ABSTRACT. We prove that the congruence lattice of a Scott-domain can be characterized as a complete lattice.

ON THE CONGRUENCE LATTICE OF A SCOTT-DOMAIN

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1. Introduction

While preparing their lecture for the Conference on mathematical foundations of programming semantics (Carnegie Melon University, 1991), see [6], A. Jung, L. Libkin, and H. Puhlmann tried to prove that the congruence lattice of a Scott-domain is an algebraic lattice. In this note we prove the following S:intro

Theorem. *Every complete lattice L can be represented as the lattice of congruence relations of a Scott-domain S. In fact, S can be constructed as a modular algebraic lattice.*

2. Preliminaries

For the notation and basic concepts of lattice theory, we refer the reader to G. Grätzer [3]. S:pre

We are using the standard definitions for domains, as in C. Gunter and D. Scott [2]. A *dcpo* (directed-complete poset) is a poset in which all directed sets have joins. In a dcpo, an element is *compact* if whenever it is below the join of a directed set, then it is below some element of the directed set. If every element of a dcpo is the join of a directed set of compact elements, then we call the dcpo *algebraic.* A dcpo is called *bounded-complete* if every bounded set has a join. A bounded-complete algebraic dcpo with zero is called a *Scott-domain.* Note that in a Scott-domain every nonempty set has a meet.

A *congruence* of a Scott-domain is an equivalence relation Θ satisfying the Substitution Property: let $x_i \equiv y_i \ (\Theta)$, $i \in I$, $I \neq \varnothing$; then

$$\bigwedge(x_i \mid i \in I) \equiv \bigwedge(y_i \mid i \in I) \quad (\Theta);$$

and if both $\bigvee(x_i \mid i \in I)$ and $\bigvee(y_i \mid i \in I)$ exist, then

$$\bigvee(x_i \mid i \in I) \equiv \bigvee(y_i \mid i \in I) \quad (\Theta).$$

The congruences of a Scott-domain S form a complete lattice denoted by $\mathrm{Con}_c\, S$.

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For a complete lattice K, the corresponding concept is called a *complete congruence*; the lattice of complete congruences is denoted by $\operatorname{Con}_c K$.

A lattice K satisfies the ACC (Ascending Chain Condition) if it contains no infinite ascending chain: $a_0 < a_1 < \cdots < a_n < \dots$ The dual condition (Descending Chain Condition) will be denoted by DCC.

The following observation is trivial:

Lemma 1. *Let the lattice K with zero satisfy the* ACC*. Then K is a Scott-domain, in fact, K is an algebraic lattice.*

Proof. Indeed, ACC implies that arbitrary joins exist, therefore K is complete. ACC also implies that all elements are compact, hence, K is algebraic. □

3. Proof

We start by referring the reader to the construction in G. Grätzer and H. Lakser [5], where for every complete lattice L a complete lattice K is constructed such that $\operatorname{Con}_c K$ is isomorphic to L. The construction is given in two steps: firstly, two well-ordered chains with unit are constructed, and their direct product is formed. Secondly, we apply in two different ways the One-point Extension construction: Let A be a lattice and let Λ be a set of nontrivial nonprime intervals in A. We define a lattice $A' = A[\Lambda]$ by adjoining the family of new pairwise distinct elements $\{m_I \mid I \in \Lambda\}$ to A, and, for each $I = [u, v] \in \Lambda$, requiring that $u \prec m_I \prec v$. S:proof

Now observe:

Lemma 2.

(1) *Let A be the direct product of finitely many well-ordered chains with unit. Then A satisfies the* DCC.
(2) *Let A satisfy the* DCC*. Then any One-point Extension A' of A also satisfies the* DCC.

Proof. Trivial. □

Now to prove the Theorem, let S be the dual of this lattice K. Since K has the DCC by Lemma 2, it follows that S has the ACC, and so by Lemma 1, S is a Scott-domain, in fact, and algebraic lattice.

This proves the Theorem except for the clause that S be modular.

If we want S to be modular, then we have to use the construction of R. Freese, G. Grätzer, and E. T. Schmidt [1] of a complete modular lattice K such that $\operatorname{Con}_c K$ is isomorphic to L. The construction of K is given in five steps: firstly, three well-ordered chains with unit are constructed, and the direct product of the first two is formed. Secondly, we apply the One-point Extension construction to the direct product. Thirdly, we take a sublattice of this direct product which is meet-complete. Fourthly, we form the direct product of this lattice and the third well-ordered chain. Finally, some sublattices of length three of this lattice are replaced by a somewhat larger lattice of length three. Obviously, the resulting lattice satisfies the DCC. Hence, again we can construct S as the dual of K. This S is modular.

References

1. R. Freese, G. Grätzer, and E. T. Schmidt, *On complete congruence lattices of complete modular lattices*, International Journal of Algebra and Computation **1** (1991), 147–160.
2. C. Gunter and D. Scott, *Semantic domains*, Handbook on Theoretical Computer Science, J. van Leeuwen, ed., North Holland, Amsterdam, 1990, pp. 633–674.
3. G. Grätzer, *General Lattice Theory*, Academic Press, New York, N. Y., Birkhäuser Verlag, Basel, Akademie Verlag, Berlin, 1978.
4. G. Grätzer and H. Lakser, *On complete congruence lattices of complete lattices*, Trans. Amer. Math. Soc. **327** (1991), pp. 385–405.
5. G. Grätzer and H. Lakser, *On congruence lattices of* $\mathfrak{m}$*-complete lattices*, J. Australian Math. Soc. (Series A) **52** (1992), pp. 57–87.
6. A. Jung, L. Libkin, and H. Puhlmann, *Decomposition of Domains*, Proceedings of the conference on mathematical foundations of programming semantics, Carnegie Melon University, 1991; extended abstract, to appear.

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