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Multipasting of lattices

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Abstract. In this paper we introduce a lattice construction, called *multipasting*, which is a common generalization of gluing, pasting, and S-glued sums. We give a Characterization Theorem which generalizes results for earlier constructions. Multipasting is too general to prove the analogues of many known results. Therefore, we investigate in some detail three special cases: strong multipasting, multipasting of convex sublattices, and multipasting with the Interpolation Property.

1. Introduction

Lattice constructions, such as direct products, prime products, formation of ideal lattices, and so on, play an important part in lattice theory.

An important group of constructions is the Hall–Dillworth gluing and its generalizations; see §2 for the definitions of some of these constructions.

Gluing constructs a lattice by identifying an ideal of a given lattice with the dual ideal of another lattice.

Ch. Herrmann [10] extends this construction to *S-glued sum*, where S is a lattice of finite length, to provide a structural tool for examining modular lattices of finite length. There is a further generalization in A. Day and Ch. Herrmann [1].

Another important generalization of gluing is *pasting*, see V. Slavík [11] and G. Grätzer [8, Exercise 12 of §V.4]. Intuitively, the lattice L is the pasting of its sublattices A and B iff L is the union of A and B and all the joins and meets in L can be computed from the joins and meets in A and B.

The main application of this notion is the following important result of A. Day and J. Ježek [2] (see also §V.4 and Problem V.11 in G. Grätzer [8]):

There are precisely three varieties of lattices satisfying the amalgamation property: the trivial one, the variety of the distributive lattices, and the variety of all lattices.

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There are many recent results on pasting; see E. Fried and G. Grätzer [4], [5], [6], and E. Fried, G. Grätzer, and H. Lakser [7].

In this paper we introduce a new lattice construction: *multipasting*; this is a common generalization of gluing, pasting, and S-glued sum.

In §2 we define the known basic constructions. *Multipasting* is introduced in §3, where we also present the Characterization Theorem. In §4 we define multipasting as a construction under the name Σ-pasting, and we prove that multipasting and Σ-pasting are "equivalent." For lattices of finite length, *strong multipasting* is introduced in §5; we show that it is a generalization of pasting but more restrictive than multipasting. It is also a generalization of S-glued sum; see §6. We examine multipasting of convex sublattices in §7; this construction is more general than pasting but different from strong multipasting for lattices of finite length. In §8 we introduce the *(Upper) Interpolation Property*; the Interpolation Property always holds for pasting. For finite lattices, multipasting with the Upper Interpolation Property preserves semimodularity, and multipasting with the Interpolation Property preserves modularity. Finally, §9 presents some concluding comments.

For the basic concepts of lattice theory and for the notation, we refer the reader to G. Grätzer [8].

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2. Earlier constructions

In this section, we define the known lattice constructions we shall need in this paper. We start with gluing:

DEFINITION 1 (Gluing). *Let D be a dual ideal of the lattice A, let I be an ideal of the lattice B, and let φ be an isomorphism $\varphi : D \to I$. In $P = A \dot{\cup} B$, identify every $d \in D$ with $d\varphi \in I$. Define the partial ordering on P as the transitive closure of the partial order of A and the partial order of B. Then P becomes a lattice, the gluing of A and B.*

Observe that the partial ordering on P has a simpler description; it is given in Condition (P) below.

Next we define pasting, see Figure 1:

DEFINITION 2 (Pasting). *Let the lattice L be the union of its sublattices A and B, and let $S = A \cap B$. Let $f_A : A \to L$ and $f_B : B \to L$ be the natural embeddings. Then*

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & A \arrow[d, "f_A"'] \arrow[dr, "g_A"] & \\
B \arrow[r, "f_B"] \arrow[rr, bend right, "g_B"'] & L \arrow[r, dashed, "h"] & K
\end{tikzcd}$$

Figure 1

L pastes A and B together over S, iff whenever K is a lattice and $g_A : A \to K$ *and* $g_B : B \to K$ *are homomorphisms satisfying* $xg_A = xg_B$ *for every* $x \in S$, *then there exists a homomorphism* $h : L \to K$ *satisfying* $f_A h = g_A$ *and* $f_B h = g_B$.

Note that in some papers this definition is given in terms of embeddings rather than homomorphisms. In [5], there is a proof of the equivalence of the two forms of this definition. See also Corollary 14 in this paper.

It seems appropriate to state the following variant of Definition 2; it is more closely related to the definition of multipasting as given in Definition 6.

DEFINITION 2′ (Pasting). *Let the lattice L be the union of its sublattices A and B, and let* $S = A \cap B$. *Let* $f_A : A \to L$ *and* $f_B : B \to L$ *be the natural embeddings. Then L pastes A and B together over S iff whenever* $h : L \to K$ *is a map of L into a lattice K such that the restrictions* $g_A : A \to K$ *and* $g_B : B \to K$ *are homomorphisms, then h is a homomorphism.*

In their paper [2], Day and Ježek give the following variant of Slavík's characterization [11] of pasting for finite lattices:

Let A and B be sublattices of the lattice L; let $L = A \cup B$ *and* $A \cap B = S$. *Then L pastes A and B together over S iff the following two conditions hold*:

(P) *If* $a \leq b$ *holds in L for* $a \in A$ *and* $b \in B$, *then there exists an* $s \in S$ *such that* $a \leq s$ *in A and* $s \leq b$ *in B; and dually for* $b \leq a$.

(Cov) *All upper covers in L of an element* $s \in S$ *belong either to A or to B; and dually for lower covers.*

E. Fried and G. Grätzer [5] generalize the characterization theorem for pasting to infinite lattices; the result is almost the same as in the finite case, except that conditions (Cov) is replaced by the following:

(Id) *If the ideals C and Y of L satisfy*

$$X - Y \subseteq A - B \quad \text{and} \quad Y - X \subseteq B - A,$$

then

$$X \subseteq Y \quad or \quad Y \subseteq X;$$

and dually for dual ideals.

Next, we define an S-glued system and S-glued sum as in Ch. Herrmann [10] ($s \prec t$ denotes that s is *covered* by t):

DEFINITION 3 (S-glued system). *Let S and L_s, $s \in S$, be lattices of finite length. The system L_s, $s \in S$, is called an* S-glued system *iff the following conditions are satisfied*:

(1) *For all s, $t \in S$, if $s \leq t$, then either $L_s \cap L_t = \varnothing$ or $L_s \cap L_t$ is a dual ideal in L_s and an ideal in L_t.*

(2) *For all s, $t \in S$ with $s \leq t$ and for all a, $b \in L_s \cap L_t$, the relation $a \leq b$ holds in L_s iff $a \leq b$ in L_t.*

(3) *For all s, $t \in S$, the covering $s \prec t$ implies that $L_s \cap L_t \neq \varnothing$.*

(4) *If s, $t \in S$, then $L_s \cap L_t \subseteq L_{s \wedge t} \cap L_{s \vee t}$.*

DEFINITION 4 (S-glued sum). *Let $L = \bigcup (L_s \mid s \in S)$, where L_s, $s \in S$, is an S-glued system. Let the partial order $\leq$ in L be defined as follows: for a, $b \in L$, let $a \leq b$ iff there exist a sequence $a = x_0, x_1, \ldots, x_n = b$ of elements of L and a sequence $s_1, \ldots, s_n$ of elements of S such that $s_i \leq s_{i+1}$ in S, $i = 1, \ldots, n-1$, and $x_{i-1} \leq x_i$ in L_{s_i}, $i = 1, \ldots, n$. Then L is a lattice, the* S-glued sum *of L_s, $s \in S$.*

Let L be the S-glued sum of L_s, $s \in S$. We call the components the *blocks* of the S-glued sum. The following facts are easy to verify (see Ch. Herrmann [10]):

(a) An S-glued sum has as many blocks as S has elements.

(b) Any block is an interval in L.

(c) If A and B are blocks indexed by comparable elements of S, then $A \cup B$ is a sublattice of L; this sublattice is the gluing of A and B, except if A and B are disjoint.

(d) If S is a two-element lattice, then an S-glued sum is the gluing of the two blocks.

(e) $a \prec b$ in L iff $a \prec b$ in some block.

(f) If s, $t \in S$, $L_s = [a, b]$, $L_t = [c, d]$, then $L_{s \vee t}$ is of the form $[a \vee c, e]$, for some $e \in L$.

(g) Every modular lattice L of finite length is the S-glued sum of its maximal complemented intervals.

3. Multipasting

There are two ways to approach multipasting: by generalizing Definition 2 to more than two lattices (internal definition) as in the following definition, or to view it as a construction somewhat analogous to Definitions 3 and 4, see §4.

To present the internal definition of multipasting of more than two lattices we need some definitions first:

DEFINITION 5. *Let L be a lattice with the sublattices L_λ, $\lambda \in \Lambda$.*

(1) *L_λ, $\lambda \in \Lambda$, is a cover of L iff $L = \bigcup (L_\lambda \mid \lambda \in \Lambda)$.*

(2) *For $a, b \in \bigcup (L_\lambda \mid \lambda \in \Lambda)$, let $a \leq_\Lambda b$ denote that there exists a sequence $a = s_0, s_1, \ldots, s_n = b$ of elements of L such that, for each i with $0 \leq i < n$, there is a $\lambda_i \in \Lambda$ satisfying $s_i, s_{i+1} \in L_{\lambda_i}$ and $s_i \leq s_{i+1}$ in L_{λ_i}.*

(3) *A cover L_λ, $\lambda \in \Lambda$, of L is full iff $a \leq b$ in L is equivalent to $a \leq_\Lambda b$.*

(4) *A cover L_λ, $\lambda \in \Lambda$, of L is compatible iff, for a, b, $c \in L$, whenever $a \leq_\Lambda c$, $b \leq_\Lambda c$, $a, b \in L_\lambda$, for some $\lambda \in \Lambda$, then $a \vee b \leq_\Lambda c$, where $a \vee b$ is the join of a and b in L_λ; and dually.*

Note that $\leq_\Lambda$ is a partial ordering on $\bigcup (L_\lambda \mid \lambda \in \Lambda)$. A full cover is a cover that induces the partial ordering of L, that is, for which the partial ordering relation $\leq$ of L equals the relation $\leq_\Lambda$. For a finite lattice (or for a lattice of finite length), a cover is full iff every covering pair is contained in some L_λ. A compatible cover is a cover that induces a relation $\leq_\Lambda$ which is compatible with the two partial operations; that is, $\leq_\Lambda$ preserves the join and the meet for any two elements in L_λ, $\lambda \in \Lambda$. Obviously, every full cover is a compatible cover. It is easy to construct covers that are not compatible, and compatible covers that are not full.

Let us examine these concepts for the case $|\Lambda| = 2$. Let $\Lambda = \{0, 1\}$. Then L_λ, $\lambda \in \Lambda$ is a cover iff $L = L_0 \cup L_1$. Now if $a \in L_0$, $b \in L_1$, and $a \leq_\Lambda b$, then obviously there exists a sequence $a = s_0, s_1, \ldots, s_n = b$ of elements of L such that $a = s_0 \leq s_1$ in L_0, $s_1 \leq s_2$ in $L_1, \ldots, s_{n-1} \leq s_n = b$ in L_1. Since $s_1, \ldots, s_{n-1} \in L_0 \cap L_1$, it follows that $a \leq s_1$ in L_0 and $s_1 \leq b$ in L_1. So we can always assume that $n = 2$, and therefore $\leq_\Lambda$ is the same as the binary relation described in (P).

Thus L_λ, $\lambda \in \{0, 1\}$, is a full cover iff (P) holds for $L_0 = A$ and $L_1 = B$.

Note also that, in this case, every cover is compatible. Indeed, let $\Lambda = \{0, 1\}$, $a \leq_\Lambda c$, $b \leq_\Lambda c$, $a, b \in L_0$. We have to prove that $a \vee b \leq_\Lambda c$, where $a \vee b$ is the join in L_0. If $c \in L_0$, then this is obvious. If $c \in L_1 - L_0$, then, by the discussion above, there are $s_a, s_b \in L_0 \cap L_1$ such that $a \leq s_a$, $b \leq s_b$ in L_0, and $s_a \leq c$, $s_b \leq c$ in L_1. Let $s = s_a \vee s_b \in L_0 \cap L_1$. Then $a \vee b \leq s$ in L_0 and $s \leq c$ in L_1, proving that $a \vee b \leq_\Lambda c$; along with the dual argument, this proves compatibility.

Now we are ready to state the definition of our central concept.

DEFINITION 6 (Multipasting). *Let L be a lattice, and let L_λ, $\lambda \in \Lambda$, be a compatible cover of L. Then L is a multipasting of L_λ, $\lambda \in \Lambda$, iff for any lattice K and any map ψ of L into K, if all the restrictions $\psi_\lambda : L_\lambda \to K$ are homomorphisms, for $\lambda \in \Lambda$, then ψ is a homomorphism of L into K.*

If $|\Lambda| = 2$, then – as we noted above – every cover is compatible. Thus for $|\Lambda| = 2$, Definition 6 is the same as Definition 2′; so for two lattices multipasting is the same as pasting.

It may appear to the reader that the natural generalization of pasting is as stated in Definition 6 but without the assumption of the compatibility of the cover. Note, however, that compatibility is used throughout our proofs. See also §9.

Many lattice properties are preserved under pasting, for instance, modularity and distributivity (see [4] and [5]; see also [7]). However, multipasting cannot be expected to preserve any lattice properties; in fact, *every* lattice is the multipasting of *Boolean* lattices. Indeed, consider any lattice L with more than one element and let $L_{a,b}$, for $a \neq b$, a, $b \in L$, denote the sublattice generated by a and b. These sublattices are either two-element or four-element Boolean lattices, and they form a full cover. Clearly, L multipastes them.

Three stronger forms of multipasting (strong multipasting, multipasting convex sublattices, and multipasting with the Interpolation Property) will be considered in this paper.

A more interesting example of multipasting is shown in Figure 2. Consider the lattice L of Figure 2(a). L is the multipasting of four sublattices, A, B, C, and D, as illustrated by Figure 2(b). This example is not very surprising because $A \cup B$ is the gluing of A and B; similarly, $C \cup D$ is the gluing of C and D; finally, L is the

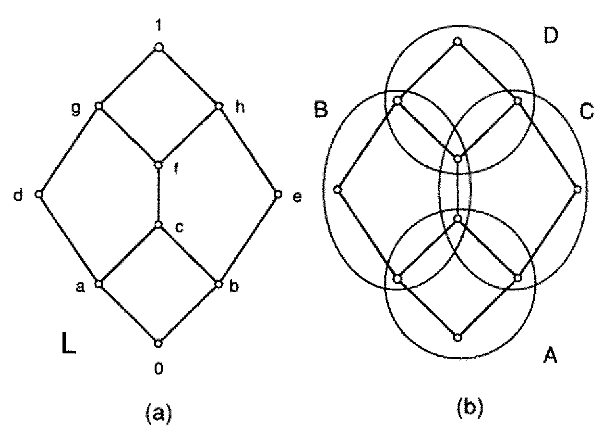

Figure 2

gluing of $A \cup B$ and $C \cup D$. Thus L is a "repeated gluing" of the four components. It is easy to see that repeated gluing is always a multipasting. However, L is also a multipasting of the following sublattices: B, C, $\{0, b, d, g\}$, $\{0, a, e, h\}$, $\{a, d, h, 1\}$, $\{b, g, e, 1\}$, and this is quite unlike any pasting or gluing.

In this section, we shall characterize multipasting in a manner similar to the characterization of pasting in E. Fried and G. Grätzer [5]. The principal tool is the construction of a structure Part Λ formed from the lattices L_λ, $\lambda \in \Lambda$, with a binary relation and two partial binary operations:

DEFINITION 7 (Part Λ). *Let L_λ, $\lambda \in \Lambda$, be a family of lattices such that for μ, $\nu \in \Lambda$, $\mu \neq \nu$, if $L_\mu \cap L_\nu \neq \varnothing$, then it is a sublattice of L_μ and L_ν. On the set $P = \bigcup (L_\lambda \mid \lambda \in \Lambda)$, define a binary relation $\leq_\Lambda$ as the transition extension of $\bigcup (\leq_\lambda \mid \lambda \in \Lambda)$, where $\leq_\lambda$ is the partial ordering on L_λ. On the set P, define two partial operations $\vee$ and $\wedge$: $a \vee b = c$ iff $a, b, c \in L_\lambda$, for some $\lambda \in \Lambda$, and $a \vee b = c$ in L_λ; and similarly for $\wedge$. The structure* Part Λ *is the set P equipped with the relation $\leq_\Lambda$ and the partial operations $\vee$ and $\wedge$.*

Note that if L_λ, $\lambda \in \Lambda$, are sublattices of a lattice, then the definition of $\leq_\Lambda$ given here agrees with the definition in (5.2); however, in general, $\leq_\Lambda$ is not a partial ordering.

As in [5], we proceed by defining ideals in Part Λ:

DEFINITION 8. *A nonempty subset I of the structure $P =$* Part Λ *is called an* ideal *iff it is closed under the partial operation $\vee$ and $a \leq_\Lambda b \in I$ implies that $a \in I$.* Dual ideal *is defined dually.*

It is obvious that $I \subseteq P$, $I \neq \varnothing$, is an ideal of $P = \text{Part } \Lambda$ iff $I \cap L_\lambda$ is an ideal or $\varnothing$, for every $\lambda \in \Lambda$.

Since, in $P = \text{Part } \Lambda$, a nonempty intersection of ideals in an ideal again, we can define an *ideal* (or *dual ideal*) *generated by a set H*, denoted by $(H]_\Lambda$, for $H \subseteq P$, $H \neq \varnothing$; *a finitely generated ideal* and a *finitely generated dual ideal*; a *principal ideal* $(a]_\Lambda = (\{a\}]_\Lambda$, and a *principal dual ideal*, in the usual manner.

The set of all ideals of Part Λ with the empty set form a lattice denoted by Id Part Λ.

The structure $P = \text{Part } \Lambda$ we obtain from a compatible cover of a lattice has some special properties:

DEFINITION 9. *The structure $P = $* Part Λ *is called smooth iff the following conditions are satisfied*:

(1) $\leq_\Lambda$ *is a partial ordering on P.*

(2) *$a, b \in L_\mu$, $c \in L_\nu$ $(\mu, \nu \in \Lambda)$, $a \leq_\Lambda c$, and $b \leq_\Lambda c$ imply that $a \vee b \leq_\Lambda c$, where $a \vee b$ is the join of a and b in L_μ; and dually.*

The crucial part in the proof of the Characterization Theorem is an understanding of how ideals are generated in structures, in general, and smooth structures, in particular. The following concept is the "structure" analogue of the "$\{a, b\}$-sequences" of [5]:

DEFINITION 10. *Let H be a nonempty subset of the structure $P = \text{Part}\,\Lambda$. Define an H-sequence as a sequence $s_0, s_1, \ldots, s_n$ of elements of P satisfying $s_0 \in H$, and, for each i with $0 < i \leq n$, one of the following conditions holds*:

(1) *$s_i \in H$.*

(2) *There is a j with $0 \leq j < i$ and $s_i \leq_\Lambda s_j$.*

(3) *There are j, k with $0 \leq j$, $k < i$ and $s_i = s_j \vee s_k$.*

Now we are ready to describe how ideals are generated:

LEMMA 11. *Let H be a nonempty subset of the structure $P = \text{Part}\,\Lambda$. Then $a \in (H]_\Lambda$ iff there exists an H-sequence $s_0, s_1, \ldots, s_n = a$.*

Proof. It follows by induction on n from the definition of an ideal that if there is an H-sequence $s_0, s_1, \ldots, s_n = a$, then $a \in (H]_\Lambda$. Conversely, let $\bar{H}$ be defined as the set of all $a \in P$ such that there is an H-sequence $s_0, s_1, \ldots, s_n = a$. By (10.1), $H \subseteq \bar{H}$. If $b \in \bar{H}$ is established by the H-sequence $s_0, s_1, \ldots, s_n = b$, and $a \leq_\Lambda b$, then by (10.2) $s_0, s_1, \ldots, s_n, a$ is an H-sequence; it establishes that $a \in \bar{H}$. Finally, let the H-sequence $s_0, s_1, \ldots, s_n = a$ establish that $a \in \bar{H}$, let the H-sequence $t_0, t_1, \ldots, t_m = b$ establish that $b \in \bar{H}$, and let $a \vee b = c$ in L_λ, for some $\lambda \in \Lambda$. Observe that $s_0, s_1, \ldots, s_n, t_0, t_1, \ldots, t_m$ is also an H-sequence; hence by (10.3) so is $s_0, s_1, \ldots, s_n, t_0, t_1, \ldots, t_m, s_n \vee t_m = c$, establishing that $\bar{H}$ is closed under joins. Thus $\bar{H}$ is an ideal and $H \subseteq \bar{H} \subseteq (H]_\Lambda$, hence $\bar{H} = (H]_\Lambda$, as claimed.

If the structure is smooth, we can say more:

LEMMA 12. *Let $P = \text{Part}\,\Lambda$ be smooth. Then*

(1) *P is partially ordered by $\leqslant_\Lambda$.*

(2) *If $a \vee b = c$, and a, b, $c \in L_\lambda$ for some $\lambda \in \Lambda$, then c is the least upper bound of a and b in P with respect to $\leq_\Lambda$; and dually.*

(3) *$(a]_\Lambda = \{x \mid x \leq_\Lambda a\}$.*

Proof. (1) and (2) just restate (9.1) and (9.2). In view of (2), Condition (3) is trivial.

Note that $\{x \mid x \leq_\Lambda a\}$ is, in general, not an ideal of Part Λ.

Now we are ready to characterize multipasting in terms of Part Λ:

THEOREM 13 (Characterization Theorem). *Let L be a lattice, and let the sublattices L_λ, $\lambda \in \Lambda$, form a compatible cover of L. Then L is a multipasting of L_λ, $\lambda \in \Lambda$, iff the following two conditions are satisfied*:

(1) *$\leq_\Lambda$ is the partial ordering of L (that is, L_λ, $\lambda \in \Lambda$, is a full cover of L).*

(2) *Every finitely generated ideal of* Part Λ *is principal, and dually.*

Proof. Let us assume that L is a multipasting of L_λ, $\lambda \in \Lambda$. We shall verify Conditions (1) and (2).

Form $P = \text{Part } \Lambda$; we claim that it is smooth (see Definition 9). Obviously, $P = L$; since each L_λ, $\lambda \in \Lambda$, is a sublattice of L, it follows that $\leq_\Lambda$ is contained in $\leq$. Hence, $\leq_\Lambda$ is a partial ordering, verifying (9.1). Since Condition (9.2) is assumed, it follows that $P = \text{Part } \Lambda$ is smooth.

Consider the embedding $\psi : x \to (x] = \{y \mid y \leq_\Lambda x\}$ of L into P^c, the MacNeille completion of $\langle P; \leq_\Lambda \rangle$. Since Part Λ is smooth, ψ preserves the $\vee$ and $\wedge$ on each L_λ, $\lambda \in \Lambda$, and so by the definition of multipasting, with $K = P^c$, we conclude that ψ is a homomorphism of L into P^c. Therefore, for $x, y \in L$, $x \leq y$ implies that $x\psi \subseteq y\psi$, and so $x \in y\psi$, that is, $x \leq_\Lambda y$, verifying Condition (1).

Now consider the lattice $K = \text{Id Part } \Lambda$ and the map $\psi : x \to (x]_\Lambda$ of L into K. Since $P = \text{Part } \Lambda$ is smooth, by Lemma 12, the map ψ restricted to any L_λ, $\lambda \in \Lambda$, is a homomorphism (in fact, an embedding). By the definition of multipasting, with $K = \text{Id Part } \Lambda$, we conclude that ψ is a homomorphism of L into Id Part Λ. Thus the principal ideals in $K = \text{Id Part } \Lambda$ form a sublattice (namely, $L\psi$). This verifies that every finitely generated ideal is principal. The dual argument completes the proof of (2).

Conversely, let us now assume that (1) and (2) hold. Form the structure $P = \text{Part } \Lambda$; recall that it is smooth.

To show that L is a multipasting, take a lattice K and a map ψ of L into K such that all the restrictions $\psi_\lambda : L_\lambda \to K$ are homomorphisms, for $\lambda \in \Lambda$. We have to prove that ψ is a homomorphism of L into K.

Since ψ_λ is a homomorphism, for $a, b \in L_\lambda$, $a \leq b$ in L_λ implies that $a\psi \leq b\psi$ in K. So, for $a, b \in L$, $a \leq_\Lambda b$ implies that $a\psi \leq b\psi$ in K. By (1), $a \leq_\Lambda b$ iff $a \leq b$ in L. We conclude that $a \leq b$ in L implies that $a\psi \leq b\psi$ in K, that is, ψ is isotone.

Let $a \vee b = c$ in L. By (2), $(a]_\Lambda \vee (b]_\Lambda = (u]_\Lambda$, for some $u \in L$. Since $P = \text{Part } \Lambda$ is smooth, Lemma 12 applies. By (12.3), $a \leq u$ and $b \leq u$, so $c \leq u$. Conversely, $(a]_\Lambda$, $(b]_\Lambda \subseteq (c]_\Lambda$, so $c = u$. By Lemma 11, there is an $\{a, b\}$-sequence $s_0, \ldots, s_n$ terminating in c. Then $s_0\psi, \ldots, s_n\psi$, is an $\{a\psi, b\psi\}$-sequence terminating in $c\psi$. Therefore, $c\psi \in (a\psi] \vee (b\psi]$ in K, proving that $a\psi \vee b\psi = c\psi$. Along with the dual argument, this shows that ψ is a homomorphism, proving that L is a multipasting.

Maybe, at this point, it is appropriate to point out that in the definition of multipasting (Definition 6) we may use embeddings in place of homomorphisms:

COROLLARY 14. *Let L be a lattice, and let L_λ, $\lambda \in \Lambda$, be a compatible cover of L. Then L is a multipasting of L_λ, $\lambda \in \Lambda$, iff for any lattice K and any map ψ of L into K, if all the restrictions $\psi_\lambda : L_\lambda \to K$ are embeddings, for $\lambda \in \Lambda$, then ψ is an embedding of L into K.*

Proof. Let L_λ, $\lambda \in \Lambda$, be a multipasting of L. Let K be any lattice and let ψ be any map of L into K such that all the restrictions $\psi_\lambda : L_\lambda \to K$ are embeddings, for $\lambda \in \Lambda$. It follows from Theorem 13 that ψ is a homomorphism. If $a \leq b$ in L and $a\psi = b\psi$, then by (13.1), $a \leq_\Lambda b$; but ψ is one-to-one on each L_λ, so $a = b$, that is, ψ is one-to-one.

Conversely, Let L satisfy the condition of Corllary 14, and let K and ψ be given as in Definition 6. Then we can define $K' = K \times L$ and the map $\psi' : L \to K'$ by

$$x\psi' = \langle x\psi, x \rangle.$$

It is clear that ψ' restricted to any L_λ is an embedding, hence ψ' is an embedding by the assumption of this corollary. Thus ψ' followed by the first projection map is a homomorphism of L into K, and this map is the same as ψ, proving that ψ is a homomorphism, completing the proof of the corollary.

4. Σ-pasting

To treat multipasting as a construction, we introduce Σ-pasting. We proceed as we did for S-glued sum; while for S-glued sum we need a lattice to "index" a family of lattices, for multipasting we start with a semilattice:

DEFINITION 15 (Σ-system). *Let Σ be a meet-semilattice. A Σ-system L_α, $\alpha \in \Sigma$, is defined by the following conditions:*

(1) *L_α is a lattice or $\varnothing$, for $\alpha \in \Sigma$.*
(2) *There is an embedding $\varphi_{\alpha,\beta} : L_\alpha \to L_\beta$,* for *$\alpha$, $\beta \in \Sigma$ with $\alpha \leq \beta$ in Σ.*
(3) *$\varphi_{\alpha,\alpha}$ is the identity map on L_α, for $\alpha \in \Sigma$.*
(4) *$\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$, for $\alpha, \beta, \gamma \in \Sigma$ with $\alpha \leq \beta \leq \gamma$ in Σ.*
(5) $\operatorname{Im} \varphi_{\alpha \wedge \beta,\gamma} = \operatorname{Im} \varphi_{\alpha,\gamma} \cap \operatorname{Im} \varphi_{\beta,\gamma}$, *for $\alpha, \beta, \gamma \in \Sigma$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$ in Σ.*

In this definition, Im φ is the image of the map φ. Given a Σ-system, we can define a Σ-pasting:

DEFINITION 16 (Σ-pasting). *A lattice L is a Σ-pasting of the Σ-system L_α, $\alpha \in \Sigma$, iff the following conditions are satisfied:*

(1) *There is an embedding $\varphi_\alpha : L_\alpha \to L$, for $\alpha \in \Sigma$.*

(2) $\varphi_{\alpha,\beta}\varphi_\beta = \varphi_\alpha$, *for* $\alpha, \beta \in \Sigma$ *with* $\alpha \le \beta$ *in* Σ.

(3) $\operatorname{Im}\varphi_\alpha \cap \operatorname{Im}\varphi_\beta = \operatorname{Im}\varphi_{\alpha\wedge\beta}$, *for* $\alpha, \beta \in \Sigma$.

(4) $\operatorname{Im}\varphi_\alpha$, $\alpha \in \Sigma$, *is a compatible cover of* L.

(5) *Let* K *be a lattice and let* ψ_α *be a homomorphism of* L_α *into* K, *for* $\alpha \in \Sigma$, *such that* $\varphi_{\alpha,\beta}\psi_\beta = \psi_\alpha$, *for* $\alpha, \beta \in \Sigma$ *with* $\alpha \le \beta$ *in* Σ. *Then there exists a homomorphism* ψ *of* L *into* K *such that* $\varphi_\alpha\psi = \psi_\alpha$, *for* $\alpha \in \Sigma$.

Note that not every Σ-system yields a Σ-pasting. For instance, let Σ be the semilattice $\{\omega, \alpha, \beta, \gamma\}$ with $\alpha \wedge \beta = \alpha \wedge \gamma = \beta \wedge \gamma = \omega$, let $L_\omega = \{0\}$, $L_\alpha = \{0, a\}$, $L_\beta = \{0, b\}$, $L_\gamma = \{0, c\}$, and let $0\varphi_{\omega,\alpha} = 0$, $0\varphi_{\omega,\beta} = 0$, $0\varphi_{\omega,\gamma} = 0$. This Σ-system has no Σ-pasting.

It is clear that if a Σ-pasting exists, then it is unique up to isomorphism. Indeed, let L and L' be Σ-pastings of a Σ-system with embeddings φ_α and φ'_α, $\alpha \in \Sigma$. Then, by (16.5), there is a homomorphism $\psi : L \to L'$, satisfying $\varphi_\alpha\psi = \varphi'_\alpha$, and a homomorphism $\psi' : L' \to L$, satisfying $\varphi'_\alpha\psi' = \varphi_\alpha$, for $\alpha \in \Sigma$. Therefore, both $\psi\psi'$ and $\psi'\psi$ must be identity maps on all $L_\alpha\varphi_a$ and $L_\alpha\varphi'_a$, respectively. It follows from (16.4) that $\psi\psi'$ and $\psi'\psi$ must be identity maps on L and L', respectively. Therefore, ψ is an isomorphism, proving the statement.

PROPOSITION 17. *The concepts of multipasting and* Σ*-pasting are naturally equivalent.*

Proof. To see the "equivalence" of Definitions 6 and 16, we start with a lattice L that is a multipasting of its sublattices L_λ, $\lambda \in \Lambda$. Let Σ be the free meet-semilattice generated by Λ. An element α of Σ is of the form $\alpha = \lambda_1 \wedge \cdots \wedge \lambda_n$; define $L_\alpha = L_{\lambda_1} \cap \cdots \cap L_{\lambda_n}$. We consider L_α, $\alpha \in \Sigma$, a Σ-system with the natural embeddings as the $\varphi_{\alpha,\beta}$ and the φ_α. Conditions (15.1)–(15.5) are obvious. Conditions (16.1)–(16.4) hold, so to show that L is a Σ-pasting of L_α, $\alpha \in \Sigma$, it is sufficient to check (16.5). Let K be a lattice and let ψ_α be a homomorphism of L_α into K, for $\alpha \in \Sigma$, as required by (16.5). Since, for $\alpha, \beta \in \Sigma$, we have $\varphi_{\alpha\wedge\beta}\psi_\alpha = \psi_{a\wedge\beta}$, it follows that $\psi = \bigcup(\psi_\alpha \mid \alpha \in \Sigma)$ is a map of L into K, and ψ obviously satisfies the assumptions of Definition 6 for ψ. The homomorphism required in (16.5) is now provided by Definition 6.

Conversely, let the lattice L be a Σ-pasting of the Σ-system L_α, $\alpha \in \Sigma$. Let $\Lambda = \{\alpha \mid \alpha \in \Sigma \text{ and } L_\alpha \ne \varnothing\}$. For $\alpha \in \Lambda$, define $A_\alpha = L_\alpha\varphi_\alpha$. By (15.1) and (16.1), the A_α are sublattices of L, and the A_α, $\alpha \in \Lambda$, form a compatible cover of L, by (16.4). Let the map ψ be given as in Definition 6. Then we can apply (16.5) to the homomorphisms $\psi_\lambda : A_\lambda \to K$, and conclude that ψ is a homomorphism of L into K. Thus, L is a multipasting of the A_α, $\alpha \in \Lambda$.

5. Strong multipasting

In this section, we consider only lattices of finite length. In this class of lattices, we introduce a stronger version of multipasting for which the analogue of the Day and Ježek characterization theorem of pasting holds (see §2). In fact, we define this new concept via the characterization theorem of pasting, and relate it then to multipasting.

We start with a lattice L *of finite length* and a cover of L by the sublattices L_λ, $\lambda \in \Lambda$.

We generalize (P) (of §2) according to Definition 6 as follows:

(Ord) For $a, b \in L$, let $a \leq b$ iff there exists a sequence $a = s_0, s_1, \ldots, s_n = b$ of elements of L such that, for each i with $0 \leq i < n$, there is a $\lambda_i \in \Lambda$ satisfying $s_i, s_{i+1} \in L_{\lambda_i}$ and $s_i \leq s_{i+1}$ in L_{λ_i}.

Note that (Ord) simply means that the cover L_λ, $\lambda \in \Lambda$, of L is full.

We rewrite the covering condition (Cov) in the following way:

(Cov_2) Any two upper covers of an element $a \in L$ belong to some L_λ, $\lambda \in \Lambda$; and dually for lower covers.

DEFINITION 18 (Strong multipasting). *Let L be a lattice of finite length, and let L_λ, $\lambda \in \Lambda$, be a cover of L. Then L is the strong multipasting of its sublattices L_λ, $\lambda \in \Lambda$, iff the conditions* (Ord) *and* (Cov_2) *are satisfied.*

To prove that this concept is stronger than multipasting, we need a property of lattice maps:

PROPOSITION 19. *Let L be a lattice of finite length, let K be any lattice, and let $\psi : L \to K$ be an order preserving map. If ψ preserves the join for any two upper covers of an element of L, then ψ is a join-homomorphism.*

Proof. Let ψ be given satisfying the assumptions of the proposition. We proceed by induction on the length of L. For lattices with less than four elements the statement is trivial.

Let L have length n, $n > 1$, and let the statement be true for any L having length n. Note that the restriction of ψ to any proper interval of L satisfies the assumption of the proposition, so by the induction hypotheses, ψ is a homomorphism on any proper interval.

Let $a, b \in L$; we want to prove that $(a \vee b)\psi = a\psi \vee b\psi$. This is obvious if a and b are comparable. So let a and b be incomparable.

Let us choose atoms p in the interval $[a \wedge b, a]$ and q in the interval $[a \wedge b, b]$; set $r = p \vee q$. Then $r\psi = p\psi \vee q\psi$ by assumption. Define $c = a \vee q$ $(= a \vee r)$ and $d = p \vee b$ $(= r \vee b)$. By the induction hypotheses, $(x \vee y)\psi = x\psi \vee y\psi$ if $p \leq x, y$ or if $q \leq x, y$. Then compute:

$$\begin{aligned}
(a \vee b)\psi &= (c \vee d)\psi = c\psi \vee d\psi \qquad (\text{since } p \leq c, d),\\
&= (a \vee r)\psi \vee (r \vee b)\psi = a\psi \vee r\psi \vee b\psi \qquad (\text{since } p \leq a, r, \text{ and } q \leq b, r),\\
&= a\psi \vee (p \vee q)\psi \vee b\psi \qquad (\text{since } r\psi = p\psi \vee q\psi),\\
&= a\psi \vee p\psi \vee q\psi \vee b\psi = a\psi \vee b\psi,
\end{aligned}$$

proving the proposition.

Now, we are ready to prove

THEOREM 20. *Let L be a lattice of finite length. Every strong multipasting of L is a multipasting, but there are multipastings that are not strong multipastings.*

Proof. Let the lattice L of finite length be the strong multipasting of its sublattices L_λ, $\lambda \in \Lambda$. Consider $P = \operatorname{Part} \Lambda$. The sets P and L coincide. By (Ord), they are equal as partially ordered sets. By Theorem 13, we only have to show that Part Λ satisfies (13.2). To this end, consider the map $\psi : L \to \operatorname{Id} \operatorname{Part} \Lambda$ defined by $x\psi = (x]_\Lambda$. By Condition (Cov_2), ψ preserves the join of any two covers of an element in L. Using Proposition 19, we get that ψ is join-preserving. The duality of the assumptions and the dual of Proposition 19 yield that ψ is meet-preserving. Hence, every two-generated ideal or dual ideal is principal. An obvious induction completes the proof.

Consider the lattice L of Figure 3, and let L_λ, $\lambda \in \Lambda$, be the following sublattices of L:

$$L_1 = \{a, b, c, d\}, \qquad L_2 = \{x, y, u, v\},$$

and the following intervals (considered as sublattices) of L:

$$[b, a \vee u], \quad [b, c \vee u], \quad [x \wedge d, v], \quad [y \wedge d, v].$$

These sublattices form a full cover of L. Condition (13.1) is obvious. To verify Condition (13.2), the only nontrivial case is the ideal (and the dual ideal) I generated by $\{a \vee u, c \vee u\}$. It is easy to see that I is generated by d; indeed, d is an

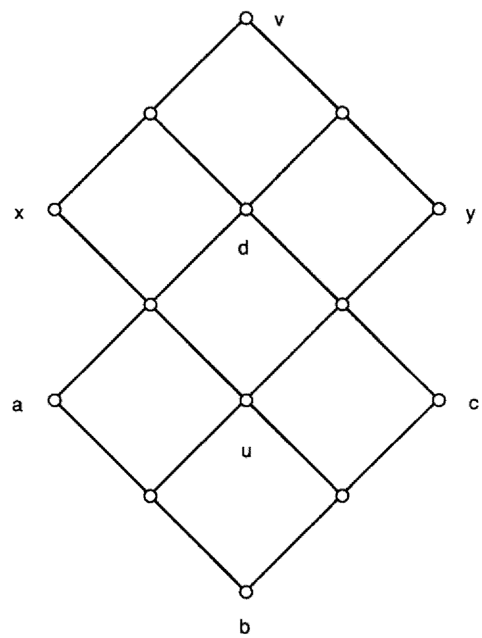

Figure 3

upper bound of $\{a \vee u, c \vee u\}$ and d belongs to I because $a \leq a \vee u$ and $c \leq c \vee u$ are in I and, in L_1, $a \vee c = d$. However, condition (Cov_2) is not satisfied, because the two upper covers of u do not belong to any one of the given sublattices.

The following is obvious without reference to Theorem 20 (and the remark concerning $|\Lambda| = 2$ following Definition 6):

PROPOSITION 21. *A strong multipasting of two lattices of finite length is a pasting of the two lattices.*

Proof. Indeed, for two lattices, Condition (Cov_2) is the same as (Cov).

An important tool used for pasting in E. Fried and G. Grätzer [5] holds true for strong multipasting:

PROPOSITION 22. *Let L be a lattice of finite length. Let L be a strong multipasting of its sublattices L_λ, $\lambda \in \Lambda$, and let K be a convex sublattice of L. Then K is the strong multipasting of $K \cap L_\lambda \neq \varnothing$, $\lambda \in \Lambda$.*

Proof. Both conditions, (Ord) and (Cov_2), remain true in K.

The most important positive result proved for pasting (namely that the variety of modular and the variety of distributive lattices are closed under pasting) trivially fails even for strong multipasting. Let N_5 denote the five-element nonmodular lattice with elements $\{0, 1, a, b, c\}$, where $a < b$, 0 is the zero, and 1 is the unit. Consider the following three sublattices: $\{a, b\}$, $\{0, 1, a, c\}$, $\{0, 1, b, c\}$. Both Conditions (Ord) and (Cov_2) are satisfied. Still, the sublattices are distributive and the result is not even modular.

6. Multipasting and S-glued sum

In this section, we prove that multipasting and strong multipasting both generalize S-glued sum.

THEOREM 23. *Let L be a lattice of finite length. An S-glued sum of L is a strong multipasting of L but not conversely.*

Proof. Suppose L is the S-glued sum of the lattices L_s, $s \in S$. Then, clearly, $L = \bigcup (L_s \mid s \in S)$. By Definition 4, $L_s, s \in S$, is a full cover. So we only have to prove (Cov_2).

Let b and c be two upper covers of a in L. Then, by the remark (e) following Definition 4, there are $s, t \in S$, such that $a, b \in L_s$ and $a, c \in L_t$. Now $a \in L_s \cap L_t$ implies, by (3.4), that $a \in L_{s \vee t}$. Thus, $a \in L_s \cap L_{s \vee t}$. By (3.1), $s \leq s \vee t$ implies that b belongs to the dual ideal $L_s \cap L_{s \vee t}$ of L_s. Hence, $b \in L_{s \vee t}$. By symmetry, $c \in L_{s \vee t}$, verifying the first clause of (Cov_2). By duality, we conclude (Cov_2).

Now we give an example of a strong multipasting that is not an S-glued sum. Consider M_3 the five-element modular nondistributive lattice, with the elements $\{0, 1, a, b, c\}$. M_3 is a strong multipasting of the three sublattices: $\{0, 1, a, b\}$, $\{0, 1, a, c\}$, $\{0, 1, b, c\}$. However, it is not an S-glued sum, since these sublattices are not intervals in M_3.

Thus, for finite lattices, strong multipasting is a common generalization of pasting and S-glued sum. Let us remark that most results proved for pasting of finite lattices also hold for lattices of finite length.

7. Multipasting and convex sublattices

It may seem that multipasting is not as well-behaved as S-glued sum because the components need not be convex sublattices. In this section, we shall investigate multipasting of convex sublattices. First of all, we shall show that

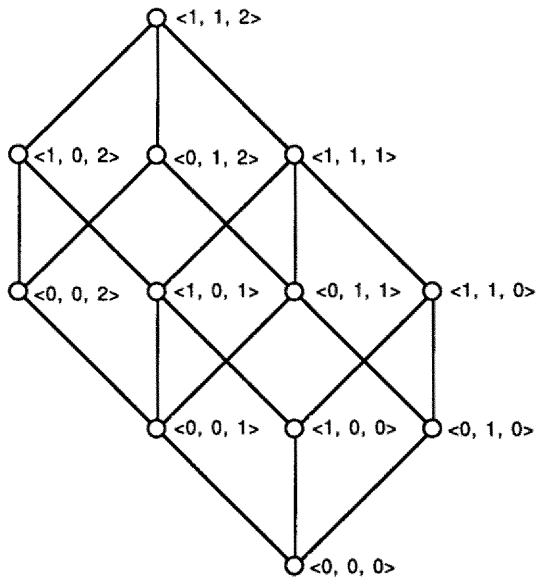

Figure 4

PROPOSITION 24. *For lattices of finite length, a multipasting of convex sublattices need not be a strong multipasting.*

Proof. We start with the direct product $L = C_2 \times C_2 \times C_3$ of two two-element chains and a three-element chain, see Figure 4. Take all convex four-element sublattices of length two, except the sublattice

$$\{\langle 0, 0, 1\rangle, \langle 1, 0, 1\rangle, \langle 0, 1, 1\rangle, \langle 1, 1, 1\rangle\}.$$

They form a full cover of L. Since L is selfdual, to show that they form a multipasting it is enough to consider ideals. There is only one nontrivial case: the ideal I generated by $\langle 0, 0, 1\rangle$ and $\langle 0, 1, 1\rangle$. Since $\langle 1, 1, 1\rangle$ is the least upper bound of $\langle 1, 0, 1\rangle$ and $\langle 0, 1, 1\rangle$, to prove that I is principal, it is enough to show that $\langle 1, 1, 1\rangle$ is in I. Since $\langle 1, 0, 0\rangle < \langle 1, 0, 1\rangle$ and $\langle 0, 1, 0\rangle < \langle 0, 1, 1\rangle$, it follows that $\langle 1, 1, 0\rangle = \langle 1, 0, 0\rangle \vee \langle 0, 1, 0\rangle \in I$; and therefore, $\langle 1, 1, 1\rangle = \langle 0, 1, 1\rangle \vee \langle 1, 1, 0\rangle \in I$ (all the joins are taken in the structure Part Λ).

However, this multpasting is not strong; indeed, (Cov_2) is not satisfied, since the two upper covers $\langle 1, 0, 1\rangle$ and $\langle 0, 1, 1\rangle$ of $\langle 0, 0, 1\rangle$ are not contained in any one of the components (the four-element sublattice generated by them is not a component).

The next result shows that, even for finite modular lattices, multipasting of convex sublattices can construct lattices that cannot be obtained by pasting.

THEOREM 25. *There exists a finite modular lattice that is a strong multipasting of more than two of its convex proper sublattices, but which is not a pasting of two of its proper sublattices.*

We shall prove this theorem in the following stronger form:

THEOREM 25′. *For every integer $m \geq 2$, there exists a finite modular lattice that can be represented as an S-glued sum of more than m proper blocks, but which is not a strong multipasting of k of its proper sublattices*, for *any $k \leq m$.*

Proof. Our construction is a simple special case of a construction in Ch. Herrmann [10].

Let S be a finite projective geometry (i.e., a finite complemented modular lattice). Let L be the sublattice of S^2 consisting of all pairs $\langle x, y\rangle$ with $x \leq y$. Obviously, L is a finite modular lattice.

For $s \in S$, let L_s be the interval $[\langle 0, s\rangle, \langle s, 1\rangle]$ of L. Observe that L_s is a complemented sublattice of L. Indeed, if $\langle x, y\rangle \in L_s$, then let u be the relative complement of x in $[0, s]$, and let v be the relative complement of y in $[s, 1]$. Since $u \leq s$ and $s \leq v$, it follows that $u \leq v$; hence, $\langle u, v\rangle \in L$, and $\langle u, v\rangle$ is the complement of $\langle x, y\rangle$ in L_s.

Next we prove that L_s is a *maximal* complemented interval of L. Indeed, let $[\langle 0, s\rangle, \langle s, 1\rangle] \subset [\langle u_1, u_2\rangle, \langle v_1, v_2\rangle]$ where $[\langle u_1, u_2\rangle, \langle v_1, v_2\rangle]$ is a complemented sublattice of L; by duality, we can assume that $\langle u_1, u_2\rangle = \langle 0, s\rangle$ and $\langle s, 1\rangle < \langle v_1, v_2\rangle$, that is, $[\langle u_1, u_2\rangle, \langle v_1, v_2\rangle] = \{\langle 0, s\rangle, \langle t, 1\rangle]$ where $s < t$. The interval $[\langle 0, s\rangle, \langle t, 1\rangle]$ contains the element $\langle 0, t\rangle$; its complement in $\{\langle 0, s\rangle, \langle t, 1\rangle]$ must be of the form $\langle t, r\rangle$, where r is a complement of t in $[s, 1]$. But $\langle t, r\rangle \in L$ and so $t \leq r$, contradicting that r is a complement of t in $[s, 1]$.

By Ch. Herrmann [10], every modular lattice of finite length is an S-glued sum of its maximal complemented intervals; we conclude that L is an S-glued sum of $L_s = [\langle 0, s\rangle, \langle s, 1\rangle]$, $s \in S$.

Now choose an integer n: let n be prime with $m < n$. Let S be a projective plane over a finite field of n elements. The lattice L constructed above is an S-glued sum of $2n^2 + 2n + 4 > m$ blocks.

The projective geometry S as a geometry is generated by any $n + 2$ points, since the lines have $n + 1$ points and S is planar. S as a lattice is generated by any $n + 3$ atoms. Indeed, let S' be the sublattice generated by $n + 3$ atoms. Since the unit element of S' is, by definition, a join of atoms, therefore, S' is a complemented modular lattice, that is, a projective geometry. If S' is directly reducible, then it is a direct product of a line and a point. Since a line contains at most $n+1$ atoms, therefore, S' has at most $n+2$ atoms, contrary to the assumption. We conclude that S' is a directly irreducible projective plane. Moreover, S' is arguesian since S is. Therefore, S' can be coordinatized with a subfield of $GF(n)$. By assumption, n is prime; therefore, $GF(n)$ has no proper subfield. Thus $S' = S$, as claimed.

Assume that L is a strong multipasting of the proper sublattices $L_1, \ldots, L_k$, $k \leq m\ (<n)$. Let p be an atom of S. Then $\langle 0, p\rangle \prec \langle p, p\rangle$ in L. By Condition (Ord), there is an L_i such that $\langle 0, p\rangle$, $\langle p, p\rangle \in L_i$. Since there are at most $m < n$ sublattices L_i, and there are n^2+n+1 atoms, there must be a sublattice, say L_1, such that $\langle 0, p\rangle, \langle p, p\rangle \in L_1$ for $n+3$ atoms p; otherwise, we obtain that $m(n+2) \geq n^2+n+1$ and $(n-1)(n+2) \geq m(n+2)$, that is, $(n-1)(n+2) \geq n^2+n+1$, a contradiction.

As we proved in the last but one paragraph, the lattice S is generated by any set of $n+3$ atoms, hence $\langle 0, p\rangle, \langle p, p\rangle \in L_1$ for all atoms p of S. Thus $\langle 0, x\rangle, \langle x, x\rangle \in L_1$ for all x in S. But if $\langle x, y\rangle \in L$, then $x \leq y$; therefore, $\langle x, y\rangle = \langle 0, y\rangle \vee \langle x, x\rangle \in L_1$, and so $L = L_1$, a contradiction.

B. Ganter suggested that we look at this proof with S the projective plane over the rational field. The resulting modular lattice L of finite length is a strong multipasting (in fact, S-glued sum) of the infinitely many convex sublattices L_s, $s \in S$, but L is not the strong multipasting of finitely many proper sublattices. Indeed, let us assume that L is the strong multipasting of the lattices $L_1, \ldots, L_n$. Choose an infinite set C of points in S such that no three elements of C are collinear. Then there must be a lattice L_i containing infinitely many of the elements $\langle 0, p\rangle$, $p \in C$. Since any four of these atoms generate all of S, we must have $L_i = L$, a contradiction. Thus we proved the following result:

THEOREM 25″. *There exists a modular lattice of finite length that is a strong multipasting of infinitely many of its convex proper sublattices, but which is not a strong multipasting of finitely many of its proper sublattices.*

8. Interpolation property

For pasting, Condition (P) (of §2) expresses the fact that $A \cap B$ is properly placed between A and B. We are looking for a similar property of strong multipasting that would make it possible to prove that modularity is preserved.

DEFINITION 26. *Let L be a lattice of finite length which is a multipasting of L_λ, $\lambda \in \Lambda$. We call this a multipasting with the* Upper Interpolation Property *iff it is a strong multipasting and the following condition holds*:

(UIP) *Let $a, b, c \in L$ and $\mu, \nu \in \Lambda$ satisfy*

$$a \leq b \leq c \text{ in } L, \quad a \in L_\mu, \quad b \in L_\nu, \quad c \in L_\mu \cap L_\nu, \ \mu \neq \nu;$$

then there exists a $d \in L_\mu \cap L_\nu$ with $a \leq d \leq b$.

A multipasting has the Interpolation Property *iff it is strong and both* (UIP) *and the dual of* (UIP) *hold.*

Observe that if all L_λ, $\lambda \in \Lambda$, are convex sublattices of L, then (UIP) trivially holds (with $d = b$). Therefore, the results of this section hold for convex multipastings.

The following theorem generalizes the main result of E. Fried and G. Grätzer [4]. Our proof is patterned after E. T. Schmidt [15].

THEOREM 27. *The class of all finite semimodular lattices is closed under multipasting with the Upper Interpolation Property.*

Proof. Let L be a finite lattice that is a multipasting with the Upper Interpolation Property of the semimodular lattices L_λ, $\lambda \in \Lambda$. Let $a, b \in L$ satisfy $a \neq b$ and $a \wedge b \prec a, b$. We wish to show that $a, b \prec a \vee b$. Since a and b both cover $a \wedge b$ and L is a strong multipasting of L_λ, $\lambda \in \Lambda$, by Condition (Cov_2), there is a $\mu \in \Lambda$ such that $a, b \in L_\mu$; obviously, $a \vee b \in L_\mu$.

By way of contradiction, assume that the semimodularity fails in L; without loss of generality we can assume that $b \prec a \vee b$ fails in L, that is, that there exists a $c \in L$ such that $b < c < a \vee b$; since L is finite, we can choose c so that $b < c \prec a \vee b$. From the semimodularity of L_μ, it follows that $c \notin L_\mu$. Choose $d \in L$ so that $a \leq d \prec a \vee b$. By Condition (Cov_2), there exists a $\nu \in \Lambda$ such that $c, d \in L_\nu$. Obviously, $a \vee b \in L_\nu$ since $c \vee d = a \vee b$.

Since $a \leq d \leq a \vee b$ in L, $a \in L_\mu$, $d \in L_\nu$, and $a \vee b \in L_\mu \cap L_\nu$, by (UIP), we conclude that there exists an $s \in L_\mu \cap L_\nu$ such that $a \leq s \leq d$. Observe that $a < s$ yields a contradiction with the semimodularity of L_μ since $a, b, a \wedge b, s, a \vee b \in L_\mu$, $a \wedge b \prec a, b \in L_\mu$, and $a < s < a \vee b$ in L_μ. Therefore, $a = s$, and so $a \in L_\nu$.

Similarly, $b \leq c \leq a \vee b$ in L, $b \in L_\mu$, $c \in L_\nu$, and $a \vee b \in L_\mu \cap L_\nu$, by (UIP), we conclude that there exists an $s \in L_\mu \cap L_\nu$ such that $b \leq s \leq c$. Again, $b < s$ contradicts the semimodularity of L_μ, so we conclude that $b = s$, and so $b \in L_\nu$.

But then $a, b, a \wedge b, c, a \vee b \in L_\nu$, a contradiction with the semimodularity of L_ν, completing the proof of the theorem.

COROLLARY 28. *The class of all finite modular lattices is closed under multipasting with the Interpolation Property.*

Let us remark that the above result does not hold for distributive lattices. Indeed, the three four-elements sublattices of M_3 form a multipasting with the Interpolation Property.

9. Concluding remarks

We have elaborated a number of variants on the theme of multipasting. Our discussions are far from complete. To illustrate this, we state the following variant on Theorem 23:

THEOREM 29. *An S-glued sum L is a multipasting satisfying the following condition*:

(Cov_∞) *For any element $a \in L$, all upper covers of a belong to some components*; *and dually for lower covers*;

but not conversely.

Proof. Let the lattice L be an S-glued sum. For $a \in L$, Let $L_s = [x_s, y_s]$, $s \in S'$, be all the blocks containing a. Define $u_a = \bigvee (x_s \mid s \in S')$ and $v_a = \bigwedge (x_s \mid s \in S')$. By note (f) following Definition 4 and by (3.4), the block L_{u_a} is of the form $[u_a, y]$, and its contains a. Now let $a \prec b$ in L. Then by (3.3), there is a block L_t containing both a and b. Since $t \leq u_a$, by (3.1), $L_t \cap L_{u_a}$ is a dual ideal of L_t; since this dual ideal contains a, it also contains b. Thus all such b are contained in L_{u_a}.

A similar argument shows that the block L_{v_a} also contains a. This is the block containing all the elements covered by a.

To show that the converse does not hold, take the three-element chain $\mathfrak{C}_3$ with elements $0, 1, 2$, and define L as $\mathfrak{C}_3^2$ with the element $\langle 0, 1\rangle$ omitted. Then the sublattices

$$\{\langle 0, 2\rangle, \langle 1, 0\rangle, \langle 1, 1\rangle, \langle 1, 2\rangle\},$$

$$\{\langle 1, 0\rangle, \langle 1, 1\rangle, \langle 2, 0\rangle, \langle 2, 1\rangle\}, \quad \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle\}$$

form a multipasting (of convex sublattices); but this is not an S-glued sum (for instance, (3.1) fails).

Thus one could define a sharper form of strong multipasting, requiring (Cov_∞) rather than (Cov_2). Even with the additional hypothesis that the components be convex sublattices, this construction is much weaker than S-glued sum.

It is an interesting question what happens if we do not assume in Definition 6 that the cover L_λ, $\lambda \in \Lambda$, of L be compatible? How would the new definition relate to the old one and to the other variants of multipasting we have studied?

Multipasting with the Interpolation Property does solve the problem of preserving modularity in the finite case. It would be of interest to generalize the Interpolation Property to infinite lattices so that it preserve modularity.

Finally, we raise the question, what is the best form of Theorem 25′? For every integer $m \geq 2$, find the smallest integer $u(m)$ such that there exists a finite modular lattice that can be represented as an S-glued sum of $u(m)$ proper blocks, but which is not a strong multipasting of m of its proper sublattices. For instance, what is $u(2)$? Our proof yields $u(m) \leq 16m^2 + 36m + 24$.

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