ON A PROBLEM OF M. H. STONE

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Pseudo-complemented lattices form an important class of (distributive) lattices. Topological distributive lattices, the lattice of all ideals of a distributive lattice with zero element, the lattice of all congruence relations of an arbitrary lattice are all pseudo-complemented. It is clear that the Boolean algebras have the same property.

Thus we might consider the distributive pseudo-complemented lattices in which $a^* \cup a^{**} = 1$ holds for all a as an immediate generalization of the Boolean algebras. The investigation of this type of lattices was proposed by M. H. STONE (it is G. BIRKHOFF's problem 70, see[1] [1], p. 149):

What is the most general pseudo-complemented distributive lattice in which $a^* \cup a^{**} = 1$ identically?

In this paper we get two solutions of STONE's problem. After this we deal with a related question.

§ 1. Preliminaries

We begin by giving some definitions.

DEFINITION 1. The (distributive) lattice L is called *pseudo-complemented* if it has a zero element and for any element a of L there exists an element a^* of L such that $a \cap x = 0$ if and only if $x \leqq a^*$. The element a^* is called the pseudo-complement of a.

DEFINITION 2. A lattice L is said to be a *Stone lattice* if it is a pseudo-complemented distributive lattice with unit element in which $a^* \cup a^{**} = 1$ for each element a of L.

DEFINITION 3. We shall call the lattice L *relative Stone lattice* if every closed interval of L is a Stone lattice.

REMARK. We mention the fact that a relative Stone lattice is a Stone lattice if and only if it has zero and unit elements. A Stone lattice is not a

[1] Numbers in brackets refer to the Bibliography given at the end of the paper.

relative Stone lattice in general, e. g. if we define a new zero and unit element for an arbitrary Boolean algebra, then this lattice is a Stone lattice which is not a relative Stone lattice.

DEFINITION 4. Let L be a lattice with zero element. The element b is said to be a *semi-complement*[2] of the element a if $a \cap b = 0$. The lattice is called *dense* if $a \cap b = 0$ implies that a or b is 0.

Now we recall a few facts on which the sequel depends.

LEMMA 1 (M. H. STONE's theorem). *Let L be a distributive lattice, I an ideal and D a dual ideal of L such that I and D are disjoint. Any maximal ideal P, for which $P \supseteq I$ further P and D are disjoint, is prime.*

The proof is well known (see [3] too).

LEMMA 2. *Let L be a distributive lattice with zero element and P a prime ideal of L. There exists a minimal prime ideal Q with $Q \subseteq P$.*

PROOF. If P is a prime ideal, then[3] $L-P$ is a dual prime ideal (see [1], p. 141). A maximal dual ideal Q which contains $L-P$ and $0 \notin Q$ is a dual prime ideal (Lemma 1), that is, $L-Q$ is a minimal prime ideal in P.

LEMMA 3. *If in a distributive lattice the meet and the join of two ideals are principal ideals, then the given ideals are also principal ideals.*

This was proved in [3].

LEMMA 4. *Under any lattice homomorphism, the complete inverse image of a prime ideal is again a prime ideal.*

This result may be found in [3].

LEMMA 5. *Let L be a distributive lattice and D a dual ideal of L. There exists a minimal congruence relation on L under which D is a congruence class. Under this congruence relation $a \equiv b$ and $a \leqq b$ are equivalent to the condition that there exists an element $d \in D$ with $b \cap d = a$.*

This is equivalent to Corollary 4 of Theorem 2 of [2].

§ 2. Characterizations of Stone lattices

The main result of this paper is

THEOREM 1. *Let L be a distributive pseudo-complemented lattice with unit element. Then L is a Stone lattice if and only if the lattice-theoretical join of any two distinct minimal prime ideals of L is L.*

[2] This notion is due to G. SZÁSZ [4].

[3] $P-Q$ denotes the set-theoretical difference and later on $P \dotplus Q$ the set-theoretical sum.

PROOF. Let L be a distributive lattice with zero and unit elements in which the join of any two distinct minimal prime ideals is L. We must prove that $a^* \cup a^{**} = 1$ for all $a \in L$. (We have supposed that a^* exists.)

We suppose that there exists an element a for which $a^* \cup a^{**} \neq 1$. By Lemma 1 there exists a dual prime ideal P with $a^* \cup a^{**} \notin P$. Let us consider the minimal congruence relation Θ on L under which P is a congruence class. We assert that in the factor lattice L/Θ the join of any two distinct minimal prime ideals is the whole lattice.

Let $\bar{Q}$ and $\bar{R}$ be minimal prime ideals of L/Θ and Q, R their complete inverse images. By Lemma 4, Q and R are prime ideals; we prove that they are minimal ones. Indeed, if $Q_1 \subset Q$ (Q_1 is a prime ideal), then the homomorphic image of Q_1 and Q coincide, hence for arbitrary $q_1 \in Q_1$ and for some $q \in Q - Q_1$ the relation $q \equiv q_1 (\Theta)$ is valid. We may suppose $q_1 < q$ so that by Lemma 5 there exists a $p \in P$ which satisfies $q \cap p = q_1$. But $p \notin Q_1$, for in case $p \in Q_1$, $\bar{p}$ would be an element common to $\bar{Q}_1$ and to $\bar{P} = \bar{1}$ which is a contradiction. Thus we get that p and q are not elements of the prime ideal Q_1, nevertheless $p \cap q \in Q_1$. This contradiction proves our assertion.

We get that in L/Θ the join of any two distinct minimal prime ideals is the whole lattice. Now we intend to show that in L/Θ there exists only one minimal prime ideal: $(\bar{0}]$.

The unit element of L/Θ is join-irreducible, for in case $\bar{x} \cup \bar{y} = \bar{1}$ and $\bar{x}, \bar{y} \neq \bar{1}$, $x \cup y \in P$ but $x, y \notin P$ which is absurd, because P is a dual prime ideal. Consequently, L contains only one minimal prime ideal, for if in L there were two minimal prime ideals, then the join of these would be the whole lattice, i
$\bar{1}$ would be join-reducible which is impossible. Finally, let $\bar{S}$ be any minimal prime ideal of L/Θ and $\bar{S} \neq (\bar{0}]$. We choose an $a \in \bar{S}$. By the *Duality Principle* and by Lemma 1 there exists a prime ideal which does not contain $[a)$. This prime ideal, by Lemma 2, contains a minimal one which is obviously different from $\bar{S}$.

We have supposed that $a^* \cup a^{**} < 1$, consequently $0 < a^*$. We assert that $a \not\equiv 0 (\Theta)$. Indeed, in case $a \equiv 0 (\Theta)$ it follows the existence of a $p \in P$ (Lemma 5) such that $a \cap p = 0$, i. e. $p \leqq a^*$, hence p is an element common to P and to $(a^* \cup a^{**}]$, which is a contradiction. Similarly, $a^* \not\equiv 0 (\Theta)$.

We get that in L/Θ $\bar{0} < \bar{a}$ and $\bar{0} < \bar{a}^*$, yet $\bar{a} \cap \bar{a}^* = \bar{0}$, in contradiction to the fact that $(\bar{0}]$ is a prime ideal.

Thus we have proved that in a pseudo-complemented lattice with unit element, if the join of any two distinct minimal prime ideals is the whole lattice, then $a^* \cup a^{**} = 1$ for all elements a of the lattice, i. e. it is a Stone lattice.

Conversely, let L be a Stone lattice, T and U distinct minimal prime ideals of L. $L-T$ and $L-U$ are maximal dual prime ideals, consequently, there exist $a \in L-U$ and $b \in L-T$ with $a \cap b = 0$. Obviously $a \in T-U$ and $b \in U-T$ is valid too, since otherwise a and b would be in the same dual prime ideal $L-T$ or $L-U$ which is impossible in view of $a \cap b = 0$. $a \in T$, so $a^* \in U-T$ and $a^{**} \in T-U$, hence from $a^* \cup a^{**} = 1$ it follows $T \cup U = (1]$.

Another — almost obvious — characterization of Stone lattices is the following

THEOREM 2. *A distributive lattice L with 0 and 1 is a Stone lattice if and only if for all $a \in L$ the ideal formed by the semi-complements of a is a direct factor of L.*

PROOF. Let L be a Stone lattice. The ideal formed by the semi-complements of a is clearly $(a^*]$, hence $(a^*] \cap (a^{**}] = (0]$ and $(a^*] \cup (a^{**}] = (1]$; thus $(a^*]$ is indeed[4] a direct factor of L.

Conversely, let I be the ideal formed by the semi-complements of an element a. If I is a direct factor, then there exists an ideal J with $I \cap J = (0]$ and $I \cup J = (1]$. By Lemma 3, it follows that I and J are principal ideals, moreover the generating elements are a^* and a^{**}. Thus the proof of Theorem 2 is complete.

If L is a Stone lattice, then it is either dense or there exists an element a $(0 < a < 1)$ such that $a^* \neq 0$. But in the latter case, by Theorem 2, L is directly factorisable. Thus, an immediate consequence of Theorem 2 is the following

COROLLARY. *A finite distributive lattice L is a Stone lattice if and only if it is the direct product of dense lattices.*

§ 3. Relative Stone lattices

The following theorem is analogous to Theorem 1 in case of relative Stone lattices:

THEOREM 3. *Let L be a distributive lattice in which every closed interval (as a sublattice) is a pseudo-complemented lattice. L is a relative Stone lattice if and only if in L for any pair of prime ideals P and Q, of which neither contains the other, $P \cup Q = L$ is valid.*

[4] It is well known that I is a direct factor of the distributive lattice L with zero and unit elements if and only if there exists an element a such that $I = (a]$ and a has a complement.

PROOF. Let us suppose that although L is a relative Stone lattice, there exists a pair of prime ideals P and Q such that $P \cup Q \subset L$, but neither $P \subseteq Q$ nor $Q \subseteq P$. We choose $a \in L-(P \cup Q)$, $b \in P-Q$ and $c \in Q-P$. By the hypothesis the interval $[b \cap c, a \cup b \cup c]$ as a sublattice is a Stone lattice. Hence, in this interval b has a pseudo-complement b^*. b^* is necessarily in $Q-P$, and $b^{**} \in P-Q$; in consequence of this fact $b^* \cup b^{**} = a \cup b \cup c$, i. e. $a \cup b \cup c \in P \cup Q$, but we have supposed $a \notin P \cup Q$. Thus the proof of the necessity of the conditions is completed.

On the other hand, assume that for any pair of prime ideals P and Q of this lattice L, none of them containing the other, $P \cup Q = L$ is valid. Now, let us consider an interval $[a, b]$ of L and two minimal prime ideals P' and Q' of $[a, b]$. There exists a pair of prime ideals P, Q of L with the property that $P \cap [a, b] = P'$ and $Q \cap [a, b] = Q'$. Indeed (see Lemma 1), let P be a maximal ideal which contains the ideal of L generated by P' and is disjoint from the dual ideal of L generated by $[a, b]-P'$; Q may be defined in a similar way. Obviously, none of P and Q contains the other, hence, by our assumption $P \cup Q = L$. It follows that $P' \cup Q' = [a, b]$. Applying Theorem 1, we get that the interval $[a, b]$ is a Stone lattice, consequently, L is a relative Stone lattice.

It is easy to characterize the relative Stone lattices if we apply the following

THEOREM 4. *If every closed interval of a distributive lattice L is pseudo-complemented and if L has no homomorphic image isomorphic to the lattice of Fig. 1, then L is a relative Stone lattice.*

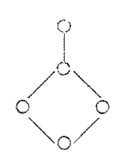

Fig. 1

PROOF. We must prove that if the distributive lattice L, in which every closed interval is as a sublattice pseudo-complemented, is not a relative Stone lattice, then it has a homomorphic image isomorphic to the lattice of Fig. 1. By Theorem 3, if L is not a relative Stone lattice, then it has a pair of prime ideals P, Q such that $P \not\subseteq Q$, $P \not\supseteq Q$ and $P \cup Q \subset L$. By Lemma 1, there exists in L a prime ideal R with $P \cup Q \subseteq R$. We define a congruence relation Θ on L as follows: let $H_1 = P \cap Q$, $H_2 = P-Q$, $H_3 = Q-P$, $H_4 = L-R$, $H_5 = R-(P+Q)$ and let $x \equiv y$ (Θ) if and only if for some $i(=1, 2, \ldots, 5)$ x and y are both in H_i. It is routine to check that Θ is a congruence relation. It is obvious that L/Θ is isomorphic to the lattice of Fig. 1, what was to be proved.

Finally, we mention the problem whether Theorem 1 is valid if pseudo-complementedness is not assumed.

The interest of this problem lies in the fact that if every minimal prime ideal is a maximal one, then the assertion is true, see e. g. [3].

(Received 2 September 1957)

Bibliography

[1] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Coll. Publ., Vol. 25, 2nd ed. (New York, 1948).

[2] G. Grätzer és E. T. Schmidt, Hálók ideáljai és kongruenciarelációi. I, *MTA Mat. és Fiz. Oszt. Közl.*, **7** (1957), pp. 93—109; Ideals and congruence relations in lattices, (to appear in these Acta).

[3] G. Grätzer and E. T. Schmidt, On ideal theory of lattices, *Acta Sci. Math. Szeged*, (under press).

[4] G. Szász, Dense and semi-complemented lattices, *Nieuw Archief vor Wiskunde*, **1** (1953) pp. 42—44.