GENERAL ALGEBRA 1988
R. Mlitz (Editor)
Elsevier Science Publishers B.V. (North-Holland), 1990

Polynomial automorphisms of lattices

E. TAMÁS SCHMIDT*

Abstract. A polynomial automorphism of a lattice is a unary lattice polynomial $f(x)$ for which the mapping $x \to f(x)$ is an automorphism. It is proved that every bounded lattice with a finite automorphism group can be embedded as an ideal in a lattice K such that each automorphism of K is polynomial and there is a bijection between the automorphism groups of L and K.

Keywords and phrases: *Polynomial automorphisms, automorphism group, lattice extension.*
AMS–MOS subject classification: 06B15

1. Introduction. Ervin Fried and Harry Lakser [1] defined the concept of a polynomial automorphism of a lattice as a unary lattice polynomial $f(x)$ for which the mapping $x \to f(x)$ is an automorphism. They proved two theorems:

THEOREM A. *Each finite lattice L can be embedded in some finite lattice K such that the following three properties hold:*

(1) *Each automorphism of K is polynomial;*
(2) *Each automorphism of L extends to a unique automorphism of K;*
(3) *Each automorphism of K is the extension of an automorphism of L.*

For infinite lattices they could prove a weaker result:

THEOREM B. *Each lattice L can be embedded as a convex sublattice in some lattice K such that every automorphism of L is the restriction of a unary polynomial function on K. If L has a 0 then this embedding is an ideal.*

In this paper first we prove a little stronger version of Theorem A; the proof is slightly shorter then the original proof.

THEOREM 1. *Each finite lattice L can be embedded as a maximal filter (or maximal ideal) in some finite simple and atomistic lattice K such that the properties (1), (2) and (3) of Theorem A hold.*

The main result of this paper is the generalization of Theorem A for arbitrary lattice having finite automorphism group:

THEOREM 2. *Let L be a bounded lattice with a finite automorphism group. L can be embedded as an ideal in some lattice K such that the properties (1), (2) and (3) of Theorem A hold.*

If $\alpha \in \operatorname{Aut} L$ then a congruence relation θ of L is called α–admissible if $x \equiv y(\theta)$ implies $x^\alpha \equiv y^\alpha(\theta)$ (x^α denotes the image of x under α). Similary if G is a subgroup of Aut L, then θ is called G– admissible if θ is α–admissible for every $\alpha \in G$. The

* Research supported by Hungarian National Foundation for Scientific Research grant. no. 1813.

G–admissible congruence relations of L from a $\{0,1\}$ – sublattice $Con_G(L)$ of $Con(L)$. Assume that L is a convex sublattice of some lattice K such that a given automorphism α of L is the restriction of a unary polynomial function $f(x)$. A polynomial function is compatible, consequently if $\theta \in Con(L)$ can be extended to K then θ is an α–admissible congruence relation of L. We prove the following:

THEOREM 3. *Let L be a lattice. L can be embedded as a convex sublattice in some lattice K such that the following properties hold:*

- (1') *Each automorphism of L is the restriction of a unary polynomial function of K;*
- (2') *Each automorphism of L extends to a unique automorphism of K;*
- (3') *Each automorphism of K is the extension of an automorphism of L;*
- (4') *Each Aut L–admissible congruence relation of L extends to a unique congruence relation of K;*
- (5') *Each proper congruence relation of K is the extension of a congruence relation of L.*

2. Proof of Theorem 1. Let L be a finite lattice, with the zero element 0. We add a new zero element $\overline{0}$ to L (i.e. $\overline{0} < x$ for all $x \in L$), the resulting lattice is denoted by $\overline{L}$. Each automorphism of L extends uniquely to an automorphism of $\overline{L}$ and each automorphism of $\overline{L}$ is the extension of an automorphism of L. For every $n \geq 1$ we consider the following lattice $S(n)$ (Fig. 1.)

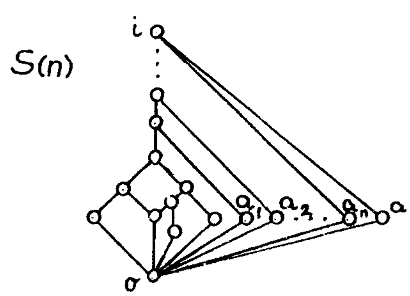

Fig. 1.

It is easy to see that $S(n)$ is an atomistic simple lattice and has no nontrivial automorphism. We fix n such that the length of $S(n)$ is greater than the length of $\overline{L}$ and we denote this $S(n)$ shortly by S. For every $u \in L$, $u \neq 0$, let S_u be a lattice isomorphic to S, with the isomorphism $\varphi_n : S \to S_u$. Assume that $S_u \cap S_v = \emptyset$ if $u \neq v$ and $S_u \cap L = \emptyset$. We construct the extension K of L by gluing the lattices $S_u (u \in L)$ and $\overline{L}$: we idenify the zero element of S_u with $\overline{0} \in \overline{L}$ and the unit element of S_u with $u \in \overline{L}$. Let K be $\overline{L} \cup \{S_u; u \in L, u \neq 0\}$. The ordering in $S_u \subseteq K$ and $\overline{L} \subseteq K$ is the original, all these are sublattices of K. For $x \in S_u$, $x \notin \overline{L}$ and $y \in \overline{L}$, $y \notin S_u$ $x \leq y$ iff $u \leq y$ in L; if $u \not\leq y$ then x and y are incomparable and $\sup\{x,y\} = u \vee y$ (the join in L) and $\inf\{x,y\} = \overline{0}$. If $x \in S_u$, $y \in S_v$, $u \neq v$ and $x, y \in \overline{L}$ then x, y are incomparable: $\sup\{x,y\} = u \vee v$, $\inf\{x,y\} = \overline{0}$. (See Fig. 2.)

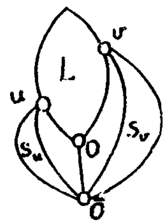

Fig. 2.

L is the filter $[0)$, therefore L is a maximal filter of K. Every element of S_u is the join of atoms, and these are atoms of K, hence K is atomistic. Assume that θ is a congruence relation of K and $a \equiv b(\theta)$, $a > -b$. We prove that $\theta = \iota$ which means that K is simple. If $a, b \in S_u$ for some u then the simplicity of S_u implies $u \equiv \overline{0}(\theta)$, therefore two different elements of $\overline{L}$ are congruent. We may assume that $a, b \in \overline{L}$. If $a = 0$ let p be the element a (in this case $b = \overline{0}$). Otherwise let p be an atom of S_a and consider an arbitrary $q \in S_1 \setminus \overline{L}$. Then $a \equiv b(\theta)$ implies $p = p \wedge a \equiv p \wedge b = \overline{0}(\overline{\theta})$, and $1 = p \vee q \equiv \overline{0} \vee q = q(\theta)$. S_1 is simple, consequently $1 \equiv \overline{0}(\theta)$, i.e. $\theta = \iota$.

By a theorem of Rudolf Wille [2] K is order – polynomialy complete, i.e. every automorphism is polynomial which proves (1).

Let α be an automorphism of L (i.e. of $\overline{L}$), and let $x \in S_u$, $x \notin \overline{L}$. Then $u = 0 \vee x$, $\overline{0} = 0 \wedge x$. Assume that α has an extension to K, we denote one of this by the same letter. We conclude that $u^\alpha = 0^\alpha \vee x^\alpha = 0 \vee x^\alpha$ and $\overline{0} = \overline{0}^\alpha = 0^\alpha \wedge x^\alpha = 0 \wedge x^\alpha$ which imply $x^\alpha \in S_{u^\alpha}$. Since S has no nontrivial automorphism x^α is uniquely determined (i.e. $(\varphi_n(x))^\alpha = \varphi_{u^\alpha}(x)$). Conversely if we define $(\varphi_n(x))^\alpha = \varphi_{n^\alpha}(x)$ for $x \in S$ then we have an extension of α to K, which proves (2).

Finally let β be an automorphism of K. 0 is an atom of K. hence 0^β must be an atom. By construction of K length$(S) >$ length$(\overline{L})$, which implies $0^\beta \in \overline{L}$, i.e. $0^\beta = 0$. If $a \in L$ then $a \geq 0$, thus $a^\beta \geq 0^\beta = 0$, i.e. $a^\beta \in L$. Consequently, the restriction of β to L is an automorphism L which proves (3).

3 Proof of Theorem 2. Let L be the given bounded lattice with the zero element u and unit element v. Let α be a fixed automorphism of L. First we construct a lattice $T_\alpha(L)$ such that L is an ideal of this lattice and L has a polynomial automorphism, which is an extension of α. We start with the following lattice T, where the principal ideals $(a_1]$ and $(a_5]$ are isomorphic to $S(2)$ resp. $S(4)$ where $S(n)$ denotes the lattice defined in the proof of Theorem 1. This lattice is a simple atomistic lattice.

It is easy to see that T has no nontrivial automorphism. We glue one–one copies of L into the prime intervals $[0, a_i]$, $[a_{i+1}, b_i]$ $i = 0, 2, 4$, i.e. we identify 0 with u and a_i with v (and similarly a_{i+1} with u,b_i with v) We fix some isomorphisms $\varphi_i : L \to [0, a_i]$, $\Psi_i : L \to [a_{i+1}, b_i]$ and we identify L with $[0, a_0]$, i.e. φ_0 is the identify map. Let $T_\alpha(L)$ be the set $T \cup \bigcup_{i=0,2,4} ([0, a_i] \cup [a_{i+1}, b_i])$. We define a partial ordering on $T_\alpha(L)$

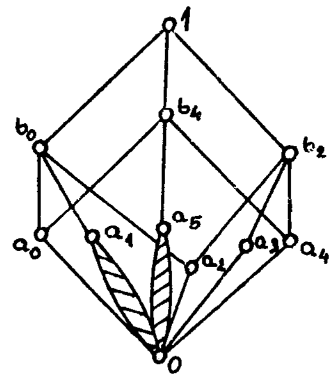

Fig. 3.

which will be an extension of the ordering of T.

$$\varphi_0(x) \leq \Psi_4(x) \quad \text{iff} \quad x \leq y^\alpha \quad \text{in} \quad L,$$

and in the other cases:

$$\left.\begin{array}{r}\varphi_i(x) \leq \Psi_i(y) \\ \varphi_{i+2}(x) \leq \Psi_i(y)\end{array}\right\} \quad \text{iff} \quad x \leq y \quad \text{in} \quad L.$$

Then the subsets $[0, a_i] \cup [a_{i+1}, b_i]$, $[0, a_{i+2}] \cup [a_{i+1}, b_i]$ are all isomorphic to $L \times 2$. It is easy to see that $T_\alpha(L)$ is a lattice and has the following schematic diagram (Fig. 4.).

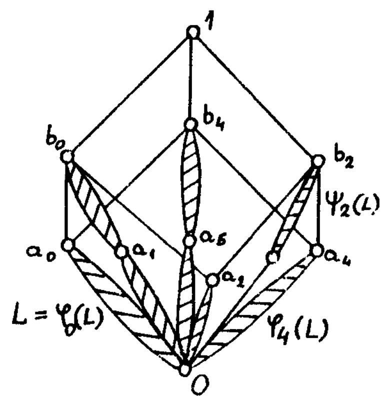

Fig. 4.

Let the unary polynomials f, g, h be defined by setting

$$f(x) = ((\ldots(x \wedge a_0) \vee a_1) \wedge a_2) \vee a_3) \vee a_5) \wedge a_0\,,$$

$$g(x) = ((\dots(x \wedge a_2) \vee a_3) \wedge a_4) \vee a_5) \vee a_1) \wedge a_2\,,$$

$$h(x) = ((\dots(x \wedge a_4) \vee a_5) \wedge a_0) \vee a_1) \vee a_3) \wedge a_4\,.$$

By the definition of $T_\alpha(L)$ for an arbitrary $x \in L$ (i.e. $x \le a_0$) $f(x) = x^\alpha$, consequently $\alpha \in$ Aut L is the restriction of the unary polynomial function $f(x)$. We extend $f(x)$ to a polynomial automorphism of $T_\alpha(L)$. Define the unary polynomial $f_\alpha(x)$ in $T_\alpha(L)$ by setting:

$$f_\alpha(x) = f(x) \vee (x \wedge a_2) \vee g(x) \vee (x \wedge a_3) \vee h(x) \vee (x \wedge a_5),$$

then $f_\alpha(x) = f(x)$ if $x \le a_0$ and similarly $f_\alpha(x) = g(x)$ if $x \le a_2$ and $f_\alpha(x) = h(x)$ for $x \le a_4$. The elements of $T \subseteq T_\alpha(L)$ are all fixelements by $f_\alpha(x)$.

Let β be an arbitrary automorphism of L. We would like to extend β to $T_\alpha(L)$, such that the elements of $T \subseteq T_\alpha(L)$ remain fixed under the extension of β. Then $(f(x))^\beta = f(x^\beta)$, consequently $x^{\alpha\beta} = x^{\beta\alpha}(x \in L)$.That means: β can be extended to $T_\alpha(L)$ if α and β commute. In the other case x^β must be a "new" element, therefore we define an extension K_α of $T_\alpha(L)$, such that every polynomial automorphism of $T_\alpha(L)$ can be extend to a polynomial automorphism of K_α.

Let C be the centralizer of α in $G =$ Aut L. $C = C_0, C_1, \dots, C_n$ denote the right cosets of C. For every C_i we consider an isomorphic copy M_i of $T_\alpha(L)$ and we fix for every i an isomorphism $\tau_i : T_\alpha(L) \to M_i$. We identify M_0 with $T_\alpha(L)$. L is an ideal of $T_\alpha(L) = M_0$. Let L_i be the image of L by τ_i, then L_i is an ideal of M_i. We glue together the lattices $M_i (i = 0, 1, \dots, n)$ by identifying the ideals L_i. Then $M_i \cap M_j = L$ if $i \neq j$. Let 1_i be the unit element of M_i and let 1 be the unit of L, then $1_i \wedge 1_j = 1$. Finally, we adjoin a new unit element I to the poset $\bigcup M_i$. Let K_α be the poset $\{I\} \cup \bigcup M_i$. This K_α is olviously a lattice, every M_i is an ideal of K_α and if $x \in M_i$, $y \in M_j$, $x, y \notin L$ then $x \vee y = I$. (see Fig. 5).

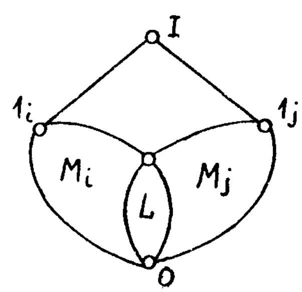

Fig. 5.

We prove that K_α satisfies the following three properties:

(i) α is the restriction of a polynomial automorphism of K_α;
(ii) Each (polynomial) automorphism of L extends to a unique (polynomial) automorphism of K_α;
(iii) Each automorphism of K_α is the extension of an automorphism of L.

Using these properties the proof of the theorem is very easy. By our assumtion $G = \mathrm{Aut}L$ is finite. Consider an $\alpha \in G$, then we have extension K_α of L. $\beta \in G$ can be extended to K_α, i.e. β is an automorphism of K_α. We apply the same construction starting with K_α and we get the extension $(K_\alpha)_\beta$. By induction we have finally an extension K of L which satisfies (1)–(3). L is an ideal of K_α, hence L is an ideal of K.

To prove (i) let f_α^i be the polynomial automorphism of M_i which corresponds to $f_\alpha = f_\alpha^0$ by the isomorphism $\tau_i : T_\alpha(L) \to M_i$. Define $F_\alpha(x) = \vee_{i=0}^{n} f_\alpha{}^i(x \wedge 1_i)$ then the restriction of $F_\alpha(x)$ to $M_i \subseteq K_\alpha$ is $f_\alpha{}^i$ consequently its restriction to $L \subseteq M_0$ is α which proves (i).

We prove (ii). Every element $y \notin \{1, r, s, t\}$ of $T_\alpha(L) = M_0$ has a unique representation in the following form:

$$y = ((\dots(x \wedge a_0) \vee a_1)\dots) \vee a_k \qquad \text{where} \quad x \in L, k \leq 5.$$

If β has an extension to K_α — we use same notation — then

$$y^\beta = ((\dots(x^\beta \wedge a_0{}^\beta) \vee a_1{}^\beta)\dots) \vee a_k{}^\beta,$$

which means that $y^\beta \in C_i$ where $\beta \in C_i$, hence $y^\beta = \tau_i(y)$. Let γ be an arbitrary automorphism of L, and let $\beta\gamma \in C_j$. Then let $(y^\beta)^\gamma = \tau_i(y)$, which is an automorphism of K_α This proves that β has a unique extension to K_α. (I is by the extension obviously a fixelement.)

Assume that β is a polynomial automorphism of L. We prove that the extension of β — which will be denoted again by β — is a polynomial automorphism of K_α. T is a sublattice of $T_\alpha(L) = M_0$, i,e, we have the embedding $\epsilon : T \to M_0$. Applying the isomorphism $\tau_i : M_0 \to M_i$ we get the sublattices $T_i = \tau_i\epsilon(T)$ of M_i. Obviously $T_i \cap T_j = \{0, a_0\}$ if $i \neq j$. Let $T_\alpha{}^*$ be the sublattice $\bigcup T_i \cup \{I\}$ of K_α. This is a simple atomistic lattice, consequently by [2] every automorphism of $T_\alpha{}^*$ is polynomial. Consider the elements $a_i (i = 0, 1, \dots, 5)$ of T (see Fig. 3.). Let γ be an arbitrary automorphism of K_α, then $a_i{}^\gamma \in T_k \subseteq M_k$ for some k. If we apply the automorphism β then $a_i{}^{\gamma\beta} \in T_l \subseteq M_l$ for some l. We discuss two different cases. First assume, that $i = 0, 2$ or 4, The intervals $[0, a_0]$ and $[0, a_i]$ are projective in $T_\alpha(L)$, consequently $[0, a_i{}^\gamma]$ and $[0, a_i{}^{\gamma\beta}]$ are projective in K_α. That means, we have a unary polynomial which transposes $[0, a_i{}^\gamma]$ onto $[0, a_i{}^{\gamma\beta}]$. Now, let $i = 1, 3$ or 5. Then the principal ideals $(a_i]$ belong to T i.e. $[0, a_i{}^\gamma]$, $[0, a_i{}^{\gamma\beta}]$ are contained in $T_\alpha{}^*$. But every automorphism of $T_\alpha{}^*$ is polynomial, i.e. we have again a unary polynomial which transposes $[0, a_i{}^\gamma]$ onto $[0, a_i{}^{\gamma\beta}]$. This proves that β is a polynomial automorphism of K_α.

Finally we prove (iii). Let γ be an arbitrary automorphism of K_α. The unit element, $1 \in L$ must be a fixelement of γ (1 is the intersection of dual atoms). Consequently the restriction of γ to $(1] = L$ is an automorphism.

4. Proof of Theorem 3. The proof is similar to the proof of Theorem 2, but we start with an other lattice T:

The principal ideals $(a_2]$ and $(a_4]$ are isomorphic to $S(2)$ resp $S(4)$ where $S(n)$ is the lattice defined in the proof of Theorem 1. T has only the trivial automorphism and

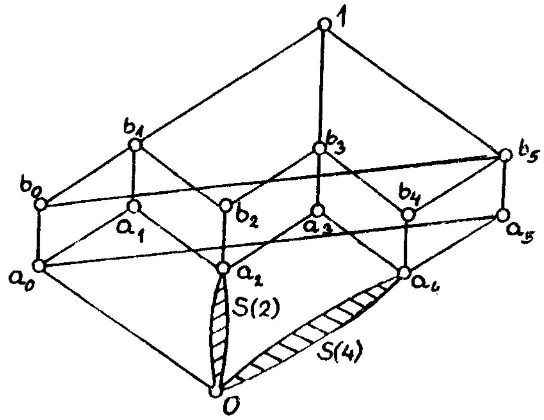

Fig. 6.

has exactly one non trivial congruence relation with the congruence classes $\{a_i, b_i\}$, $i = 0, \dots, 5$. Let L be the given bounded lattice, with zero element u and unit element v. We glue one–one copies of L into the prime intervals $[a_i, b_i]$ ($i = 0, \dots .5$) identifying u with a_i and v with b_i. We fix the isomorphisms $\varphi_i : L \to [a_i, b_i]$, φ_0 is the identify map i.e. $L = [a_0, b_0]$. Let α be an automorphism of L_γ. The ordering relation is defined as follows

$$\begin{array}{lll} \varphi_i(x) \leq \varphi_{i+1}(y) & i = 0,2,4 & \text{iff} \ \ x \leq y \ \ \text{in} \ \ L \\ \varphi_i(x) \geq \varphi_{i+1}(y) & i = 1,3 & \text{iff} \ \ y \leq x \ \ \text{in} \ \ L \\ \varphi_6(x) \leq \varphi_5(y) & & \text{iff} \ \ x \leq y^\alpha \ \ \text{in} \ \ L \end{array}$$

We denote this poset by $T_\alpha(L)$. It is easy to see that $T_\alpha(L)$ is a lattice. If

$$f(x) = ((\dots(x \wedge b_0) \vee a_1) \wedge b_2) \vee a_3 \cdots \vee a_5 \wedge b_0$$

then its restriction to L is α. This f is obviously not a polynomial automorphism of $T_\alpha(L)$.

As in the constuction given in the proof of Theorem 2 we extend $T_\alpha(L)$ to a lattice K_α. Let C be the centralizer of α in $G = Aut L$, and let denote $C = C_0, \dots, C_n$ the right cosets. For every i we consider an isomorphic copy M_i of $T_\alpha(L)$ with a fixed isomorphism $\tau_i : T_\alpha(L) \to M_i$. Finally we identify M_0 with $T_\alpha(L)$ i.e. τ_0 is the identify map. L is a convex subset of M_0. The image of L by τ_i is a convex sublattice L_i of M_i. We glue together the lattices $M_i (i = 0.1. \dots, n)$ by identifying the sublattices L_i, then $M_i \cap M_j = L$ if $i \neq j$. Finally we adjoin a new unit I. and a new zero 0 to the poset $\bigcup M_i$. Let K_α be the poset $\{I, 0\} \cup \bigcup M_i$. It is easy to see that K_α is a lattice and every M_i is a convex sublattice of K_α. Similar to the proof of Theorem 2 we can show:

(i') α is the restriction of a unary polynomial function;

(ii') Each automorphism of L extends to an automorphism of K_α.

(iii') Each automorphism of K_α is the extension of an automorphism of L.

In K_α every non unit congruence relation is determined by its restriction to $L = [a_0, b_0]$ which proves (4') and (5') in the theorem.

References

1. E. Fried and H. Lakser, *Polynomial automorphisms of lattices*, Preprint.
2. R. Wille, *Eine Charecterisieruny endlicher, ordnungspolynomvollständiger Verbände*, Arch. Math. **28** (1977), 557–560.