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# COVER-PRESERVING EMBEDDING

E. T. SCHMIDT* Budapest

## 1. Introduction

A finite lattice $K$ has the cover-preserving embedding property, abbreviated as CPEP with respect a variety $V$ of lattices, if whenver $K$ can be embedded into a finite lattice $L$ in $V$, then $K$ has a cover-preserving embedding into $L$, that is an embedding $\varphi$ with the property that if $a$ covers $b$ in $K$ then $\varphi(a)$ covers $\varphi(b)$ in $L$. This concept was introduced by E. Fried, G. Grätzer and H. Lakser in [2], and it was proved that a finite projective geometry $P$ (i.e. a simple complemented modular lattice) has the cover-preserving embedding property with respect to the variety $M$ of all modular lattices if and only if one of the following three conditions hold:

(i) the length of $P$ is 1;
(ii) the length of $P$ is 2 and $P$ is isomorphic to $M_3$;
(iii) the length of $P$ is greater than 2 and either $P$ is non-arguesian or $P$ is arguesian and for some prime $p$, each interval of $P$ of length 2 contains $p+1$ atoms.

In this note we prove the following:

THEOREM 1.. *If a finite modular lattice $L$ has the CPEP with respect to $M$ then $L$ is the subdirect product of projective geometries of type (i)-(iii).*

## 2. Preliminaries

All lattices considered in this paper are finite modular lattices. First we need some definitions (see G. Grätzer [3]). We shall say that $a/b$ is perspective to $c/d$, and write $a/b \sim c/d$ if either $b = a \wedge d, c = a \vee d$ or $a = b \vee c, d = b \wedge c$. In the first case we write $a/b \nearrow c/d$ and we write $a/b \searrow c/d$ in the second case. If for some natural number $n$ there exist $a/b = e_0/f_0 \sim e_1/f_1 \sim \cdots \sim e_n/f_n = c/d$, we shall say that $a/b$ is projective to $c/d$. This will be denoted by $a/b \approx c/d$. We write $c/d \searrow_w a/b$ iff $b \leq d$ and $c = a \vee d$; similarly, $c/d \nearrow_w a/b$ iff $c \leq a$ and $d = b \wedge c$. If $c/d \nearrow_w a/b$ or $c/d \searrow_w a/b$, then $c/d$ is weakly perspective into $a/b$, in symbols $c/d \sim_w a/b$. If for some natural number $n$ and $c/d = e_0/f_0, e_1/f_1, \ldots, e_n/f_n = a/b$ we have $e_i/f_i \sim_w e_{i+1}/f_{i+1}, i = 0, \ldots, n-1$ then $c/d$ is weakly projective into $a/b$, in symbol, $c/d \approx_w a/b$.

The well-known Hall-Dilworth gluing of a filter and ideal was extended in [4] by Ch. Herrmann. This construction is called the $S$-glued sum, where $S$ is a lattice of finite length. The definition is based on the following concept. Let $S$ be a lattice.

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An $S$-connected system is an $S$-indexed system of bounded lattices $(L_x; x \in S)$ together with lattice isomorphisms $\gamma_{xy}$ for each $x \leq y$ in $S$ satisfying

(1) $\gamma_{xy}$ has as domain, a (possibly empty) filter $G_{yx} \subseteq L_x$ and as codomain, a (possibly empty) ideal $J_{xy} \subseteq L_y$;
(2) $\gamma_{xx} = 1_{L_x}$, and for $x \leq y \leq z$ $\gamma_{yz}\gamma_{xy} = \gamma_{xz}$;
(3) $G_{x,x\wedge y} \cap G_{y,x\wedge y} \subseteq G_{x\vee y,x\wedge y}$ and $J_{x,x\vee y} \cap J_{y,x\vee y} \subseteq J_{x\wedge y,x\vee y}$ for all $x, y \in S$;
(4) for all $x \leq y$ in $S$ there exists a sequence $x = z_0 \leq \cdots \leq z_n = y$ with $G_{z_i,z_{i+1}} \neq 0$.

Let $L$ be a lattice, and let $(L_x; x \in S)$ be a system of sublattices of $L$ such that $(L_x; x \in S)$ is an $S$-connected system, where for $x \leq y$, $L_x \cap L_y = G_{yx} \subseteq L_x, L_x \cap L_y = J_{xy} \subseteq L_y$ and $\gamma_{xy}$ is the identy map on $L_x \cap L_y$. We say that $L$ is the $S$-glued sum of $L_s$, $s \in S$. If $(L_x; x \in S)$ is an $S$-connected system then $L = \cup L_s$ is a lattice, the $S$-glued sum of $L_s$. A maximal complemented interval of a finite modular lattice is called a block. The minimal elements of the blocks form a lattice $S$, we may assume that the blocks $B_x$ are indexed by its minimal elements, i.e. $x \in S$. The main result of [4] asserts that a finite (or of finite length) modular lattice is the $S$-glued sum of the blocks $B_x$.

Let $L$ be a lattice. $L^{(2)}$ is the set of all pairs $(t_1, t_2)$ such that $t_1, t_2 \in L$ and $t_1 \leq t_2$. $L^{(2)}$ is a sublattice of $L^2$, it is a subdirect power of $L$. If $t \in L$ then $t \to \tilde{t} = (t,t)$ is the canonical embedding of $L$ into $L^{(2)}$. If $[b, a]$ is a prime interval of $L$ then the corresponding interval $[\tilde{b}, \tilde{a}]$ of $L^{(2)}$ is the three element chain. A maximal irreducible, complemented interval is called a *projective block*.

The length of an interval $[b, a]$ will be denoted by $\ell[b, a]$. A cover-preserving sublattice is a sublattice for wich the identy map is a cover-preserving embedding.

LEMMA 2.. *Let $P$ be a finite non-distributive projective geometry. An interval $[v, u]$ of $P^{(2)}$ is a projective block if and only if*

$$[v, u] = \begin{cases} [(0, s), (s, s)] & \text{where } \ell[0, s] > 1 \text{ in } P, \text{ or} \\ [(t, t), (t, 1)] & \text{where } \ell[t, 1] > 1 \text{ in } P. \end{cases}$$

PROOF.. The mapping $\varphi_1 : x \to (0, x)$ is of course a cover-preserving embedding of $P$ into $P^{(2)}$ and $\varphi_1(P)$ is the interval $[(0,0),(0,1)]$, i.e. this interval is isomorphic to P. $(t,t) \wedge (0,1) = (0,t)$ and $(t,t) \vee (0,1) = (t,1)$ imply that $[(t,t),(t,1)] \sim [(0,t),(0,1)]$. Since $[(0,t),(0,1)]$ is an interval of $\varphi_1(P)$ of length greater than 1, it is a non distributive projective geometry. Similarly, $\varphi_2 :\to (x, 1)$ is an embedding of $P$ into $P^{(2)}$ and $\varphi_2(P) = [(0,1),(1,1)]$. Then $[(0,s),(s,s)]$ is weakly perspective into $[(0,1),(1,1)]$, i.e. it is a projective geometry. Let $[v, u]$ be a non-distributive projective geometry, $P^{(2)}$ is the subdirect square of $P$ and $[v, u]$ is subdirect irreducible, i.e. if $v = (v_1, v_2)$, $u = (u_1, u_2)$ $(v_1 \leq v_2, u_1 \leq u_2)$ then either $v_1 = u_1$ or $v_2 = u_2$. Then the maximal irreducible complemented intervals which contains $[v, u]$ are just the intervals given in the lemma. ■

LEMMA 3.. *Let $L$ be the $S$-glued sum of the sublattices $L_s$, $s \in S$. If the interval $[v, u]$ of $L$ is a projective geometry then there exists an $L_s$ such that $[v, u] \subseteq L_s$.*

PROOF.. By the definition of the $S$-glued sum $v \leq u$ iff there exist a sequence $v = x_0, x_1, \ldots, x_n = u$ of elements of $L$ and a sequence $s_0, \ldots, s_n$ of elements of $S$ such that $s_i \leq s_{i+1}$ in $S$ and $x_{i-1} \leq x_i$ in $L_{s_i}, i = 1, 2, \ldots, n$. Then $L_{s_i} \cup L_{s_{i+1}}$ is a sublattice of $L$, the Hall-Dilworth gluing of $L_{s_i}$ and $L_{s_{i+1}}$. Obviously, $[v, u] \cap (L_{s_i} \cup L_{s_{i+1}})$ is a projective geometry. Therefore it is enough to prove that projective geometries are indecomposable with respect the Hall-Dilworth gluing. Let $I$ be a proper ideal of the projective geometry $P$. Then $P$ has an atom $p$ which is not contained in $I$. Let $r$ be an arbitrary atom from $I$. Then exists a third atom $q$ such that $p \vee q = p \vee r = q \vee r$. $q \in I$ would imply that $p \leq q \vee r \in I$ in contradiction to $p \notin I$. Consequently $p, q \in I$. Now, if $F$ is a filter such that $P = I \cup F$ then $p, q \in F$ and therefore $0 = p \wedge q \in F$, i.e. $F = P$. This proves that $P$ is indecomposable. ∎

Finally, we need a lemma from G. Grätzer [3] (see p. 166).

LEMMA 4.. *Let $x_0/y_0 \nearrow x_1/y_1 \searrow x_2/y_2$. Then the sublattice generated by $x_0$, $x_1$, $x_2$, $y_0$, $y_1$, $y_2$ is isomorphic to one of the lattices of Figure 1, Figure 2, or Figure 3.*

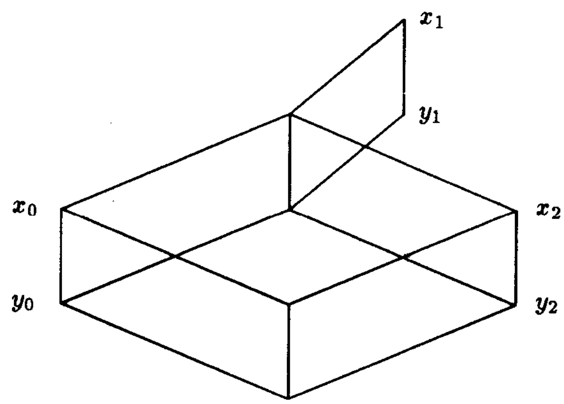


*Figure 1*

## A necessary condition

In this section we prove the following

THEOREM 5.. *If the intervals $[b, a]$ and $[d, c]$ are projective blocks of $L$ and $b < b \vee c < a, d < b \wedge c < c$ then the CPEP fails for L.*

PROOF.. Assume that $P_1 = [b, a]$ and $P_2 = [d, c]$ are projective blocks and $b < b \vee c < a, d < b \wedge c < c$. Under these circumstances we use the symbol $P_1 \succ P_2$ and define the distance of $P_1$ and $P_2$ as follows:

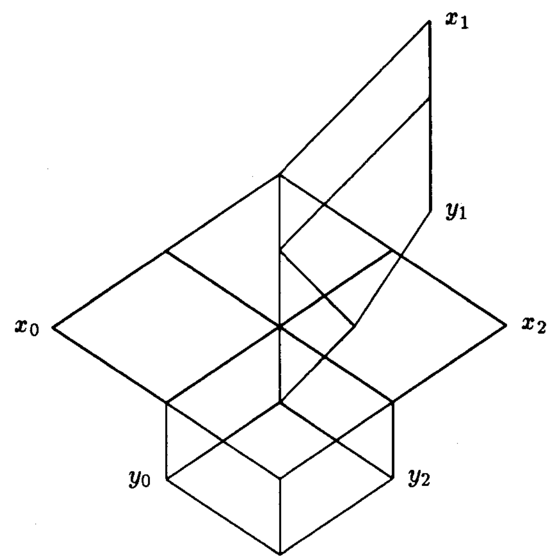


*Figure 2*

$$\partial(P_1, P_2) = \ell[c, b \vee c](= \ell[b \wedge c, b]).$$

We may assume that the distance $\partial(P_1, P_2)$ is minimal, i.e. for arbitrary pair $P_1' \succ P_2', \delta(P_1, P_2) \leq \delta(P_1', P_2')$.

We are going to define a lattice $L^*$ and we prove that $L$ is isomorphic to a sublattice of $L^*$ but $L$ has no cover-preserving embedding into $L^*$.

$L$ is the $S$-glued sum of their blocks $B_x, x \in S$, where $S$ is the lattice of the minimal elements of the blocks. If $x$ is covered by $y$ in $S$ then the identy mapping $\gamma_{xy}$ of $B_x \cap B_y$ has the domain $B_x \cap B_y \subseteq B_x$ and codomain $B_x \cap B_y \subseteq B_y$. Consider the lattice $L^{(2)}$, then $B_x^{(2)}$ is an interval of $L^{(2)}$. The intersection of $B_x^{(2)}$ and $B_y^{(2)}$ for $x \prec y$ is just $(B_x \cap B_y)^{(2)}$, consequently we can extend $\gamma_{xy}$ to an isomorphism $\bar{\gamma}_{xy} : B_x^{(2)} \to B_y^{(2)}$ with the domain $(B_x \cap B_y)^{(2)} \subseteq B_x^{(2)}$. An easy computation shows that the system $(B_x^{(2)}, x \in S)$ with the isomorphisms $\bar{\gamma}_{xy}$ from again an $S$-connected system. Then $L^* = \cup B_x^{(2)}$ is a lattice, the $S$-glued sum of $B_x^{(2)}$. Every $B_x^{(2)}$ is an interval of $L^{(2)}$ and $L^*$ is a sublattice of $L^{(2)}$. By the canonical embedding $L \to L^{(2)}$ the image of $L$ is a sublattice of $L^*$. We prove that $L$ has no cover-preserving embedding into $L^*$. Assume that we have a cover-preserving embedding $\tau : L \to L^*$. $P_1 = [b, a]$ and $P_2 = [d, c]$ are projective geometries, i.e. simple lattices. Then the conditions $b < b \vee c < a, d < b \wedge c < c$ yield that $\Theta(a, b) = \Theta(c, d)$ in $L$. Then, the same holds for $\tau(a)$, $\tau(b)$, $\tau(c)$ and $\tau(d)$ in $L^*$. $L^*$ is a subdirect square of $L$, we denote by $\pi_1$ and $\pi_2$ the corresponding projections into the components. By Lemma 3 there exist $x, y \in S$ such that $\tau(P_1) \subseteq B_x^{(2)}$ and $\tau(P_2) \subseteq B_y^{(2)}$. Then we have two projective blocks $[v, u] \subseteq B_x, [z, w] \subseteq B_y$ such that $\tau(P_1) \subseteq [v, u]^{(2)}$ and $\tau(P_2) \subseteq [z, w]^{(2)}$.

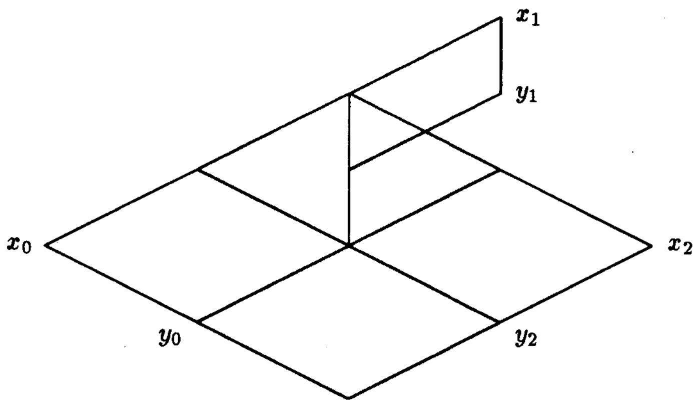


*Figure 3*

Then either

$$\pi_1(\tau(a)) = \pi_1(\tau(b)),\ \pi_1(\tau(c)) = \pi_1(\tau(d)),\ \pi_2(\tau(a)) \neq \pi_2(\tau(b)),\ \pi_2(\tau(c)) \neq \pi_2(\tau(d))$$

or conversely. By Lemma 2 we may assume that $\tau(P_1)$ and $\tau(P_2)$ are the following intervals:

$$Q_1 = \tau(P_1) = [(t,t),(t,u)] \subseteq B_x^{(2)} \text{ for some } t,\ v \leq t \leq u,$$

$$Q_2 = \tau(P_2) = [(s,s),(s,w)] \subseteq B_y^{(2)} \text{ for some } s,\ z \leq s \leq w.$$

The condition $Q_1 \succ Q_2$ means:

$$(t,t) < (t,t) \vee (s,w) < (t,u) \text{ and } (s,s) < (t,t) \wedge (s,w) < (s,w)$$

where $s < t$ and $w < v$. Then $(t,t) \vee (s,w) = (t, t \vee w) < (t,u)$ gives $t \vee w < u$ and $(t,t) \wedge (s,w) = (s, t \wedge w) > (s,s)$ implies $t \wedge w > s$. From $(t,t) < (t, t \vee w)$ we get $t < t \vee w$. The conditions $t < t \vee w < u, s < t \wedge w$ imply that the intervals $P_1' = [v,u]$ and $P_2' = [z,w]$ satisfy $P_1' \succ P_2'$ in $L$.

Now, determine $\partial(Q_1,Q_2)$. By the definition of $\delta$, $\partial(Q_1,Q_2) = \ell[(s,w),(t,t \vee w)]$. Since $(s,w) \leq (s,t \vee w) \leq (t,t \vee w)$ we have $\ell[(s,w),(t,t \vee w)] = \ell[(s,w),(s,t \vee w)] + \ell[(s,t \vee w),(t,t \vee w)] = \ell(w,t \vee w) + \ell(s,t)$ (the last expression is in $L$). But $\ell[s,t] = \ell[s,t \wedge w] + \ell[s,t \wedge w]$. From $s < t \wedge w$ we obtain $\ell[s,t \wedge w] > 0$, i.e. $\partial(Q_1,Q_2) = 2\ell[w,t \vee w] + \ell[s,t \wedge w] > \ell[w,t \vee w] = \partial([v,u],[z,w]) = \partial(P_1',P_2')$.

By the minimality of $\delta(P_1,P_2), \partial(P_1,P_2) \leq \partial(P_1',P_2') < \partial(Q_1,Q_2)$, i.e. $\tau$ is not a cover-preserving embedding. This contradiction proves the theorem. ■

COROLLARY 6.. *If $L$ has the CPEP and $b \prec a$ then there exists exactly one projective block which contains $a$ and $b$.*

PROOF.. Let us call the elements $a$ and $b$ perspective, in symbol $a \sim b$, iff they have a common complement in the interval $[a \wedge b, a \vee b]$. It is well-known that a finite complemented modular lattice is a projective geometry iff any two atoms are perspective. Define

$$b^* = \begin{cases} \sup\{p, p \succ b, p \sim a\} & \text{if } b < 1 \\ 1 & \text{if } b = 1 \end{cases}$$

and

$$b^{*+} = \begin{cases} \inf\{q, q \prec b^*, q \sim r \text{ where } b \leq r \prec b^*\} & \text{if } b^* > 0 \\ 0 & \text{if } b^* = 0 \end{cases}$$

Similarly, we define $a^+$ and $a^{+*}$ (see Ch. Herrmann [4]). Then $[a^+, a^{+*}]$ and $[b^{*+}, b^*]$ are two projective blocks, which contain $a$ and $b$. It is easy to see that $[a^+, b^*]$ is the Hall-Dilworth gluing of these two projective blocks. By Theorem 5 these blocks coincide, which proves that the projective block containing $a$ and $b$ is uniquely determined. ■

Corollary 7.. *Assume that $L$ has the CPEP. Let $[b,a]$ and $[d,c]$ be projective blocks. Let us assume that $b \leq b_1 \prec a_1 \leq a, d \leq d_1 \prec c_1 \leq c$ and $a_1/b_1 \sim c_1/d_1$. Then either $[b,a] \sim_w [d,c]$ or $[d,c] \sim_w [b,a]$.*

Proof.. If $c_1/d_1 \nearrow a_1/b_1$ then $b \vee c = b$, i.e. $b \geq c$ would imply that $b_1 \geq b \geq c \geq c_1$, i.e. $b_1 \vee c_1 = b_1$ in contradiction to $b_1 \neq a_1 = b_1 \vee c_1$. Similarly $b \wedge c < c$. Then by Theorem 5 either $b \vee c = a$ or $b \wedge c = c$. This is equivalent either to $[b,a] \sim_w [d,c]$ or to $[d,c] \sim_w [b,a]$ . ■

## 4. Proof of Theorem 1

Assume that $L$ has the CPEP with respect to $M$. By congruence distributivity it is enough to prove that all $L/\Theta'$ are projective geometries, where each $\Theta'$ is the complement of some atom $\Theta$ of the congruence lattice Con $(L)$. For any given atom $\Theta$ consider a projective block of maximal length contained in some $\Theta$-class. With other words we consider an interval $[b,a]$ of $L$ which satisfies the following properties: (1) $a \equiv b(\Theta)$; (2) $[b,a]$ is a projective block; (3) if $[d,c]$ is a projective block and $c \equiv d(\Theta)$ then $\ell[d,c] \leq \ell[b,a]$.

We are going to prove that $L/\Theta' \simeq [b,a]$. It is enough to show that the complement of $\Theta$ is just $\Theta' = \Theta(0,b) \vee \Theta(a,1)$. $\Theta$ is an atom and $[b,a]$ as a projective geometry is a simple lattice, which imply $\Theta = \Theta(a,b)$. We have to prove that $\Theta(a,b) \wedge \Theta(0,b) = \Theta(a,b) \wedge \Theta(a,1) = \omega$.

Consider a prime quotient $u/v$ such that $u/v \approx a'/b', u, v \notin [b,a]$ for a subquotient $a'/b'$ of $a/b$. Let $[b_2, a_2]$ be the projective block which contains $u$ and $v$. We prove that either $[b_2, a_2] \sim_w [b,a]$ or there exists a projective block $[b_1, a_1]$ such that the sublattice $\bar{A}$ generated by the elements $a$, $a_1$, $a_2$, $b$, $b_1$, $b_2$ is the lattice of Figure 4. $a'/b' \approx u/v$ means that there exists a normal sequence of perspectivites: $a'/b' = x_0/y_0 \sim x_1/y_1 \sim \cdots \sim x_n/y_n = u/v$ such that for each $i$ with $0 < i < n$ either $x_{i-1}/y_{i-1} \nearrow x_i/y_i \searrow x_{i+1}/y_{i+1}$ or $x_{i-1}/y_{i-1} \searrow x_i/y_i \nearrow x_{i+1}/y_{i+1}$ and $x_i = x_{i-1} \vee x_{i+1}$ in the first case and $y_i = y_{i-1} \wedge y_{i+1}$ in the second (see G. Grätzer [3], p.164). We prove the previous statement by induction on $n$. If $n = 1$, then either $u/v \nearrow a'/b'$ or $u/v \searrow a'/b'$ and $u/v$ is a subquotient of $a_2/b_2$.

By Corollary 7 we have $[b, a] \sim_w [b_2, a_2]$ or $[b_2, a_2] \sim_w [b, a]$. But by the choice of $a, b$ $\ell[b, a] \geq \ell[b_2, a_2]$, i.e. in the first case we have $[b, a] \sim [b_2, a_2]$. We obtain in both cases $[b_2, a_2] \sim_w [b, a]$.

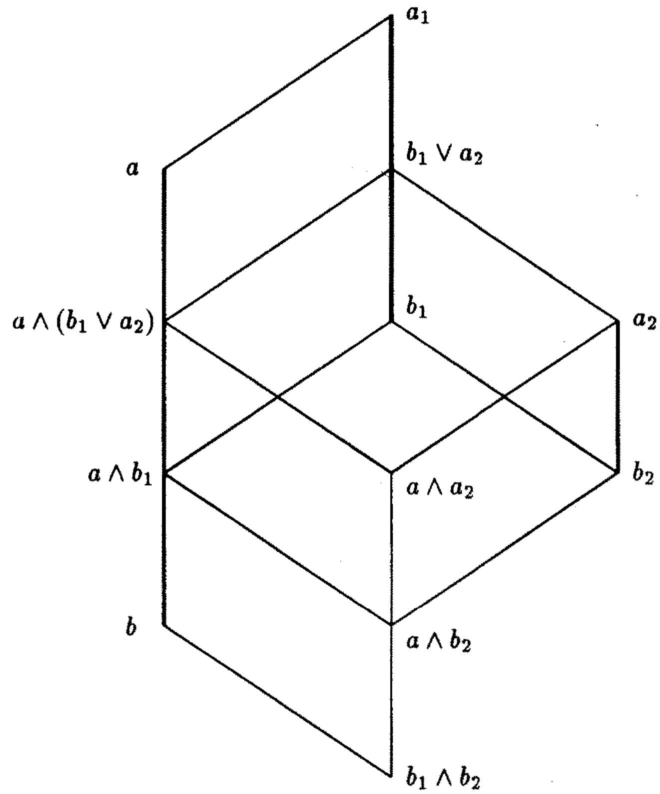


*Figure 4*

Let us consider the $n = 2$ case. We may assume that $[b_2, a_2]$ is not weakly perspective into $[b, a]$. Let $[b_1, a_1]$ be the projective block which contains $x_1, y_1$. Then by Corollary 7 and by the $n = 1$ case we have $[b_1, a_1] \searrow_w [b, a]$ (or $[b_1, a_1] \nearrow_w [b, a]$) and either (i) $[b_1, a_1] \searrow_w [b_2, a_2]$ or (ii) $[b_2, a_2] \nearrow_w [b_1, a_1]$. We discuss these two cases.

(i) By our assumption $x_0/y_0 \sim x_1/y_1 \sim x_2/y_2$ is a normal sequence, hence we have $x_1 = x_0 \vee x_2$. This gives $a \vee a_2 = (a \vee x_0) \vee (x_2 \vee a_2) = a \vee (x_0 \vee x_2) \vee a_2 = a \vee x_1 \vee a_2 = (a \vee x_1) \vee (x_1 \vee a_2) = a_1 \vee a_1 = a_1$; which proves that $a/a \wedge b_1 \sim a_1/b_1 \sim a_2/a_2 \wedge b_1$ is again a normal sequence of perspectivites. Now, we apply Theorem 5. If $a$, $a_1$, $a_2$, $a \wedge b_1$, $b_1$, $a_2 \wedge b_1$ would generate a non distributive sublattice (Figure 2 or Figure 3) then $[b_1, a_1]$ and the diamond would generate a projective geometry $P$ such that $[b_1, a_1] \neq P$, in contradiction to our assumption $[b_1, a_1]$ is a projective block. This proves that $a$, $b_1$, $a_2$ generate the eight element boolean algebra (see Figure 1), consequently $a/a \wedge b_1 \searrow a \wedge a_2/a \wedge a_2 \wedge b_1 \nearrow a_2/a_2 \wedge b_1$.

We prove that $b_2 = a_2 \wedge b_1$, i.e. $[b_2, a_2] \nearrow_w [b_1, a_1]$ as in case (ii). Indeed, if $b_2 < a_2 \wedge b_1$ then $a \wedge b_2 < a \wedge a_2 \wedge b_1$. $[a \wedge b_2, a \wedge a_2]$ and $[b \wedge a_2, a \wedge a_2]$ are projective geometries and their intersection contains the projective geometry $[a \wedge a_2 \wedge b_1, a \wedge a_2]$. This implies that $[b \wedge b_2, a \wedge a_2]$ is again a projective geometry. The definition of $[b, c]$ and Corollary 7 imply $[b \wedge b_2, a \wedge a_2] \nearrow_w [b, a]$. We claim that, for $x, y \in [b \wedge b_2, a \wedge a_2]$ $b \vee x \neq b \vee y$. Since $b \vee (a \wedge b_2) = a \wedge (b \vee b_2) = a \wedge b_1 = b \vee (a \wedge a_2 \wedge b_1)$ and $a \wedge a_2 \wedge b_1 \neq a \wedge b_2$, we have a contradiction.

(ii) We have $[b_2, a_2] \nearrow_w [b_1, a_1]$. Then obviously $a \wedge (b_1 \vee a_2)/a \wedge a_1 \sim b_1 \vee a_2/b_1 \sim a_2/b_2$ is a normal sequence. The same argument as in case (i) yields that $a \wedge (b_1 \vee a_2)$, $b_1$ and $b_2$ generate the eight element boolean algebra, and $a_2/b_2 \searrow a \wedge a_2/a \wedge b_2 \nearrow a \wedge (b_1 \vee a_2)/a \wedge b_1$. This proves that the lattice generated by $a$, $a_1$, $a_2$, $b$, $b_1$, $b_2$ is just $\bar{A}$ (see Figure 4).

If $n = 3$, then either $[b_3, a_3] \searrow_w [b_2, a_2]$ or $[b_2, a_2] \nearrow_w [b_3, a_3]$. Observe that $a_2/b_2 \sim a \wedge a_2/a \wedge a_2 \wedge a_1$, i.e. we have $[b_3, a_3] \searrow_w [a \wedge a_2 \wedge a_1, a \wedge a_2]$ or $[a \wedge a_2 \wedge a_1, a \wedge a_2] \nearrow [b_3, a_3]$ which is the $n = 2$ case.

The given characterization of $\Theta = \Theta(a, b)$ yields that if $a \leq v \leq u$ (or similarly if $v \leq u \leq b$) then $u \equiv v(\Theta(a, b))$ implies $u = v$, i.e. $L/\Theta' \cong [a, b]$.

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MATHEMATICAL INSTITUTE,
HUNGARIAN ACADEMY OF SCIENCES,
H-1364 BUDAPEST, PF. 127.