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Pasting and semimodular lattices

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Pasting is a special lattice construction (see A. Slavík [5] and G. Grätzer [4], Exercise 12 of Section V.4), more general than gluing but less general than amalgamation. Let L be a lattice. Let A, B, and S be sublattices of L, $A \cap B = S$, and $A \cup B = L$. Then L *pastes* A and B together over S, if every amalgamation of A and B over S contains L as a sublattice. (For a more formal definition, see E. Fried and G. Grätzer [2], [3].) A. Day and J. Ježek [1] (see also A. Slavík [5]) gave the following characterization of pasting:

LEMMA. *Let A, B, and S be sublattices of the finite lattice L, $A \cap B = S$, and $A \cup B = L$. Then L pastes A and B together over S if and only if the following two conditions hold*:

(1) *For $a \in A$ and $b \in B$, if $a < b$, then there exists an $s \in S$ satisfying $a \leq s \leq b$; and dually.*

(2) *For $s \in S$, all the covers of s in L are in A or all are in B; and dually.*

Note that Conditions (1) and (2) imply the following stronger form of Condition (2):

(2′) For $x \in L$, all the covers of x in L are in A or all are in B; and dually.

We prove the following:

THEOREM. *The class of all finite semimodular lattices is closed under pasting of finite lattices.*

Proof. We have to prove that $a \neq b$ and $a \wedge b \prec a, b$ imply that $a, b \prec a \vee b$. Since a and b are covers of $a \wedge b$, therefore by (2′), both a and b are in A or both

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are in B; we may assume that $a, b \in A$. The case $a, b \in A \cap B = S$ is trivial, hence, we can assume that a or b belongs to $A - B$.

By way of contradiction, assume that $b \prec a \vee b$ fails, that is, that there exists a $c \in L$ such that $b < c \prec a \vee b$. Since A is semimodular, it follows that $c \in B - A$.

Let $d \in [a, a \vee b]$ be covered by $a \vee b$. By condition (2′), $c \in B - A$ implies that $d \in B$. if $a < d$, then by the semimodularity of A, we have that $d \in B - A$, and by (1) there exist elements $s_a, s_b \in S = A \cap B$ such that $a \leq s_a < d$ and $b \leq s_b < c$. Since now the conditions are symmetric in a and b, we can assume that $b \in A - B$. This implies $a \vee b > s_b > b$ and $a, b, s_b \in A$, contradicting the semimodularity of A. Finally, if $a = d$, then $a \in A \cap B$. This implies that $b \in A - B$, and again by (1), there exists an $s \in S$ with $b < s < c$, contradicting the semimodularity of A.

COROLLARY (E. Fried and G. Grätzer [2]). *The variety of all modular lattices is closed under the pasting of finite lattices.*

In E. Fried and G. Grätzer [3], this Corollary is proved for infinite lattices. It would be interesting to see whether the method of this note could be extended to cover the infinite case.

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