A REPRESENTATION OF $\mathfrak{m}$-ALGEBRAIC LATTICES

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ABSTRACT. In this paper, we give a short proof of the following result of G. Grätzer and E. T. Schmidt: every $\mathfrak{m}$-algebraic lattice can be represented as the lattice of $\mathfrak{m}$-complete congruence relations of some $\mathfrak{m}$-complete modular lattice.

1. Introduction

In 1983, R. Wille raised the following question (see, e.g., K. Reuter and R. Wille [7]): Is every complete lattice L isomorphic to the lattice of complete congruence relations of a suitable complete lattice K? S.-K. Teo [8] solved this problem for a finite lattice L. An affirmative solution to the question was provided in G. Grätzer [5], where the background of this field was also discussed. In a series of papers, various authors obtained sharper results (see [6] for a detailed accounting), culminating in the following result of G. Grätzer and E. T. Schmidt [6]:

Theorem . *Let $\mathfrak{m}$ be a regular cardinal $> \aleph_0$. Every $\mathfrak{m}$-algebraic lattice L is isomorphic to the lattice of $\mathfrak{m}$-complete congruence relations of a suitable $\mathfrak{m}$-complete modular lattice K.*

In this paper, we shall present a new proof of this result. Given L, a suitable lattice K is constructed in a short and direct way which allows easy study of its $\mathfrak{m}$-complete congruences. It can be shown—although we will not do this—that the construction of [6] yields the same lattice K.

2. Notation

A cardinal $\mathfrak{m}$ is *regular* if whenever J is a set with $|J| < \mathfrak{m}$ and $(I_j \mid j \in J)$ is a family of sets satisfying $|I_j| < \mathfrak{m}$, for all $j \in J$, then $|\bigcup(I_j \mid j \in J)| < \mathfrak{m}$. For instance, all cardinals of the form $\aleph_{\alpha+1}$ are regular. In this paper, $\mathfrak{m}$ stands for a fixed *uncountable regular cardinal.* A set of cardinality less than $\mathfrak{m}$ is said to be *small.* We refer the reader to [3] and [4] for standard lattice-theoretic notation and for proofs of elementary facts about algebraic lattices; natural extensions to $\mathfrak{m}$-algebraic lattices, which first appeared in [2], are restated below.

Some definitions and remarks have analogues for $\mathfrak{m} = \aleph_0$, but this is not the case for our later results. In the Theorem itself, the requirement that $\mathfrak{m} > \aleph_0$ is essential

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because the lattice $\operatorname{Con} K$ of $\aleph_0$-complete congruences of a lattice K is distributive, and so a non-distributive algebraic lattice L cannot be represented in that form.

A lattice M is $\mathfrak{m}$-*complete* if $\bigvee X$ and $\bigwedge X$ exist in M for every small nonempty subset X of M. A congruence relation Θ of an $\mathfrak{m}$-complete lattice M is an $\mathfrak{m}$-*complete congruence relation* if the Substitution Property holds for fewer than $\mathfrak{m}$ elements; that is, if $x_j \equiv y_j \ (\Theta)$ for all j in some nonempty small set J, then

$$\bigvee(\, x_j \mid j \in J\,) \equiv \bigvee(\, y_j \mid j \in J\,) \quad (\Theta)$$

and

$$\bigwedge(\, x_j \mid j \in J\,) \equiv \bigwedge(\, y_j \mid j \in J\,) \quad (\Theta).$$

An element c of a lattice M is $\mathfrak{m}$-*compact* if $c \leq \bigvee X$ implies that $c \leq \bigvee X_1$ for some small subset X_1 of X. A lattice M is $\mathfrak{m}$-*algebraic* if it is complete (not just $\mathfrak{m}$-complete) and every element is a join of $\mathfrak{m}$-compact elements.

The lattice $\operatorname{Con}_{\mathfrak{m}} M$ of all $\mathfrak{m}$-complete congruence relations of an $\mathfrak{m}$-complete lattice M is an $\mathfrak{m}$-algebraic lattice (see [2]); its lattice operations are denoted by $\wedge$, $\vee_{\mathfrak{m}}$, and the non-binary variants by $\bigwedge$, $\bigvee_{\mathfrak{m}}$. Note that $\wedge$ and $\bigwedge$ are set intersection.

We always let L denote the $\mathfrak{m}$-algebraic lattice in the Theorem; K is the lattice we construct to prove the Theorem. If L has only one element, we take $K = L$. We shall henceforth assume that $|L| \geq 2$.

Let C denote the set of *nonzero* $\mathfrak{m}$-*compact* elements of L. Since $|L| \geq 2$, it follows that $C \neq \varnothing$. The set C inherits a partial order from L and it is closed under small nonempty joins in L. Since C need not have a least element, it is not, in general, an $\mathfrak{m}$-complete join-semilattice.

An $\mathfrak{m}$-*complete ideal* of C is a nonempty subset I of C with the property that, for each small nonempty subset X of C, $\bigvee X \in I$ iff $X \subseteq I$. In particular, if $c \in I$ and $c' \in C$ with $c' < c$, then $c' \in I$, as can be seen by using $X = \{c, c'\}$. Let $\operatorname{Id}_{\mathfrak{m}} C$ denote the set, ordered by inclusion, of all $\mathfrak{m}$-complete ideals of C, together with the "empty ideal" $\varnothing$. It is a complete lattice, since $\operatorname{Id}_{\mathfrak{m}} C$ is closed under arbitrary intersections.

The importance of C in the proof of the Theorem stems in part from the following easy result (see [2]).

Lemma 1. *The map $x \mapsto \{c \in C \mid c \leq x\}$ is an isomorphism between L and $\operatorname{Id}_{\mathfrak{m}} C$.*

When discussing posets (partially ordered sets), we usually let $\leq$ denote the (partial) order, with $<$ standing for the corresponding strict order. An *ordinal* is a poset with the structure of a well-ordered chain. The *sum* of the posets A_1 and A_2 will be denoted by $A_1 + A_2$; we place A_2 on top of A_1. Define $2A_1 = A_1 + A_1$. The *lexicographic product* of the posets A_1 and A_2 is the set $A_1 \times A_2$, ordered by

$$\langle\, a_1, a_2 \,\rangle \leq \langle\, a_1', a_2' \,\rangle \text{ iff } a_1 < a_1' \text{ or } a_1 = a_1' \text{ and } a_2 \leq a_2'.$$

For any poset A, let A^* denote the lexicographic product of the chain $\omega = \{0, 1, 2, \ldots\}$ with A; it has no largest element. If A is a well-ordered chain, then so is A^*.

Let $\{\, A_\gamma \mid \gamma < \chi \,\}$ be the set of all small nonempty subsets of C, indexed by the ordinal χ. Henceforth, χ always stands for this ordinal; χ is not 0, since C is nonempty. We assume that each A_γ, $\gamma < \chi$, has been well-ordered.

In preparation for in the construction of the lattice K, we define the chains B_γ, $\gamma < 2\chi$, as follows:

For $\gamma < \chi$, let $B_\gamma = A^*_\gamma$.

For $\chi \leq \gamma < 2\chi$, let $B_\gamma = \omega \times \{ \gamma \}$ be a chain isomorphic to ω, with the elements

$$\langle 0, \gamma \rangle < \langle 1, \gamma \rangle < \langle 2, \gamma \rangle < \dots .$$

Let B be the disjoint union $\bigcup(B_\gamma \mid \gamma < 2\chi)$. From the strict linear orders on the B_γ, $\gamma < 2\chi$, we define a strict partial order on B, denoted by $\ll$ (to avoid confusion with other partial orders). For b_1, $b_2 \in B$, say $b_1 \in B_\gamma$ and $b_2 \in B_{\gamma'}$,

$$b_1 \ll b_2 \text{ iff } \gamma = \gamma' \text{ and } b_1 < b_2 \text{ in } B_\gamma. \tag{1}$$

This gives B the structure of a collection of mutually incomparable small well-ordered chains, each having no largest element.

There is a natural pairing of these chains, and of the ordinals less than 2χ, given by the operation $\tilde{\ }$:

For $\gamma < \chi$, let $\tilde{\gamma} = \chi + \gamma$.

For $\chi \leq \gamma < 2\chi$, let $\tilde{\gamma}$ be the unique ordinal satisfying $\chi + \tilde{\gamma} = \gamma$.

So, for any $\gamma < 2\chi$, the operation $\tilde{\ }$ interchanges γ and $\tilde{\gamma}$.

The elements of C are called *colors*, and a *coloring* of a set X is a function from X onto C. A coloring of X partitions X into *color classes* $X^c = \{ b \in B \mid \varphi(b) = c \}$, where $c \in C$. Coloring is a technique introduced by S.-K. Teo [8].

The construction of K uses the following coloring $\varphi : B \to C$. Let $b \in B$, say $b \in B_\gamma$, where $\gamma < 2\chi$. Then define

$$\varphi(b) = \begin{cases} a, & \text{if } \gamma < \chi \text{ and } b = \langle i, a \rangle; \\ \bigvee A_{\tilde{\gamma}}, & \text{if } \chi \leq \gamma < 2\chi. \end{cases} \tag{2}$$

Recall that any $A_{\gamma'}$, $\gamma' < \chi$, has a join in C.

Henceforth, the coloring of B will always be this coloring; the color classes B^c, $c \in C$, are the color classes under this coloring. Note that $B^c \neq \varnothing$, for all $c \in C$.

3. The lattice K

Let V be a vector space. The set of all subspaces of V, ordered by inclusion, is denoted by $\mathrm{PG}(V)$. It is a complete lattice in which meet is set intersection and the join of the two subspaces s and t is $s + t$. A lattice isomorphic to a $\mathrm{PG}(V)$ is a *projective geometry*. (There are other projective geometries which we need not consider here.) Note that any interval sublattice $[s, t]$ of a projective geometry is again a projective geometry. We also use the lattice-theoretic concept of *projectivity*, and the following easily established fact: *In a projective geometry, any two prime intervals are projective to each other.*

For a set X of vectors of V, let $[X]$ denote the subspace of V spanned by X. If S is a set of of subspaces of V such that $[\bigvee(S \setminus \{ s \})] \cap s = \{ 0 \}$, for all $s \in S$, then $[\bigcup S]$ will be denoted by $\bigoplus S$.

We shall construct the lattice K as a sublattice of a projective geometry $\mathrm{PG}(W)$, where W is any vector space with basis B (the set B was defined in Section 2). The choice of the underlying field (or division ring) is immaterial; one may use the two-element field, as in [6].

Any $v \in W$ can be represented in the form $v = \sum_b \lambda_b b$, where $b \in B$ and λ_b is an element of the underlying field. The set

$$\operatorname{supp} v = \{\, b \in B \mid \lambda_b \neq 0 \,\}$$

is finite; it is called the *support* of v.

For any linear combination $v = \sum(\lambda_i v_i \mid 1 \le i \le n)$ of vectors in W, we have:

$$\operatorname{supp} v \subseteq \bigcup(\operatorname{supp} v_i \mid 1 \le i \le n). \tag{3}$$

As always, C is the set of nonzero $\mathfrak{m}$-compact elements of L. The variables b, c, and v range over B, C, and W, respectively, while s, t, and so on, denote subspaces of W.

The decomposition of B into color classes induces the decompositions

$$W = \bigoplus(\,[B^c] \mid c \in C)$$

and

$$v = \sum(\, v^c \mid c \in C),$$

where $v^c \in [B^c]$ and all but finitely many of the vectors v^c are 0. For any subspace s of W, we also write $s^c = s \cap [B^c]$. Note that $s \supseteq \bigoplus(s^c \mid c \in C)$, where the containment is, in general, strict.

Let K be the set of all subspaces s of W satisfying the following conditions:

(K1) $\dim s < \mathfrak{m}$.
(K2) $s = \bigoplus(s^c \mid c \in C)$.
(K3) If $v \in s$, $b_1 \in \operatorname{supp} v$, and $b_2 \in B$ with $b_2 \ll b_1$, then $b_2 \in s$.
(K4) For $\gamma < 2\chi$, $B_\gamma \subseteq s$ iff $B_{\tilde{\gamma}} \subseteq s$.

The lattice K has a zero element 0_K, namely, the subspace $\{0\}$ of W.

If the elements s_i, $i \in I$, of $\mathrm{PG}(W)$ satisfy (K2) (in particular, if they lie in K), then

$$(\bigvee(\, s_i \mid i \in I))^c = \bigvee(\, s_i^c \mid i \in I) \text{ and } (\bigwedge(\, s_i \mid i \in I))^c = \bigwedge(\, s_i^c \mid i \in I), \tag{4}$$

for all $c \in C$.

Lemma 2. *K is a sublattice of* $\mathrm{PG}(W)$*; it is an $\mathfrak{m}$-complete lattice.*

Proof. It is obvious that K is closed under arbitrary nonempty meets formed in $\mathrm{PG}(W)$. Hence, to prove the lemma, we must verify two statements: K is closed under finite joins formed in $\mathrm{PG}(W)$; and small joins exist in K.

To verify the first statement, let $s, t \in K$. We show that $s \vee t$ satisfies (K1)–(K4). (K1) is obvious. By (4), (K2) holds for $s \vee t$. Let $v \in s \vee t$, say $v = v_1 + v_2$, where $v_1 \in s$ and $v_2 \in t$. Since s and t satisfy (K3), so does $s \vee t$, by (3).

Finally, suppose that $B_\gamma \subseteq s \vee t$ for some $\gamma < 2\chi$; we wish to show that $B_{\tilde{\gamma}} \subseteq s \vee t$. We claim that $B_\gamma \subseteq s$ or $B_\gamma \subseteq t$. Assume to the contrary that b_1, $b_2 \in B_\gamma$ and $b_1 \notin s$, $b_2 \notin t$. Then we can choose $b \in B_\gamma$ with $b_1 \ll b$, $b_2 \ll b$. By (3), b is in the support of an element of s or of t, contradicting (K3) for s or t. So $B_\gamma \subseteq s$ or $B_\gamma \subseteq t$. Since s and t satisfy (K4), we conclude that $B_{\tilde{\gamma}} \subseteq s$ or $B_{\tilde{\gamma}} \subseteq t$, which implies that $B_{\tilde{\gamma}} \subseteq s \vee t$. This proves (K4) for $s \vee t$. Therefore $s \vee t \in K$.

To verify the second statement, let S be a small subset of K. In $\mathrm{PG}(W)$, let $u' = \bigvee S$. Define

$$, _{u'} = \{\, \gamma \mid \gamma < 2\chi,\ B_\gamma \nsubseteq u',\ B_{\tilde\gamma} \subseteq u' \,\}. \tag{5}$$

Since the B_γ are pairwise disjoint and $\dim u' < \mathfrak{m}$, it follows that $, _{u'}$ is small. Therefore, the set

$$u = [u' \cup \bigcup(\, B_\gamma \mid \gamma \in\, , _{u'})] \tag{6}$$

satisfies (K1) as well as (K4). Using (3) and (4), it is easy to see that u satisfies (K2).

To verify (K3), let $v \in u$, $b_1 \in \operatorname{supp} v$, and $b_2 \ll b_1$ in B. We have at least one of: (i) $b_1 \in \operatorname{supp} v'$, for some $v' \in u'$, or (ii) $b_1 \in B_\gamma$, for some $\gamma \in\, , _{u'}$.

If (i) holds, then we apply (3) to obtain $b_1 \in \operatorname{supp} v''$, where $v'' \in s$, for some $s \in S$. Then (K2) and (K3) imply that $b_2 \in s$, and therefore $b_2 \in u$.

If (ii) holds and b_1 lies in B_γ, then so does b_2, by the definition of the relation $\ll$ in (1). Hence $b_2 \in u$.

This completes the verification of (K1)–(K4) for u. Therefore $u \in K$. Moreover, u is clearly the join in K of S. Thus K is an $\mathfrak{m}$-complete lattice. □

Next, we show that certain intervals in K are projective geometries.

Lemma 3. *Let $s \in K$. For some color c, let X be a small subset of B^c satisfying the condition:*

$$\text{If } b_1 \in X \text{ and } b_2 \ll b_1, \text{then } b_2 \in s. \tag{7}$$

Then $t \in K$ for any subspace t of W such that $s \subseteq t \subseteq s \vee [X]$.

Proof. By modularity, $t = s \vee (t \cap [X])$. Of course t satisfies (K1). As s satisfies (K2), it has a basis consisting of elements of $\bigcup(\, [B^c] \mid c \in C)$. Now t has the same property, since $t \cap [X] \subseteq [B^c]$, so t satisfies (K2). Let b_1 be an arbitrary element of B lying in the support of a vector of t. We must verify (K3) with t in place of s and with the b_1 just defined. Note that b_1 is either an element of X or else it lies in the support of a vector of s. In the first case, (K3) holds by (7); in the second case, it holds because s satisfies (K3). Since B_γ has no greatest element, it follows from (7) that $B_\gamma \subseteq s$ iff $B_\gamma \subseteq t$, for all $\gamma < 2\chi$. As s satisfies (K4), so does t. Thus $t \in K$. □

Lemma 4. *Every prime interval $[s, t]$ of K is prime in $\mathrm{PG}(W)$, and there is a unique color c with $s^c \neq t^c$.*

Proof. Let $[s, t]$ be a prime interval of K. If $s \cap B \neq t \cap B$, then choose v as an element in $\langle\, B; \ll \,\rangle$ which is minimal subject to $v \in t \setminus s$. If $s \cap B = t \cap B$, then we can, by (K2), choose $v \in t \setminus s$ with the property that $v \in [B^c]$ for some color c. Note that t satisfies (K3). Therefore, $X = \operatorname{supp} v$ is a nonempty finite set satisfying (7). Define $t_1 = [s \cup \{\, v \,\}]$. By Lemma 3, $t_1 \in K$ and hence $t_1 = t$. Evidently, $[s, t]$ is a prime interval of $\mathrm{PG}(W)$ and $s^c \neq t^c$. As s satisfies (K2) and $v \in [B^c]$, we also have $s^{c'} = t^{c'}$ whenever $c' \neq c$. □

The previous lemma allows us to color the prime intervals of K in a natural way: the color of $[s, t]$ is the unique color c with $s^c \neq t^c$. Given $b \in B$, we can define a prime interval $[s_b, t_b]$ of K whose color is $\varphi(b)$, as follows:

$$s_b = [\{\, b_1 \in B \mid b_1 \ll b \,\}],\ t_b = [s_b \cup \{\, b \,\}]. \tag{8}$$

Such prime intervals play a crucial role in the proof, starting with the next two lemmas.

Lemma 5. *Every prime interval of K is projective to one of the form $[s_b, t_b]$, where $b \in B$.*

Proof. In K, let $[s, t]$ be a prime interval of color c. Let $X = \operatorname{supp} v$, where $v \in t \setminus s$ is chosen as in the proof of Lemma 4. Then X satisfies condition (7), and $X \nsubseteq s$. Choose $b \in X \setminus s$ and set $t' = [s \cup \{ b \}]$. Note that $s_b \leq s$ and $t_b \leq t'$, by (7) and the definitions of s_b, t_b, and t'.

Lemma 3 implies that all subspaces of W between s and $[s \cup X]$ lie in K; therefore the prime intervals $[s, t]$ and $[s, t']$ are projective in K. The latter interval, and hence the former, is projective to $[s_b, t_b]$ because $s \vee t_b = t'$ and $s \wedge t_b = s_b$. ☐

Lemma 6. *Two prime intervals of K are projective iff they have the same color.*

Proof. If s, t, and u are elements of K and $s^c = t^c$, for some color c, then $(s \vee u)^c = (t \vee u)^c$ and $(s \wedge u)^c = (t \wedge u)^c$, by (4). In particular, projective prime intervals of K have the same color.

To establish the converse, it suffices, by Lemma 5, to prove that if b and $b' \in B$ are of the same color c, then the prime intervals $\mathfrak{p} = [s_b, t_b]$ and $\mathfrak{p}' = [s_{b'}, t_{b'}]$ are projective. Let $b \in B_\gamma$, $b' \in B_{\gamma'}$, where γ, $\gamma' < 2\chi$. We can reduce this proof to the case $\gamma \neq \gamma'$ by using the transitivity of projectivity and by choosing some $\mathfrak{p}'' = [s_{b''}, t_{b''}]$ of color c, where $b'' \in B_{\gamma''}$ and $\gamma'' \neq \gamma$. Such a $\mathfrak{p}''$ always exists.

Define $s = s_b \vee s_{b'}$, $t = s \vee t_b$ and $t' = s \vee t_{b'}$. Note that $t = [s \cup \{ b \}]$, and $t' = [s \cup \{ b' \}]$. Since $\gamma \neq \gamma'$, $\mathfrak{p}$ and $\mathfrak{p}'$ are projective in K to the prime intervals $[s, t]$ and $[s, t']$, respectively. By applying Lemma 3 with $s = s_b \vee s_{b'}$ and $X = \{ b, b' \}$, we conclude that $[s, t \vee t']$ is a projective geometry that lies in K. It follows that $[s, t]$ and $[s, t']$ are projective in K. Therefore, so are $\mathfrak{p}$ and $\mathfrak{p}'$. ☐

Lemma 7. *In K, every well-ordered chain with an upper bound is small.*

Proof. Let S be a well-ordered chain in K bounded from above by $u \in K$. For each $s \in S$ other than the greatest element of S (if it exists), let s^+ be the cover of s in S, and choose $v_s \in s^+ \setminus s$. It is clear that this produces an independent set of vectors contained in u. Since $\dim u < \mathfrak{m}$, by (K1), it follows that $|S| < \mathfrak{m}$. ☐

4. The proof of the Theorem

Having constructed K, we need to study its lattice of $\mathfrak{m}$-complete congruences. As before, c ranges over the set C of colors (nonzero $\mathfrak{m}$-compact elements of K). We also let x range over the elements of L.

For each $x \in L$, define an equivalence relation Φ^x on $\operatorname{PG}(W)$ by

$$s \equiv t \quad (\Phi^x) \text{ iff } s^c = t^c \text{ for all } c \nleq x. \tag{9}$$

Let Θ^x be the restriction of the relation Φ^x to K. We shall prove, in several steps, that L is isomorphic to $\operatorname{Con}_{\mathfrak{m}} K$; and this is established by the isomorphism $x \mapsto \Theta^x$.

Lemma 8. *For $x \in L$, Θ^x is an $\mathfrak{m}$-complete congruence of K.*

Proof. Property (4) and Lemma 2 imply that Θ^x is a congruence on K. To prove that Θ^x is $\mathfrak{m}$-complete, assume that $s_j \equiv t_j$ (Θ^x), for all j in some small nonempty set J. Since the meet in K is set intersection, which is trivial to handle, we discuss only the join. Let s' (resp. s) be the join of $(s_j \mid j \in J)$ in $\mathrm{PG}(W)$ (resp. in K). Define t' and t, similarly. Then $s' \equiv t'$ (Φ^x), by (4) and (9).

To prove that $s \equiv t$ (Θ^x), we must consider the process by which s and t are obtained from s' and t', respectively. Define $\Gamma_{s'}$ and $\Gamma_{t'}$ as $\Gamma_{u'}$ was defined in (5). Elements of these sets are ordinals less than 2χ. Also recall from (6) in the proof of Lemma 2 that

$$s = [s' \cup \bigcup(\, B_\gamma \mid \gamma \in \Gamma_{s'})], \tag{10}$$

and similarly for t.

If y, $z \in \mathrm{PG}(W)$ satisfy (K2) and $y \equiv z$ (Φ^x), then for any $\Gamma \subseteq 2\chi$ we have:

$$y \vee \bigvee(\, [B_\gamma] \mid \gamma \in \Gamma\,) \equiv z \vee \bigvee(\, [B_\gamma] \mid \gamma \in \Gamma\,) \quad (\Phi^x). \tag{11}$$

A similar relation holds if, in addition, we join another family $(\, [B_\gamma] \mid \gamma \in \Gamma_1)$ to one side but not to the other, provided every $\gamma \in \Gamma_1$ satisfies:

$$B_\gamma \subseteq \bigcup(\, B^c \mid c \leq x). \tag{12}$$

So to show that $s' \equiv t'$ (Φ^x), it suffices to verify that γ satisfies condition (12) whenever γ lies in exactly one of the sets $\Gamma_{s'}$ and $\Gamma_{t'}$. If one of γ and $\tilde{\gamma}$ satisfies (12), then so does the other, because of the way in which the chains were colored. Indeed, if $\gamma < \chi$ and A is the set of colors of elements of B_γ, then the elements of $B_{\tilde{\gamma}}$ have color $\bigvee A$. Therefore, we need only verify (12) for either γ or $\tilde{\gamma}$.

Let us suppose that $\gamma \in \Gamma_{t'} \setminus \Gamma_{s'}$. Then $B_\gamma \nsubseteq t'$ but $B_{\tilde{\gamma}} \subseteq t'$. Also, either B_γ and $B_{\tilde{\gamma}}$ are contained in s', or else neither of them is contained in s'. Therefore, possibly after interchanging the roles of s', t' and/or γ, $\tilde{\gamma}$, we may assume that $B_\gamma \subseteq s'$ but $B_\gamma \nsubseteq t'$. Choose $b \in B_\gamma$ with $b \in s' \setminus t'$.

Now suppose that there is an element b' of B_γ of color $c' \nleq x$. By (2), the definition of the coloring, we can choose such a b' with $b \ll b'$. Since $s' \equiv t'$ (Φ^x), we have $s' \cap [B^{c'}] = t' \cap [B^{c'}]$; therefore $b' \in t'$, as $b' \in s' \cap B^{c'}$. It follows that b' is a linear combination of vectors lying in subspaces of the form t_j, $j \in J$. By (3), there exists a $j \in J$ and a vector $v \in t_j$ such that $b' \in \operatorname{supp} v$. But now, (K3) applied to t_j shows that b lies in t_j, and hence also in t'. This contradicts $b \notin t'$. Thus no element b' of B_γ has color $c' \nleq x$.

This verifies the condition (12), as claimed. It follows that the congruence Θ^x of K is $\mathfrak{m}$-complete. ☐

For $\Theta \in \mathrm{Con}_{\mathfrak{m}} K$, let $\mathrm{I}(\Theta)$ denote the set of all colors of prime intervals $[s, t]$ for which $s \equiv t$ (Θ). By Lemma 6, Θ collapses all prime intervals whose color is in $\mathrm{I}(\Theta)$.

Lemma 9. *Let $\Theta \in \mathrm{Con}_{\mathfrak{m}} L$, and let A be a small nonempty subset of C. Then $\bigvee A \in \mathrm{I}(\Theta)$ iff $A \subseteq \mathrm{I}(\Theta)$.*

Proof. For $\gamma < 2\chi$, let $C_\gamma = \{\, s_b \mid b \in B_\gamma \,\}$, where s_b was defined in (8). Evidently C_γ, with the ordering inherited from K, is isomorphic to B_γ, and it is therefore a small well-ordered chain in K. Its prime intervals are of the form $[s_b, t_b]$, $b \in B_\gamma$.

Now choose the $\gamma < \chi$ for which $A_\gamma = A$. Let $i_\gamma = [B_\gamma \cup B_{\chi+\gamma}]$. It is routine to prove that $i_\gamma \in K$ and, in view of (K4),

$$i_\gamma = \bigvee C_\gamma = \bigvee C_{\chi+\gamma}.$$

A proof by transfinite induction shows that if $s_b \equiv t_b\ (\Theta)$, for all $b \in B_\gamma$, then $0_K \equiv i_\gamma\ (\Theta)$. The converse of this result is trivial. A similar equivalence holds for $B_{\chi+\gamma}$.

The set of colors of the prime intervals of the chain C_γ is A, whereas the prime intervals of $C_{\chi+\gamma}$ all have the same color, namely, $\bigvee A$. By considering the way in which the prime intervals $[s_b, t_b]$ were colored, we can now see that $\bigvee A \in \mathrm{I}(\Theta)$ iff $0_K \equiv i_\gamma\ (\Theta)$, which in turn is equivalent to $A \subseteq \mathrm{I}(\Theta)$. □

Recall that $\mathrm{Id}_{\mathfrak{m}} C$ is the lattice of all $\mathfrak{m}$-complete ideals of C, together with $\varnothing$.

Lemma 10. *The map* $\mathrm{Con}_{\mathfrak{m}} K \to \mathrm{Id}_{\mathfrak{m}} C$ *defined by* $\Theta \mapsto \mathrm{I}(\Theta)$ *is an isomorphism.*

Proof. By Lemma 9 and the definition of $\mathfrak{m}$-complete ideals of C, $\Theta \mapsto \mathrm{I}(\Theta)$ maps $\mathrm{Con}_{\mathfrak{m}} K$ into $\mathrm{Id}_{\mathfrak{m}} C$. Moreover, this map is order-preserving. Now let us assume that Θ and Ψ are distinct $\mathfrak{m}$-complete congruences of K, say $\Theta \nsubseteq \Psi$. Choose an interval $[s, t]$ of K collapsed by Θ but not by Ψ. By Lemma 7 and Zorn's Lemma, there is in K a maximal well-ordered chain from s to t, of order type less than $\mathfrak{m}$. In this chain, let u be the least element which satisfies $s \not\equiv u\,(\Psi)$. Since Ψ is $\mathfrak{m}$-complete, u has an immediate predecessor u^- in the chain. Obviously, Θ but not Ψ collapses the prime interval $[u^-, u]$. Thus $\mathrm{I}(\Theta) \nsubseteq \mathrm{I}(\Psi)$. It also follows that the map is injective.

Next observe that, for $x \in L$, $\mathrm{I}(\Theta^x) = \{\, c \in C \mid c \leq x \,\}$, as can be seen from the definition of Θ^x and the fact that every $c \in C$ is the color of some prime interval. Thus by Lemma 1 the map is onto. □

The Theorem is an immediate consequence of the previous lemma and Lemma 1. The isomorphism $L \cong \mathrm{Con}_{\mathfrak{m}} K$ is given by $x \mapsto \Theta^x$.

5. Concluding comments

This lattice K we constructed for the Theorem has many special properties.

(i) The lattice $K + 1$ (K with a unit element adjoined) is a complete lattice.

(ii) K is not only modular, it is a sublattice of the lattice of subspaces of a vector space; therefore it is arguesian.

(iii) K has a type 1 representation (see, for instance, [3], p. 198).

(iv) K has a zero element 0, and there is a one-to-one correspondence $\Theta \mapsto [0]\Theta$ between $\mathfrak{m}$-complete congruences of K and $\mathfrak{m}$-complete congruence kernel ideals.

In [1], R. Freese, W. A. Lampe, and W. Taylor proved an interesting result on the representation of algebraic lattices as congruence lattices: *For every cardinal* $\mathfrak{n}$, *there is an algebraic lattice* L *which cannot be represented as the congruence lattice of a finitary algebra with fewer than* $\mathfrak{n}$ *operations.*

We would like to point out that the case of infinitary algebras is different.

Theorem . *Let* $\mathfrak{m}$ *be a cardinal of the form* $\aleph_{\alpha+1}$. *Every* $\mathfrak{m}$-*algebraic lattice* L *is isomorphic to the congruence lattice of an algebra with a single operation, of arity* $\aleph_\alpha$.

Proof. We can regard the lattice K constructed in Section 3 as an algebra

$$\langle\, K; \{\, \textstyle\bigvee_{\mathfrak{m}}, \bigwedge_{\mathfrak{m}} \,\}\,\rangle,$$

where the arity of the operations $\bigvee_{\mathfrak{m}}$ and $\bigwedge_{\mathfrak{m}}$ is $\aleph_\alpha$. This algebra is polynomially equivalent to an algebra $\langle\, K; f\,\rangle$, where the operation f, of arity $\aleph_\alpha$, is defined below.

We can write $I = \{\, \gamma \mid \gamma < \aleph_\alpha \,\}$ as a disjoint union $I = \dot{\bigcup}(\, I_i \mid i < \aleph_\alpha)$, where each I_i is a set of cardinality $\aleph_\alpha$. Now define

$$f(\mathbf{x}) = \bigvee\nolimits_{\mathfrak{m}} (\bigwedge\nolimits_{\mathfrak{m}} (\mathbf{x_j} \mid \mathbf{j} \in \mathbf{I_i}) \mid \mathbf{i} < \aleph_\alpha),$$

where $\mathbf{x} = \langle\, \mathbf{x_0}, \mathbf{x_1}, \dots, \mathbf{x}_\gamma, \dots \rangle$, $\gamma < \aleph_\alpha$. □

Obviously, the arity $\aleph_\alpha$ is the best possible cardinal in this theorem. In fact, for any cardinal $\mathfrak{n} < \aleph_\alpha$, there is an $\mathfrak{m}$-algebraic lattice L that cannot be represented as the congruence lattice of an algebra $\langle\, A; F \,\rangle$ where each $f \in F$ has arity $\leq \mathfrak{n}$.

For example, take a set J that is not small and let L be the lattice of all subsets of J whose complement is small, together with the empty set.

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