ), and so

$$\bigvee(x_i \mid i \in I) \equiv \bigvee(y_i \mid i \in I) \quad (\Theta),$$

as claimed. The dual argument completes the proof. □

Let H_0 and H_1 be lattices; let G_0 be a dual ideal of H_0, and let G_1 be an ideal of H_1. Let G_0 and G_1 be isomorphic lattices. Then we can form the lattice H, the *gluing* of H_0 and H_1 over G_0 and G_1 in which $G_0 = G_1$.

The following statements are trivial:

Lemma 4. 1. *If H_0 and H_1 are $\mathfrak{m}$-complete lattices, G_0 is an $\mathfrak{m}$-dual ideal of H_0, and G_1 is an $\mathfrak{m}$-ideal of H_1, then H is an $\mathfrak{m}$-complete lattice.*

2. *Let Φ_0 be a congruence relation on H_0, and let Φ_1 be a congruence relation on H_1. Let us assume that Φ_0 restricted to G_0 corresponds to Φ_1 restricted to G_1 under the isomorphism between H_0 and H_1. Then there is a congruence Φ on H such that Φ restricted to H_0 is Φ_0 and Φ restricted to H_1 is Φ_1.*
3. *If Φ_0 is an $\mathfrak{m}$-congruence relation on H_0 and Φ_1 is an $\mathfrak{m}$-congruence relation on H_1, then Φ is an $\mathfrak{m}$-congruence on H.*

6. First constructions

The following definition is generalized from [12] (recall that $\mathfrak{m}$ stands for a fixed uncountable regular cardinal):

Definition 1. *Let L be a lattice with 0 and 1.*

1. *L is called a* J-lattice *if the lattice is complete, distributive, and the following condition is satisfied:*

   (J) 1 *is join-irreducible and completely join- reducible.*

   If, in addition, the following condition holds:

   (M) 0 *is meet-irreducible and completely meet- reducible,*

   then we shall call L a JM-lattice.
2. *L is called a* $\mathrm{J}_{\mathfrak{m}}$-lattice *if the lattice is $\mathfrak{m}$-complete, distributive, and the following condition is satisfied:*

   ($\mathrm{J}_{\mathfrak{m}}$) 1 *is join-irreducible and $\mathfrak{m}$-completely join-reducible.*

   If, in addition, the following condition holds:

   ($\mathrm{M}_{\mathfrak{m}}$) 0 *is meet-irreducible and $\mathfrak{m}$-completely meet-reducible,*

   then we shall call L a $\mathrm{JM}_{\mathfrak{m}}$-lattice.

The first construction we need (see Figure 3, depicting the construction for a chain) was formulated in [12] (Definition 3):

Definition 2. *Let D be a $\mathrm{JM}_{\mathfrak{m}}$-lattice with 0 and 1. Then define the following subset of D^2:*

$$D^{\langle 2\rangle} = \{\langle x_0, x_1\rangle \mid x_0, x_1 \in D,\ x_0 = 0 \Rightarrow x_1 = 0, \textit{and } x_1 = 1 \Rightarrow x_0 = 1\}.$$

Equivalently,

$$D^{\langle 2\rangle} = (D^2 - ((\{0\} \times D) \cup (D \times \{1\}))) \cup \{\langle 0,0\rangle, \langle 1,1\rangle\}.$$

Note that $\langle 0,0\rangle$, $\langle 1,1\rangle \in D^{\langle 2\rangle}$; they are the zero and unit of $D^{\langle 2\rangle}$, respectively. In general, $\langle d,0\rangle$, $\langle 1,d\rangle \in D^{\langle 2\rangle}$, for any $d \in D$.

We shall utilize the following properties of this construct (generalizing Lemma 4 of [12]):

Lemma 5. *Let D be a* $\mathrm{JM}_{\mathfrak{m}}$*-lattice. Then*

1. $D^{\langle 2\rangle}$ *is a* $\mathrm{JM}_{\mathfrak{m}}$*-lattice.*
2. *Let Θ be an $\mathfrak{m}$-congruence relation of $D^{\langle 2\rangle}$ such that*
$$\langle d, 1\rangle \equiv \langle 1, 1\rangle \quad (\Theta),$$
for some $d \in D$, $d < 1$, or
$$\langle e, 0\rangle \equiv \langle 0, 0\rangle \quad (\Theta),$$
for some $e \in D$, $e > 0$. Then $\Theta = \iota$.
3. *Let $\Theta < \iota$ be an $\mathfrak{m}$-congruence relation of $D^{\langle 2\rangle}$. Then both 0 and 1 are isolated under Θ.*
4. *If D is $\mathfrak{m}$-simple, then so is $D^{\langle 2\rangle}$.*

Proof.

1. By the first clause of Condition ($\mathrm{J}_{\mathfrak{m}}$) and by the first clause of Condition ($\mathrm{M}_{\mathfrak{m}}$), $D^{\langle 2\rangle}$ is a sublattice of D^2. Hence, $D^{\langle 2\rangle}$ is a lattice. Since $D^{\langle 2\rangle}$ is a sublattice of a distributive lattice, $D^{\langle 2\rangle}$ is a distributive lattice.

   Obviously, $D^{\langle 2\rangle}$ has a zero and a unit element, namely, $\langle 0, 0\rangle$ and $\langle 1, 1\rangle$. To show that $D^{\langle 2\rangle}$ is $\mathfrak{m}$-complete, let A be a small nonempty set in $D^{\langle 2\rangle}$, and let $a = \bigvee A$ in D^2. If $a \in D^{\langle 2\rangle}$, then $a = \bigvee A$ in $D^{\langle 2\rangle}$. Otherwise, a is of the form $\langle b, 1\rangle$ for some $b \in D$, $b < 1$. Then $\bigvee A = \langle 1, 1\rangle$ in $D^{\langle 2\rangle}$. By duality, $D^{\langle 2\rangle}$ is $\mathfrak{m}$-complete.

   Conditions ($\mathrm{J}_{\mathfrak{m}}$) and ($\mathrm{M}_{\mathfrak{m}}$) obviously hold for $D^{\langle 2\rangle}$. It follows that $D^{\langle 2\rangle}$ is a $\mathrm{JM}_{\mathfrak{m}}$-lattice.
2. Let us assume that $\langle 1, d\rangle \equiv \langle 1, 1\rangle$ (Θ). Let a, $c \in D$ with $d \leq c < 1$ and $a > 0$. Compute:
$$\langle a, c\rangle = \langle a, c\rangle \wedge \langle 1, 1\rangle \equiv \langle a, c\rangle \wedge \langle 1, d\rangle = \langle a, d\rangle \quad (\Theta).$$
Using the second clause of Condition ($\mathrm{J}_{\mathfrak{m}}$), forming the $\mathfrak{m}$-join for a small set X_1 of $c < 1$ joining to 1 in D, we obtain that
$$\langle 1, 1\rangle = \bigvee(\langle a, c\rangle \mid c \in X_1) \equiv \bigvee \langle a, d\rangle = \langle a, d\rangle \quad (\Theta),$$
that is, $\langle 1, 1\rangle \equiv \langle a, d\rangle$ (Θ). Now forming the complete meet for a small set X_2 of $a > 0$ meeting to 0, using the second clause of Condition ($\mathrm{M}_{\mathfrak{m}}$), we obtain that
$$\langle 1, 1\rangle \equiv \bigwedge(\langle a, d\rangle \mid a \in X_2) = \langle 0, 0\rangle \quad (\Theta);$$
hence $\Theta = \iota$.
3. This is obviously equivalent to (2).
4. This trivially follows from (2).

□

Corollary . *Let D be a complete lattice. If D is complete-simple, then so is $D^{\langle 2\rangle}$.*

Now we introduce the second preliminary construction:

Definition 3. *Let D_i, $i \in I$, be* $\mathrm{J}_{\mathfrak{m}}$*-lattices. Then the $\Pi^*_{\mathfrak{m}}$* product *of D_i, $i \in I$, is the following lattice:*
$$\Pi^*_{\mathfrak{m}}(D_i \mid i \in I) = \Pi_{\mathfrak{m}}(D_i^- \mid i \in I) + 1;$$
that is, $\Pi^*_{\mathfrak{m}}(D_i \mid i \in I)$ is $\Pi_{\mathfrak{m}}(D_i^- \mid i \in I)$ with a new unit element.

In Figure 4, we illustrate $\Pi^*_{\mathfrak{m}}(D_i \mid i \in I)$ with $\Pi^*_{\mathfrak{m}}(D_i \mid i \in I)$ where $I = \{0, 1\}$.

Notation . If $i \in I$ and $d \in D_i^-$, then

$$\langle \dots, 0, \dots, \overset{i}{d}, \dots, 0, \dots \rangle$$

denotes the element of $\Pi^*_{\mathfrak{m}}(D_i \mid i \in I)$ whose i-th component is d and all the other components are 0.

Lemma 6. *If D_i, $i \in I$, are $\mathrm{J}_{\mathfrak{m}}$-lattices, then so is $\Pi^*_{\mathfrak{m}}(D_i \mid i \in I)$.*

Proof. To see that $\Pi^*_{\mathfrak{m}}(D_i \mid i \in I)$ is $\mathfrak{m}$-complete, let $\mathbf{X}$ be a small subset of $\Pi^*_{\mathfrak{m}}(D_i \mid i \in I)$. If the new unit element 1 is in $\mathbf{X}$, then $\bigvee \mathbf{X} = 1$. Otherwise, let $\mathbf{X}(i) = \{\mathbf{x}(i) \mid i \in I\}$. Define $v_i = \bigvee \mathbf{X}(i)$ in D_i. If for some $i \in I$, v_i is the unit element of D_i, then $\bigvee \mathbf{X} = 1$ in $\Pi^*_{\mathfrak{m}}(D_i \mid i \in I)$. Otherwise, $\bigvee \mathbf{X} = \mathbf{v}$ in $\Pi^*_{\mathfrak{m}}(D_i \mid i \in I)$. The existence of $\bigwedge \mathbf{X}$ is even more trivial.

It remains to prove Condition ($\mathrm{J}_{\mathfrak{m}}$). 1 is obviously join-irreducible. Pick any $j \in I$. By assumption, 1_j is $\mathfrak{m}$- completely join-reducible, so there is a small set X in D_j^- so that $\bigvee X = 1_j$ in D_j. Let $\mathbf{X} = \{\langle \dots, 0, \dots, \overset{j}{x}, \dots, 0, \dots \rangle \mid x \in X\}$. Then obviously $X \subseteq \Pi^*_{\mathfrak{m}}(D_i \mid i \in I)^-$ and $\bigvee \mathbf{X} = 1$ in $\Pi^*_{\mathfrak{m}}(D_i \mid i \in I)$. ☐

Definition 4. *We define a map of D_i into $\Pi^*_{\mathfrak{m}}(D_i \mid i \in I)$ as follows: if $d \in D_i$ and d is not the unit element, then d is mapped into $\langle \dots, 0, \dots, \overset{i}{d}, \dots, 0, \dots \rangle$; the unit element of D_i is mapped into the unit element of $\Pi^*_{\mathfrak{m}}(D_i \mid i \in I)$. This mapping is an isomorphism.*

The image of D_i under this isomorphism will be called the *canonical image* of D_i in $\Pi^*_{\mathfrak{m}}(D_i \mid i \in I)$. Note that the canonical image of D_i with the unit element removed is an ideal of $\Pi^*_{\mathfrak{m}}(D_i \mid i \in I)$.

Lemma 7. *Let D_i, $i \in I$, be $\mathrm{J}_{\mathfrak{m}}$-lattices, $|I| > 1$. Let Θ be an $\mathfrak{m}$-congruence relation on $D = \Pi^*_{\mathfrak{m}}(D_i \mid i \in I)$. Then*

1. *If there exists an $i \in I$ and an element $d \in D_i^-$ such that the congruence*

$$\langle \dots, 0, \dots, \overset{i}{d}, \dots, 0, \dots \rangle \equiv \langle \dots, 0, \dots, \overset{i}{c}, \dots, 0, \dots \rangle \quad (\Theta)$$

   holds for all $d \leq c \in D_i^-$, then $\Theta = \iota$.
2. *If $\Theta < \iota$, then there are $\mathfrak{m}$-congruences Θ_i on D_i^-, for all $i \in I$, such that the Θ-classes on D are the unit element of D as a singleton and the $\Pi_{\mathfrak{m}}(\Theta_i \mid i \in I)$ classes on D.*

Proof. Pick $i \in I$ and a $d \in D_i^-$ as in the lemma. $|I| > 1$, so we can pick a $j \in I$, $j \neq i$. The congruence

$$\langle \dots, 0, \dots, \overset{i}{d}, \dots, 0, \dots \rangle \equiv \langle \dots, 0, \dots, \overset{i}{c}, \dots, 0, \dots \rangle \quad (\Theta),$$

holds for all $d \leq c \in D_i^-$. By Condition ($\mathrm{J}_{\mathfrak{m}}$) the unit element of D_i is $\mathfrak{m}$-complete reducible, hence we can choose a small set $X \subseteq D_i^-$ of elements $c \geq d$ whose join in D_i is the unit element in D_i. Form the $\mathfrak{m}$-join of both sides for all $c \in X$. We obtain that

$$\langle \dots, 0, \dots, \overset{i}{d}, \dots, 0, \dots \rangle \equiv 1 \quad (\Theta),$$

where 1 is the unit element of $\Pi^*(D_i \mid i \in I)$. For any $x \in D_j^-$, meet both sides with $\langle \dots, 0, \dots, \overset{j}{x}, \dots, 0, \dots \rangle$ to get

$$0 \equiv \langle \dots, 0, \dots, \overset{j}{x}, \dots, 0, \dots \rangle \quad (\Theta).$$

Again, by Condition ($\mathrm{J}_{\mathfrak{m}}$) we find in D_j^- a small set with join 1, and form the $\mathfrak{m}$-join for these $x \in D_j^-$ to obtain

$$0 \equiv 1 \quad (\Theta),$$

that is, $\Theta = \iota$. This proves clause (1). Clause (2) trivially follows. □

Observe that in this lemma the lattices D_i^- are not necessarily $\mathfrak{m}$-complete, so $\mathfrak{m}$-congruences have to be understood as defined in Section 5.

Finally, we state the main result of [11] (see also [12]):

Theorem 1. *There exists a self-dual complete-simple* JM*-lattice S of cardinality $\mathfrak{c}$.*

Observe that in the construction of S the following property obviously holds: If X is any subset of S, then there exists a countable subset $X' \subseteq X$ such that $\bigvee X = \bigvee X'$, and dually. Therefore, from Theorem 1 we immediately conclude

Theorem 2. *Let $\mathfrak{m}$ be a regular cardinal $> \aleph_0$. Then there exists a self-dual $\mathfrak{m}$-simple* $\mathrm{JM}_{\mathfrak{m}}$*-lattice S of cardinality $\mathfrak{c}$.*

In this paper, we fix an S as in this theorem, with zero: 0_S, and unit: 1_S.

7. The tower construction

In this section, let A be a small set with at least two elements. We build a distributive lattice $T(A)$, the "tower over A," as a sublattice of $(S^A)^\omega$, where S is given as in Theorem 2.

Let $D = S^A$ with zero $\mathbf{o}_D$ and unit $\mathbf{i}_D$. We regard the elements of D as vectors $\mathbf{v}$; for $a \in A$, we have $\mathbf{v}(a) \in S$. Since A is small, D is the complete product as well as the $\mathfrak{m}$-weak product of A copies of S.

For $a \in A$, let $\mathbf{a} \in D$ be defined as follows: $\mathbf{a}(a) = 1_S$; and $\mathbf{a}(b) = 0_S$, for all $b \in A$, $b \neq a$. Let $\mathbf{a}'$ be the complement of $\mathbf{a}$ in D, that is, $\mathbf{a}'(a) = 0_S$ and $\mathbf{a}'(b) = 1_S$, for all $b \in A$, $b \neq a$. In D, denote the interval $[\mathbf{o}_D, \mathbf{a}]$ by S_a, and $[\mathbf{a}', \mathbf{i}_D]$ by S'_a. Observe that $S \cong S_a \cong S'_a$. Note also that if a, $b \in A$ and $a \neq b$, then $\mathbf{a} \wedge \mathbf{b} = \mathbf{o}_D$.

The elements of $D^\omega = (S^A)^\omega$ are vectors of type ω. For every natural number n, we define the element $\mathbf{o}_n$ of D^ω as follows:

$$\mathbf{o}_n = \langle \underbrace{\mathbf{i}_D, \dots, \mathbf{i}_D}_{n\text{- times}}, \mathbf{o}_D, \dots \rangle;$$

then

$$\mathbf{o}_0 = \langle \mathbf{o}_D, \mathbf{o}_D, \mathbf{o}_D, \dots \rangle < \mathbf{o}_1 = \langle \mathbf{i}_D, \mathbf{o}_D, \mathbf{o}_D, \dots \rangle < \mathbf{o}_2 = \langle \mathbf{i}_D, \mathbf{i}_D, \mathbf{o}_D, \dots \rangle < \dots$$

Let $D_n = [\mathbf{o}_n, \mathbf{o}_{n+1}] \subseteq D^\omega$, for $n < \omega$.

Obviously, there is a natural isomorphism between D and D_n. Let $\mathbf{o}_n$ and $\mathbf{i}_n$ be the image of $\mathbf{o}$ and $\mathbf{i}$, respectively, under this isomorphism; of course, $\mathbf{i}_n = \mathbf{o}_{n+1}$. Let $\mathbf{a}_n$ and $\mathbf{a}'_n$ be the image of $\mathbf{a}$ and $\mathbf{a}'$, respectively, under this isomorphism. Finally, let $S_{a,n} = [\mathbf{o}_n, \mathbf{a}_n]$ and $S'_{a,n} = [\mathbf{a}'_n, \mathbf{i}_n]$. Of course, $S_{a,n} \cong S'_{a,n} \cong S$.

In D^ω, let $\mathbf{r}_n^a$ be the relative complement of $\mathbf{o}_{n+1}$ in $[\mathbf{a}'_n, \mathbf{a}_{n+1}]$; since $\mathbf{a}'_n < \mathbf{o}_{n+1} < \mathbf{a}_{n+1}$, and the i-th component of the three vectors form a one- or two-element chain in D, it follows that $\mathbf{r}_n^a$ exists. The intervals $[\mathbf{a}'_n, \mathbf{r}_n^a]$ and $[\mathbf{a}'_n, \mathbf{o}_{n+1}]$ of D^ω are isomorphic to S, hence in D^ω the interval $[\mathbf{a}'_n, \mathbf{a}_{n+1}]$ is isomorphic to S^2. It follows that $[\mathbf{a}'_n, \mathbf{a}_{n+1}]$ contains an isomorphic copy of $S^{\langle 2\rangle}$, denoted by $S_{a,n}^{\langle 2\rangle}$, such that $[\mathbf{a}'_n, \mathbf{o}_{n+1}] \subseteq S_{a,n}^{\langle 2\rangle}$ and $[\mathbf{o}_{n+1}, \mathbf{a}_{n+1}] \subseteq S_{a,n}^{\langle 2\rangle}$.

For $x \in S_{a,n}^{\langle 2\rangle}$, there is a largest element $\underline{x}$ of D_n satisfying $\underline{x} \leq x$. Indeed, $\underline{x} = x \wedge \mathbf{i}_n$. Dually, there is a smallest element $\overline{x}$ of D_{n+1} satisfying $x \leq \overline{x}$.

Now we are ready to define the tower $T(A)$:

Definition 5. *Let $D = S^\omega$, and let D_n and $S_{a,n}^{\langle 2\rangle}$ be defined as above. Let A be a small set with at least two elements. We define the* tower $T(A)$ *as a subset of D^ω:*

$$T(A) = \bigcup(D_n \mid n < \omega) \cup \bigcup(S_{a,n}^{\langle 2\rangle} \mid n < \omega \text{ and } a \in A).$$

D and $T(A)$ is illustrated in Figure 5 with $A = \{a, b, c\}$.

The zero of $T(A)$ will be denoted by $0_{T(A)}$. Let $\widehat{T}(A)$ be $T(A)$ with a unit adjoined.

Lemma 8. *$T(A)$ is a sublattice of D^ω. Furthermore, $\widehat{T}(A)$ is a complete lattice. It is a* J*-lattice. In fact, it is a* $\mathrm{J}_{\mathfrak{m}}$*-lattice.*

Proof. Let x, $y \in T(A)$, let x and y be incomparable. We prove that $x \vee y$ formed in D^ω is also the join in $T(A)$. For x and y (by possibly interchanging the two elements) one of the following cases hold for some $n < \omega$:

1. x, $y \in D_n$;
2. x, $y \in S_{a,n}^{\langle 2\rangle}$;
3. $x \in S_{a,n}^{\langle 2\rangle}$ and $y \in D_n$;
4. $x \in S_{a,n}^{\langle 2\rangle}$ and $y \in S_{b,n}^{\langle 2\rangle}$, a, $b \in A$, $a \neq b$;
5. $x \in S_{a,n}^{\langle 2\rangle}$ and $y \in S_{b,n+1}^{\langle 2\rangle}$, a, $b \in A$, $a \neq b$;
6. $x \in S_{a,n}^{\langle 2\rangle}$ and $y \in D_{n+1}$.

The existence of $x \vee y$ for Cases (1) and (2) is trivial; for Cases (3) and (5) it follows from Lemma 4. Finally, for Case (4), $x \vee y = x \vee \underline{x} \vee y \vee \underline{y}$; since $\underline{x} \vee \underline{y} \geq \mathbf{a}'_n \vee \mathbf{b}'_n = \mathbf{o}_{n+1}$, it follows that $x \vee y = x \vee \mathbf{o}_{n+1} \vee y \vee \mathbf{o}_{n+1} = \overline{x} \vee \overline{y}$, reducing this case to Case (1); Case (5) is handled similarly. Along with the dual argument, this proves that $T(A)$ is a sublattice of D^ω.

It follows from Lemma 4 that $T(A)$ is closed under complete meets and for complete joins of bounded subsets. If a subset in $\widehat{T}(A)$ is not bounded from above, its complete join will be the newly adjoined unit. Hence $\widehat{T}(A)$ is a complete lattice. It is obviously a J-lattice and also a $\mathrm{J}_{\mathfrak{m}}$-lattice. ☐

Let $a \in A$. We define an equivalence relation Θ_A^a on $\widehat{T}(A)$. The nontrivial classes of Θ_A^a are of the following three types:

1. the congruence class of $\Theta(\mathbf{o}_0, \mathbf{a}_0)$ on D_0 containing $\mathbf{o}_0$;
2. the congruence classes of $\Theta(\mathbf{o}_n, \mathbf{a}_n)$ on D_n not containing $\mathbf{o}_n$ or $\mathbf{i}_n$;
3. the intervals $[\mathbf{a}'_n, \mathbf{a}_{n+1}]$, for all $n < \omega$.

Lemma 9. *Θ_A^a is a complete congruence on $\widehat{T}(A)$, for all $a \in A$. In fact, $\Theta_A^a = \Theta_c(\mathbf{o}_0, \mathbf{a}_0) = \Theta_{\mathfrak{m}}(\mathbf{o}_0, \mathbf{a}_0)$, and the interval $[\mathbf{o}_0, \mathbf{a}_0]$ of $\widehat{T}(A)$ is isomorphic to S.*

Proof. It is routine to compute that Θ_A^a is a congruence. Since every congruence class of Θ_A^a is an interval, it follows that Θ_A^a is a complete congruence by Lemma 2; hence it is also an $\mathfrak{m}$-congruence. □

Since S is ($\mathfrak{m}$-)complete-simple and A is small, it follows that the ($\mathfrak{m}$-)complete congruences of $D = S^A$ are the factor congruences. Let Θ be an ($\mathfrak{m}$-)complete congruence of $\widehat{T}(A)$. Then Θ on each D_n is a factor congruence, that is, a complete join of Θ_A^a-s. However, on each level, it is the complete join of the same Θ_A^a-s since $S_{a,n}^{\langle 2\rangle}$ is ($\mathfrak{m}$-)complete-simple by Lemma 5.3. It follows that on $\widehat{T}(A)$, Θ is a complete join of Θ_A^a-s. Hence the lattice of ($\mathfrak{m}$-)complete congruence relations of $\widehat{T}(A)$ is isomorphic to the lattice of all subsets of A.

Lemma 10. *There is a one-to-one correspondence between the subsets of A and the ($\mathfrak{m}$-)complete congruences of $\widehat{T}(A)$, given by*

$$B \to \bigvee(\Theta_A^a \mid a \in B),$$

where $B \subseteq A$.

8. Two special towers

Let L be the $\mathfrak{m}$-algebraic lattice we want to represent. We can assume that $|L| > 2$. C will denote the set of nonzero $\mathfrak{m}$-compact elements of L.

For every small $A \subseteq C$, $|A| > 1$, we build a tower $T(A)$. The ($\mathfrak{m}$-)complete congruences of $\widehat{T}(A)$ are described in Lemma 10.

For every $a \in C$, we "double" a to obtain the two- element set $\{\dot{a}, \ddot{a}\}$, and carry out the tower construction for this set; the resulting lattice will be denoted by $\widehat{T}(a)$. The zero of $\widehat{T}(a)$ will be denoted by $0_{T(a)}$. Note that in this case $D = S^2$, and the lattice of ($\mathfrak{m}$-)complete congruence relations of $\widehat{T}(A)$ is isomorphic to $(\mathfrak{C}_2)^2$:

Lemma 11. *There is a one-to-one correspondence between the subsets of $\{\dot{a}, \ddot{a}\}$ and the ($\mathfrak{m}$-)complete congruences of $\widehat{T}(a)$.*

9. Merging the towers

The next couple of constructions bring all the towers of Section 8 together into one $\mathrm{J}_{\mathfrak{m}}$-lattice.

Let $A \subseteq C$, $|A| > 1$ be a small set; first we merge the towers $\widehat{T}(A)$ and $\widehat{T}(\bigvee A)$ by forming their $\Pi_{\mathfrak{m}}^*$ product; let $M(A)$ denote this lattice with zero: $0_{M(A)}$. Observe that $\widehat{T}(A)$ and $\widehat{T}(\bigvee A)$ are $\mathrm{J}_{\mathfrak{m}}$- lattices, hence by Lemma 6, $M(A)$ can be formed and it is a $\mathrm{J}_{\mathfrak{m}}$-lattice. See Figure 6.

Lemma 12. *The $\mathfrak{m}$-congruence lattice $\operatorname{Con}_{\mathfrak{m}} M(A)$ of $M(A)$ is* atomic *(that is, every element is a complete join of atoms). The atoms of $\operatorname{Con}_{\mathfrak{m}} M(A)$ are of the form*

$$\Theta_{\mathfrak{m}}(0_{M(A)}, \langle \mathbf{a}_0, 0_{T(\bigvee A)}\rangle),$$

where $a \in A$, or of the form

$$\Theta_{\mathfrak{m}}(0_{M(A)}, \langle 0_{T(A)}, \mathbf{b}_0\rangle),$$

where $b = \dot{u}$ or $b = \ddot{u}$ and $u = \bigvee A$.

Proof. This follows from Lemmas 10 and 11 in combination with the definition and properties of $\Pi_{\mathfrak{m}}^*$ products, see Definition 3 and Lemma 7. □

We merge all the towers, by forming the lattice:

$$M(L) = \Pi_{\mathfrak{m}} (M(X) \mid X \subseteq C \text{ and } 1 < |X| < \mathfrak{m}),$$

with zero 0_M, see Figure 7; observe that, in general, $M(L)$ does not have a unit element.

The following lemma trivially follows from Lemma 12:

Lemma 13. *The $\mathfrak{m}$-congruence lattice $\operatorname{Con}_{\mathfrak{m}} M(L)$ is atomic. The atoms θ of $\operatorname{Con}_{\mathfrak{m}} M(L)$ are in one-to-one correspondence with principal ideals $(\mathbf{x}_\theta]$ isomorphic to S, where $\mathbf{x}_\theta$ is obtained as follows: it has a zero component in all but one factor $M(A_\theta)$ of $M(L)$; its $M(A_\theta)$ component is*

$\langle (\mathbf{a}_\theta)_0, 0_{T(u)} \rangle$ where $a_\theta \in A_\theta \subseteq C$ and $u = \bigvee A_\theta$,

or

$\langle 0_{T(A_\theta)}, \mathbf{b}_0 \rangle$, where $u = \bigvee A_\theta \in C$ and $b = \dot{u}$ or $b = \ddot{u}$.

This motivates the following definition:

Definition 6. *Let I denote the family of all principal ideals of $M(L)$ described in Lemma 13. If $I \in I$, then I is associated with an atom θ of $\operatorname{Con}_{\mathfrak{m}} M(L)$ and an element $a = a_\theta \in C$. We shall call a the* color *of I, and write $\operatorname{col} I = a$.*

Observe that if I, $J \in I$ and $I \neq J$, then $I \wedge J = \{0_M\}$.

10. The construction of K

For every $a \in C$, we define $I_a = \{I \mid I \in I \text{ and } \operatorname{col} I = a\}$, where I was introduced in Definition 6. For every $I \in I$, we take a copy S_I of S. For every $a \in C$, we construct the lattice $B_a = \Pi^*_{\mathfrak{m}}(S_I \mid I \in I_a)$; we can do this since S is a $\mathrm{J}_{\mathfrak{m}}$-lattice by Theorem 2. By Lemma 6, B_a is again a $\mathrm{J}_{\mathfrak{m}}$-lattice. Next we form the $\mathfrak{m}$-weak product $B = \Pi_{\mathfrak{m}} (B_a \mid a \in C)$. The dual of this lattice we shall denote by F; let $x \to x^*$ denote this dual isomorphism between B and F. Under this dual isomorphism, the image of B_a will be denoted by F_a, with zero 0_{F_a}.

Lemma 14. *Every $I \in I$ naturally corresponds to a sublattice I^* of F such that I^* with its zero removed is a dual ideal of F and I is dually isomorphic to I^*.*

Proof. Indeed, form the canonical image I' (see Definition 4) of I in B_a. Then I' is a sublattice of B_a and $(I')^-$ is an ideal of B_a. Since B_a is an $\mathfrak{m}$-direct factor of B, it follows that I' can be regarded as a sublattice of B and $(I')^-$ is an ideal of B. Let I^* be the image of I' under the duality that maps B to F. It is now obvious that I^* has the properties stated in the lemma. □

The zero of I^* is 0_{F_a} where $a = \operatorname{col} I$.

Now we are ready to construct the lattice K of the Theorem. We form the lattice $F \times M(L)$. We identify F with the sublattice $\{\langle f, 0_M \rangle \mid f \in F\}$ and $M(L)$ with the sublattice $\{\langle 1_F, m \rangle \mid m \in M(L)\}$, where 1_F is the unit element of F. Observe that after the identification, $1_F = 0_M$.

Let $I \in I$; we define a subset A_I of $F \times M(L)$.

Definition 7. *A_I consists of the following elements:*

1. *the elements in I;*
2. *the elements in I^*;*
3. *the relative complements of 1_F in intervals of the form $[r^*, s]$, for r, $s \in I$.*

Lemma 15. *A_I is a sublattice of $F \times M(L)$, and A_I is isomorphic to S^2.*

Proof. Observe that if r, $s \in I$, then r^* is of the form $\langle u, 0_M \rangle$ and s is of the form $\langle 1_F, v \rangle$. Hence $\langle u, v \rangle \in F \times M(L)$ is the relative complement of 1_F in $[r^*, s]$. It follows that A_I is isomorphic to $I^* \times I$. Since S is self-dual (see Theorem 2), A_I is isomorphic to S^2, as claimed. □

Let $I \in I$. Since A_I is isomorphic to S^2, it has a unique sublattice isomorphic to $S^{\langle 2 \rangle}$ containing I and I^*. We name this sublattice $S_I^{\langle 2 \rangle}$.

Definition 8. *The subset K of $F \times M(L)$ is defined as the union:*

$$K = F \cup M(L) \cup \bigcup (S_I^{\langle 2 \rangle} \mid I \in I).$$

Lemma 16. *K is a sublattice of $F \times M(L)$, hence K is a distributive lattice. Furthermore, K is an $\mathfrak{m}$-complete lattice.*

Proof. The proof is similar to that of Lemma 8. □

By this lemma, we have the $\mathfrak{m}$-complete distributive lattice K of the Theorem.

11. The proof of the Theorem

For $a \in C$, we define an equivalence relation Θ^a on K in several steps.

Firstly, let A be a small subset of C with $|A| > 1$, and we define $\Theta^a_{M(A)}$ on $M(A)$. If $A \subseteq (a]$ (or equivalently, if $\bigvee A \leq a$), then we define $\Theta^a_{M(A)}$ on $M(A)$ as $\iota_{M(A)}$. Otherwise, using Lemma 10, we take the $\mathfrak{m}$-congruence Φ of $\widehat{T}(A)$ that corresponds to the set $A \cap (a]$, and let $\Theta^a_{M(A)}$ be the $\mathfrak{m}$-congruence on $M(A)$ that is Φ on $\widehat{T}(A)$ and $\omega_{\widehat{T}(\bigvee A)}$ on $\widehat{T}(\bigvee A)$; by Lemma 7, this describes the $\mathfrak{m}$-congruence $\Theta^a_{M(A)}$.

Secondly, we form the $\mathfrak{m}$-weak direct product of these $\Theta^a_{M(A)}$, and obtain $\Theta^a_{M(L)}$ on $M(L)$.

Thirdly, observe that, for every $I \in I$, the ideal I is collapsed by $\Theta^a_{M(L)}$ iff $\operatorname{col} I \leq a$; otherwise, $\Theta^a_{M(L)}$ is discrete on I. Since F is the dual of the $\mathfrak{m}$-weak direct product $B = \Pi_{\mathfrak{m}} (B_b \mid b \in C)$, we can define Θ^a_F on F as the unique $\mathfrak{m}$-congruence with the property that the dual of Θ^a_F restricted to B_b is ω_{B_b} iff $b \nleq a$, otherwise, it is ι_{B_b}.

Fourthly, on $S_I^{\langle 2 \rangle}$, observe that either I is discrete under $\Theta^a_{M(L)}$ and I^* is discrete under Θ^a_F, in which case, we define $\Theta^a_{S_I^{\langle 2 \rangle}}$ on $S_I^{\langle 2 \rangle}$ as $\omega_{S_I^{\langle 2 \rangle}}$; or I is collapsed under $\Theta^a_{M(L)}$ and I^* is collapsed under Θ^a_F, in which case we define $\Theta^a_{S_I^{\langle 2 \rangle}}$ on $S_I^{\langle 2 \rangle}$ as $\iota_{S_I^{\langle 2 \rangle}}$.

Observe that for u, $v \in K$, if u, $v \in M(L)$ and u, $v \in S_I^{\langle 2 \rangle}$, $I \in I$, then

$$u \equiv v \quad (\Theta^a_{M(L)})$$

iff

$$u \equiv v \quad (\Theta^a_{S_I^{\langle 2 \rangle}}).$$

Similarly, for u, $v \in F$ and u, $v \in S_I^{\langle 2 \rangle}$.

Definition 9. *We define Θ^a on K as the transitive closure of the relation*

$$\Theta^a_{M(L)} \cup \Theta^a_F \cup \bigcup (\Theta^a_{S_I^{\langle 2 \rangle}} \mid I \in I).$$

Lemma 17. *For $a \in C$, Θ^a is an $\mathfrak{m}$-congruence of K.*

Θ^a can be described as follows. Let $u, v \in K$ and $u \leq v$. Then $u \equiv v \pmod{\Theta^a}$ iff:

1. *$u, v \in F$ and $u \equiv v \ (\Theta^a_F)$.*
2. *$u, v \in M(L)$ and $u \equiv v \ (\Theta^a_{M(L)})$.*
3. *$u, v \in S_I^{\langle 2\rangle}$, for some $I \in I$, and $u \equiv v \ (\Theta^a_{S_I^{\langle 2\rangle}})$.*
4. *$u \in F$, $v \in M(L)$, $u \equiv 1_F \ (\Theta^a_F)$ and $1_F = 0_{M(L)} \equiv v \ (\Theta^a_{M(L)})$.*
5. *$u \in F$, $v \in S_I^{\langle 2\rangle}$, for some $I \in I$, and*
$$u \equiv v \wedge 1_F \quad (\Theta^a_F), \quad v \wedge 1_F \equiv v \quad (\Theta^a_{S_I^{\langle 2\rangle}}).$$
6. *$u \in S_I^{\langle 2\rangle}$, for some $I \in I$, $v \in M(L)$, and*
$$u \equiv u \vee 0_{M(L)} \quad (\Theta^a_{S_I^{\langle 2\rangle}}), \quad u \vee 0_{M(L)} \equiv v \quad (\Theta^a_{M(L)}).$$

Proof. By Lemma I.3.8 of [5], it is enough to prove the transitivity of Θ^a for comparable elements. So assume that $u \equiv v \ (\Theta^a)$, $v \equiv w \ (\Theta^a)$, $u \leq v \leq w$; we wish to prove that $u \equiv w \ (\Theta^a)$. There are 10 cases to distinguish according to where u, v, and w lie: in $M(L)$, in an $S_I^{\langle 2\rangle}$, for some $I \in I$, in F. We discuss in detail the most complicated case: $u \in F$, $v \in S_I^{\langle 2\rangle}$, for some $I \in I$, and $w \in M(L)$.

If $v = 1_F$, then $u \equiv w \ (\Theta^a)$ by (4). So let $v \neq 1_F$. By (5), $u \equiv v \ (\Theta^a)$ implies that
$$u \equiv v \wedge 1_F \quad (\Theta^a_F), \quad v \wedge 1_F \equiv v \quad (\Theta^a_{S_I^{\langle 2\rangle}}).$$
By (6), $v \equiv w \ (\Theta^a)$ implies that
$$v \equiv v \vee 0_{M(A)} \quad (\Theta^a_{S_I^{\langle 2\rangle}}), \quad v \vee 0_{M(A)} \equiv w \quad (\Theta^a_{M(L)}).$$
Since $v \neq 1_F$, it follows that $v \wedge 1_F \neq v$ or $v \neq v \vee 0_{M(A)}$. Therefore, two distinct elements of $S_I^{\langle 2\rangle}$ are congruent; since S is $\mathfrak{m}$-simple , so is $S_I^{\langle 2\rangle}$ by Lemma 5.3. We conclude that $\Theta^a_{S_I^{\langle 2\rangle}} = \iota_{S_I^{\langle 2\rangle}}$. Hence $v \wedge \mathbf{1}_F \equiv 1_F \ (\Theta^a_{S_I^{\langle 2\rangle}})$; therefore,
$$v \wedge \mathbf{1}_F \equiv 1_F \quad (\Theta^a_F).$$
It follows that $u \equiv 1_F \ (\Theta^a_F)$. Similarly, $1_F = 0_{M(A)} \equiv w \ (\Theta^a_{M(L)})$. By (1),
$$u \equiv w \quad (\Theta^a),$$
as required.

All the other cases are either similar or trivial, and the details are left to the reader.

Now again by Lemma I.3.8 of [5], to prove the Substitution Property for Θ^a, it is sufficient to take $u, v, t \in K$, $u \equiv v \ (\Theta^a)$, and prove $u \vee t \equiv v \vee t \ (\Theta^a)$ only under the assumptions that $u \leq v$ and $u \leq t$, and dually. It is routine to check that all these cases follow from Lemma 4.

It follows from (1)–(6), that Θ^a restricted to $M(L)$ is $\Theta^a_{M(L)}$, Θ^a restricted to F is Θ^a_F, and for $I \in I$, Θ^a restricted to $S_I^{\langle 2\rangle}$ is $\Theta^a_{S_I^{\langle 2\rangle}}$. Therefore, every Θ^a class is $\mathfrak{m}$-complete, and Θ^a is $\mathfrak{m}$-complete by Lemma 3. ☐

Lemma 18. *Let $a \in C$. Take the intervals $I = [0_M, \mathbf{x}] \in I_a$ and $J = [0_M, \mathbf{y}] \in I_a$ of K. Then, in K,*
$$\Theta_{\mathfrak{m}}(0_M, \mathbf{x}) = \Theta_{\mathfrak{m}}(0_M, \mathbf{y}).$$

Proof. Since I, $J \in I_a$, both I and J are factors of the $\Pi^*_{\mathfrak{m}}$ product B_a. By Lemma 5, B_a is $\mathfrak{m}$-simple , hence in B_a,

$$\Theta_{\mathfrak{m}}(0_M, \mathbf{x}) = \Theta_{\mathfrak{m}}(0_M, \mathbf{y}) = \iota_{B_a}.$$

Therefore, in F,

$$\Theta_{\mathfrak{m}}(\mathbf{x}^*, 1_F) = \Theta_{\mathfrak{m}}(\mathbf{y}^*, 1_F).$$

Moreover, in $S_I^{\langle 2\rangle}$, which is $\mathfrak{m}$-simple by Lemma 5,

$$\Theta_{\mathfrak{m}}(0_M, \mathbf{x}) = \Theta_{\mathfrak{m}}(\mathbf{x}^*, 1_F)$$

and in $S_J^{\langle 2\rangle}$,

$$\Theta_{\mathfrak{m}}(0_M, \mathbf{y}) = \Theta_{\mathfrak{m}}(\mathbf{y}^*, 1_F).$$

Hence by transitivity, we obtain that the statement of the lemma holds in K. □

Lemma 19. *For $a \in C$, choose a small set $A \subseteq C$ such that $a \in A$, $|A| > 1$. Let* $\mathbf{a}$ *denote the element of K that corresponds to the element $\mathbf{a}_0 \in T(A)$. Then $\Theta^a = \Theta_{\mathfrak{m}}(0_M, \mathbf{a})$, and $[0_M, \mathbf{a}] \in I_a$.*

Proof. This is obvious from Lemmas 9, Lemma 10, and 18. □

Obviously, the map $a \to \Theta^a$ is one-to-one and isotone. We shall need the following statement (recall that $\bigvee_{\mathfrak{m}}$ is the join in $\mathrm{Con}_{\mathfrak{m}}$):

Lemma 20. *For every small nonempty X in C,*

$$\bigvee\nolimits_{\mathfrak{m}} (\, \Theta^x \mid x \in X) = \Theta^{\bigvee X}$$

holds in $\mathrm{Con}_{\mathfrak{m}} K$*.*

Proof. Since $x \to \Theta^x$ is isotone, it follows that

$$\bigvee\nolimits_{\mathfrak{m}} (\, \Theta^x \mid x \in X) \leq \Theta^{\bigvee X}.$$

To prove the reverse inequality, we can assume that $|X| > 1$, and consider the lattice $M(X)$. Let $y = \bigvee X$ in C. By definition, in $M(X)$ all $\mathbf{x}_0$ $(x \in X)$ are collapsed with 0_M by $\bigvee_{\mathfrak{m}} (\Theta^x \mid x \in X)$. It follows from Lemmas 10–12 that in $M(X)$:

$$\bigvee\nolimits_{\mathfrak{m}} (\, \Theta^x \mid x \in X) = \iota_{M(X)}.$$

Therefore, $0_M \equiv \mathbf{b}_0$ $(\bigvee_{\mathfrak{m}} (\, \Theta^x \mid x \in X))$, where $b = \dot{y}$ and $\mathbf{b}_0$ is formed in $\widehat{T}(\bigvee X)$. Now take any small $Y \subseteq C$, such that $y \in Y$, $|Y| > 1$. By Lemma 18, $\Theta_{\mathfrak{m}}(0_M, \mathbf{b}) = \Theta_{\mathfrak{m}}(0_M, \mathbf{y})$, and $\Theta_{\mathfrak{m}}(0_M, \mathbf{y}) = \Theta^{\bigvee X}$ by Lemma 19, completing the proof of the lemma. □

Hence to prove the Theorem, by Lemma 1 it is sufficient to verify the following

Lemma 21. *Every $\mathfrak{m}$-compact $\mathfrak{m}$-congruence $\Theta \neq \omega$ of K is of the form $\Theta = \Theta^a$, for some $a \in C$.*

Proof. S is $\mathfrak{m}$-simple by Lemma 5. Therefore $S_I^{\langle 2\rangle}$ is $\mathfrak{m}$-simple , for all $I \in I$. It follows that Θ is completely determined by its restriction to $M(L)$, which by Lemma 13 satifies the conclusion of this lemma. Therefore, so does Θ. □

12. Concluding Remarks

While preparing their lecture for the Conference on mathematical foundations of programming semantics (Carnegie Melon University, 1991), see [15], A. Jung, L. Libkin, and H. Puhlmann raised the question whether the congruence lattice of a Scott-domain is an algebraic lattice. (By definition, a congruence of a Scott-domain is a complete congruence.) In response, in [14], we proved the following:

Theorem 3. *Every* complete lattice L *can be represented as the lattice of congruence relations of a* Scott-domain S*. In fact,* S *can be constructed as a* modular algebraic lattice.

The question naturally arises whether this result can be strengthened by requiring that S be a distributive algebraic lattice. The answer to this is in the negative:

Theorem 4. *Let* L *be a* complete lattice *with more than two elements and with a meet-irreducible zero. Then* L *cannot be represented as the lattice of complete congruence relations of a* distributive algebraic lattice K*.*

Proof. Let us assume that L can be represented as the lattice of complete congruence relations of a distributive algebraic lattice K. Then K cannot be finite, because the complete congruence lattice of a finite distributive lattice is a Boolean lattice, contradicting that the zero of L is meet-irreducible. Now let K be infinite; then $K = \mathrm{Id}_{\mathfrak{m}}\, C$, where C is the join-semilattice of nonzero compact elements of K. Let a, b, $c \in C$, $a < b < c$. Let A be a maximal proper ideal of C in $(b]$, and let B be a maximal proper ideal of C in $(c]$. Then $[A, (b]]$ and $[B, (c]]$ correspond to prime intervals in K, one on top of the other. By Lemma 2, the congruence relation generated by any prime interval is a complete congruence relation. Hence the congruence relations generated by these two prime intervals of K are complete congruence relations, and they are disjoint. Therefore the complete congruence lattice of K has a meet-reducible zero, a contradiction. □

This result suggest the following problem:

Problem 2. *Under what conditions can an* ($\mathfrak{m}$-)*algebraic lattice be represented as the* ($\mathfrak{m}$-)*complete congruence lattice of a distributive* ($\mathfrak{m}$-)*algebraic lattice?*

Since the variety of distributive lattices is the smallest nontrivial variety, the Theorem of this paper is the best possible from this point of view. However, since we deal with complete distributive lattices, we can further restrict this class using infinitary identities.

The two best known infinitary identities are the Join Infinite Distributive Identity:

$$a \wedge \bigvee X = \bigvee (a \wedge x \mid x \in X), \tag{JID}$$

and its dual, the Meet Infinite Distributive Identity:

$$a \vee \bigwedge X = \bigwedge (a \vee x \mid x \in X). \tag{MID}$$

We shall denote by ($\mathrm{JID}_{\mathfrak{m}}$) the condition that (JID) holds for small sets X, where $\mathfrak{m}$ is a regular cardinal, $\mathfrak{m} > \aleph_0$. We define ($\mathrm{MID}_{\mathfrak{m}}$) dually.

It is easy to see that the lattice K we construct for the Theorem fails both ($\mathrm{JID}_{\mathfrak{m}}$) and ($\mathrm{MID}_{\mathfrak{m}}$).

So we can raise the following:

Problem 3. *Characterize the complete congruence lattices of complete distributive lattices satisfying* (JID) *and/or* (MID).

Or more generally:

Problem 4. *Characterize the* $\mathfrak{m}$*-congruence lattices of* $\mathfrak{m}$*-complete distributive lattices satisfying* ($\mathrm{JID}_{\mathfrak{m}}$) *and/or* ($\mathrm{MID}_{\mathfrak{m}}$).

Finally, we mention the problem of prime intervals. In Section 2, we outlined how previous proofs were based on the existence of many prime intervals, specifically, on the weak atomicity of K. The best result on weakly atomic lattices is the result of R. Freese, G. Grätzer, and E. T. Schmidt [3] and G. Grätzer and E. T. Schmidt [13] in which weakly atomic complete modular lattices are constructed.

Problem 5. *Find "small" varieties* $\mathbf{M}$ *of modular lattices such that for every regular cardinal* $\mathfrak{m} > \aleph_0$*, every* $\mathfrak{m}$-algebraic lattice L *can be represented as the lattice of* $\mathfrak{m}$*-congruence relations of an* $\mathfrak{m}$-complete weakly atomic modular lattice $K \in \mathbf{M}$.

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