Periodica Mathematica Hungarica Vol. 19 (1), (1988), pp. 25—31

ON A REPRESENTATION OF DISTRIBUTIVE LATTICES

E. T. SCHMIDT (Budapest)

§ 1. Introduction

The characterization problem of congruence lattices of lattices can be reduced to the representation of distributive join-semilattices as special join-homomorphic images of Boolean lattices. First, we formulate this problem. Let D be an arbitary finite distributive lattice and consider a finite Boolean lattice B containing D. Then D defines a closure operation s: $B \to B$ as follows:

$$s(x) = \wedge \{y \in D;\ y \geq x\}.$$

This closure operation has the additional property that for $x, y \in B$

(1) $$s(x \vee y) = s(x) \vee s(y)$$

which means that s is a topological closure operation. The sublattice D is the set of all closed elements, i.e.,

$$D = s(B) = \{s(x);\ x \in B\}.$$

Conversely, if s is a topological closure operation on B then the closed elements form a sublattice. We would like to represent all distributive (join-) semilattices on a "similar" way. If s is a topological closure operation on an arbitary Boolean lattice B then it is easy to prove that $s(B)$ is a dual Heyting algebra, i.e., we cannot represent all distributive semilattices in this form. To overcome on this difficulties we follow a little modificated way. (1) means that s is a join-homomorphism from B onto D. Let h: $B \to D$ be a join-homomorphism from the Boolean lattice B into a distributive semilattice D. Then h is called a *distributive join-homomorphism* if there is a family $\{s_i;\ i \in I\}$ of topological closure operations on B such that the congruence kernel *ker* h of h is the join of the congruence kernels $\ker s_i$, i.e.,

$$\ker h = \vee \ker s_i.$$

AMS (MOS) subject classifications (1980). Primary 06D05; Secondary 06A12, 06E99.
Key words and phrases. Distributive homomorphic images of Boolean lattices, distributive lattices, distributive semilattices.

Akadémiai Kiadó, Budapest
Kluwer Academic Publishers, Dordrecht

In [2] I have proved the following:

THEOREM. *Every bounded distributive lattice is the distributive join-homomorphic image of a Boolean lattice.*

In other words, if D is a bounded distributive lattice, then there exist a Boolean lattice B and a family $\{s_i;\ i \in I\}$ of topological closure operations on B such that

$$D \cong B/\bigvee \ker s_i. \tag{2}$$

The theorem is true if D does not have a unit element, in this case B denotes a generalized Boolean lattice. The related representation for distributive semilattices is still open. H. Dobbertin [1] has proved that every locally countable distributive semilattice can be represented in the form (2). The given representation has his own interest, but the most important consequence is the following: if D can be represented in form (2) then the ideal lattice of D is isomorphic to the congruence lattice of a lattice.

In this paper I shall give a new, relative short proof of this theorem. The most important part is the construction of a Boolean algebra **B** which will be called the decomposition Boolean algebra of the given distributive lattice D.

§ 2. Some properties of free Boolean algebras

The decomposition Boolean algebra **B** of D is a special subalgebra of a free Boolean algebra which satisfies the property (2). First of all, we list some elementary, wellknown properties of the free Boolean algebras.

The free Boolean algebra $F(G)$ on a set G of n elements $g_1, \dots, g_n$ is isomorphic to 2^{2^G} and each element may be expressed uniquely in disjunctive normal form, i.e., as a finite join of so called minimal terms:

$$\bigvee_e (g_1^{e_1} \wedge \dots \wedge g_n^{e_n}), \tag{3}$$

where $g_i^{e_i}$ is g_i or g_i' (the complement of g_i) and e is a selection from the 2^n different distributions of dashes on the g's. The minimal terms are the atoms of $F(G)$. I resp. 0 denote the unit resp. zero element. Let $G' = \{g' \colon g \in G\}$ where $G \cap G' = \emptyset$ and $g \to g'$ is a bijection between G and G'. Let $(g')' = g'' = g$. For every natural number $k \leq n$ we define a subset of $F(G)$: $G_0 = \{\mathrm{I}\}$, $G_1 = G \cup G'$ and

$$G_k = \{x \in F(G),\ x = g_{i_1}^{e_1} \wedge \dots \wedge g_{i_k}^{e_k}\}$$

where $g_{i_1}, \ldots, g_{i_k}$ are different elements of G. For simplicity we can write $x = g_1 \wedge \ldots \wedge g_k$, where $g_1, \ldots, g_k \in G_1$ and assume that, for each i, j, $i \neq j$, $g_j \neq g_i \neq g'_j$. Further, let

$$H = \bigcup_{i=0}^{n} G_i \cup \{0\}.$$

Then H is closed under the meet operation of $F(G)$. If we restrict the $\vee$ operation to H on the usual way (i.e., for $u, v, w \in H$ if $u \vee v = w$ then we say that $u \vee v$ is in H defined), we get a relative sublattice $\langle H; \vee, \wedge \rangle$ of $F(G)$.

It is easy to show that for incomparable $u, v \in H$, $u \vee v$ is defined if and only if there exist $k \in N$, $w \in G_k$ and $g \in G_1$ such that $u = w \wedge g$, $v = w \wedge g'$. Then

$$u \vee v = (w \wedge g) \wedge (w \wedge g') = w \wedge \mathrm{I} = w.$$

Obviously $u, v \in G_{k-1}$. An ideal of a partial lattice is a nonvoid subset I such that

(i) if $a, b \in I$ and $a \vee b$ exists then $a \vee b \in I$,
(ii) $x \leq a \in I$ implies $x \in I$.

It is easy to prove that $F(G)$ is isomorphic to the lattice of all ideals of H.

The description of the free Boolean algebra $F(G)$ generated by an arbitrary (not necessarily finite) set G is similar, but in the infinite case there are no minimal terms (atoms) and therefore

$$H = \bigcup_{i=0}^{\infty} G_i \cup \{0\}.$$

In this case $F(G)$ is the lattice of all finitely generated ideals of H, and for every $x \in F(G)$ there exists a smallest natural number n such that x has a uniquely representation in the from (3) with suitable $g_1, \ldots, g_n \in G$. Obviously $x \in F(\{g_1, \ldots, g_n\})$.

We define a Boolean subalgebra of $F(G)$ with a subset K of H. Now, let us assume that K satisfies the following properties:

(4) $0 \in K$ and if $x, y \in K$ then $x \wedge y \in K$.

(5) If $u \in K \cap G_{k+1}$ then there exists a $v \in K \cap G_{k+1}$, $v \neq u$ such that $u \vee v$ is defined in H and $u \vee v \in K$.

Let A be the set of all those elements of $F(G)$ which have a representation as a finite join of elements of K. Then A is obviously a sublattice of $F(G)$. If $u \in K \cap G_{k+1}$ then by (5) we have a $v \in K \cap G_{k+1}$ such that $u \vee v$ exists in H and $u \vee v \in K \cap G_k$. Consequently, there exists a $g \in G_1$ satisfying $u = (u \vee v) \wedge g$, $v = (u \vee v) \wedge g'$ which involves

$$u \wedge v = (u \vee v) \wedge g \wedge g' = 0.$$

This means that v is the relative complement of u in the interval $[0, u \vee v]$. Similarly to $u \vee v \in K \cap G_k$, there exists a relative complement $z \in K$ in $[0, u \vee v \vee z]$ such that $u \vee v \vee z \in K \cap G_{k-1}$. Then $v \vee z \in A$ is the relative complement of u in $[0, u \vee v \vee z]$. After a finite number of steps this process breaks of by $[0, I]$, i.e., every $u \in K$ has a complement in A say $u' = \vee k_i$, $k_i \in K$. Let v be another element of K then $v' = \vee h_j$ with suitable $h_j \in K$. Thus

$$(u \vee v)' = u' \wedge v' = \vee k_i \wedge \vee h_j = \vee (k_i \wedge h_j).$$

By our assumption, K is closed under $\wedge$, hence $k_i \wedge h_j \in K$, i.e., $u' \wedge v' \in A$. This proves that A is complemented, i.e., we have the following:

LEMMA 1. *Let K be a subset of H ($\subseteq F(G)$) which satisfies* (4) *and* (5). *Then the sublattice A generated by K is a $\{0, 1\}$-Boolean sublattice of $F(G)$.*

Any pair (p, q), $p \not\geq q$ of elements of a Boolean lattice B defines a closure operation $C_{p,q}$ as follows:

$$C_{p,q}(x) = \begin{cases} x \vee q \text{ if } x \geq p \\ x \text{ otherwise.} \end{cases}$$

It is easy to show that $C_{p,q}$ is topological if and only if p is an atom. In a finite Boolean lattice a topological closure operation $s(x)$ is determined by the closures of the atoms, consequently $s(x)$ is the join of closure operations in the form $C_{p,q}(x)$, where p and q are atoms. We need a special topological closure operation on $F(G)$ which replace $C_{p,q}$. Let $p, q \in G_n$ for some n and let $g \in G_1$. Then

$$p = (p \wedge g) \vee (p \wedge g'), \ q = (q \wedge g) \vee (q \wedge g')$$

imply that

$$C_{p,q} = C_{p\wedge g, q\wedge g} \vee C_{p\wedge g', q\wedge g'}, \tag{6}$$

hence

$$C_{p,q}(x) \geq C_{p\wedge g, q\wedge g}(x)$$

for all x with $x \neq C_{p,q}(x)$. This implies that the operation on $F(G)$ defined by

$$s_{p,q} = \bigvee_{t\in H} C_{p\wedge t, q\wedge t} \tag{7}$$

where $p \wedge t, q \wedge t \neq 0$ is a closure operation. (6) implies that $s_{p,q}$ is topological.

§ 3. The decomposition Boolean algebra

Let D be a bounded distributive lattice. We construct from D a Boolean algebra $\mathbf{B}$, the decomposition Boolean algebra of D. First of all we define the set G of generators of a free Boolean algebra $F(G)$ and then we define a subset K of the corresponding relative sublattice $H \subseteq F(G)$.

1 resp. 0 denote the unit resp. zero element of D. Let G be a subset of $D \times D$ which contains no pairs (a, a) and which contains to each two elements $a \neq b$ exactly one of the two pairs (a, b) and (b, a). If $g = (a, b) \in G$ then $g' = (b, a) \in G'$. Further, G_i and H are defined as before, in § 2.

Let h be a mapping from G_1 onto D defined by

$$h((a, b)) = a \in D.$$

Every nonunit and nonzero element of H has a unique representation as a meet of elements from G_1. Therefore we can extend h to H as follows: if $0 \neq x = g_1 \wedge \ldots \wedge g_k$ then

$$h(x) = h(g_1) \wedge \ldots \wedge h(g_k).$$

Further, let $h(\mathrm{I}) = 1$ and $h(0) = 0$. In general, h is not a join-homomorphism of the partial lattice H, e.g., if $(a, b) \in G_1$ and $a \vee b \neq 1$ in D then $\mathrm{I} = (a, b) \vee \vee (b, a)$ in H but

$$h(\mathrm{I}) = 1 \neq a \vee b = h((a, b)) \vee h((b, a)).$$

We shall define the "greatest" relative-sublattice K of H such that the restriction of h to K will be a join-homomorphism.

The DEFINITION of K: First of all $0, \mathrm{I} \in K$. Let $u \in G_k$, $k > 0$. Then $u \in K$ if and only if there exist $w \in K \cap G_{k-1}$ and $g = (a, b) \in G_1$ such that $u = w \wedge g$ and $h(w) \leq a \vee b$.

This definition implies that $g = (a, b) \in K$ iff $a \vee b = 1$.

First, we show that K satisfies (4), i.e., it is closed under the $\wedge$-operation. Let $T_k = K \cap \bigcup\limits_{i=0}^{k} G_i$. We prove by induction on k that $u_1, u_2 \in T_k$ implies $u_1 \wedge u_2 \in K$. If $u_1, u_2 \in T_1$ then we may assume that $u_1 \neq \mathrm{I} \neq u_2$, i.e., $u_1 = (a_1, b_1)$, $u_2 = (a_2, b_2)$. Then $u_1, u_2 \in K$ means that $a_1 \vee b_1 = a_2 \vee b_2 = 1$. Consequently, $h(u_1) = a_1 \leq a_2 \vee b_2$, i.e., $u_1 \wedge u_2 \in K$. Assume that our statement for T_k is proved. If $u_1, u_2 \in T_{k+1}$ then $u_1 = w_1 \wedge g_1$, $u_2 = w_2 \wedge g_2$ where $g_i = (a_i, b_i)$ and w_1, w_2 are suitable elements of T_k such that $h(w_i) \leq a_i \vee b_i$. By the assumption $w_1 \wedge w_2 \in K$. Then

$$h(w_1 \wedge w_2) \leq h(w_1) \leq a_1 \vee b_1$$

yields $w_1 \wedge w_2 \wedge g_1 \in K$. Similarly,

$$h(w_1 \wedge w_2 \wedge g_1 \wedge g_2) \leq h(w_2) \leq a_2 \vee b_2$$

implies

$$w_1 \wedge w_2 \wedge g_1 \wedge g_2 = u_1 \wedge u_2 \in K.$$

K satisfies (5). Let $u \in K \cap G_{k+1}$. By the definition of K there exist $w \in K \cap G_k$ and $g = (a, b) \in G_1$ such that $u = w \wedge g$ and $h(w) \leq a \vee b$. Let v be the element $w \wedge g'$ where $g' = (b, a)$. Then again by the definition of K we have $v \in K \cap G_{k+1}$ and $u \vee v = w$ is defined in K. This proves (5).

K can be considered as a relative sublattice of H. The mapping $h\colon H \to D$ can be restricted to K, $h|_K\colon K \to D$. We prove:

LEMMA 2. *$h|_K\colon K \to D$ is a join-homomorphism onto D.*

PROOF. If u and v are incomparable elements of K and $u \vee v$ is defined then

$$u = g_1 \wedge \ldots \wedge g_{n-1} \wedge g,\ v = g_1 \wedge \ldots \wedge g_{n-1} \wedge g',\ w = u \vee v = g_1 \wedge \ldots \wedge g_{n-1}$$

where $g_1, \ldots, g, g' \in G_1$. We have to prove that $h(u \vee v) = h(u) \vee h(v)$. By the definition of K there is a permutation of the elements $g_1, \ldots, g_{n-1}, g$, say

$$g_1, g_2, \ldots, g_i, g, g_{i+1}, \ldots, g_{n-1}$$

such that if $g = (a, b)$ then

$$a \vee b \geq h(g_1 \wedge \ldots \wedge g_i).$$

Consequently,

$$a \vee b \geq h(g_1 \wedge \ldots \wedge g_{n-1})$$

and hence we conclude:

$$\begin{aligned} h(u) \vee h(v) &= (h(g_1 \wedge \ldots \wedge g_{n-1}) \wedge h(g)) \vee (h(g_1 \wedge \ldots \wedge g_{n-1}) \wedge h(g')) = \\ &= (h(g_1 \wedge \ldots \wedge g_{n-1}) \wedge a) \vee (h(g_1 \wedge \ldots \wedge g_{n-1}) \wedge b) = \\ &= h(g_1 \wedge \ldots \wedge g_{n-1}) \wedge (a \vee b) = h(g_1 \wedge \ldots \wedge g_{n-1}) = h(u \vee v). \end{aligned}$$

This proves our lemma.

By Lemma 2 we have a join-homomorphism $h|_K\colon K \to D$. $(a, 1) \in K$ and $h((a, 1)) = a$ which means that $h|_K$ is onto mapping. On the other hand the conditions of Lemma 1 are satisfied for K, thus the sublattice of $F(G)$ generated by K is a Boolean lattice. We denote this by $\mathbf{B}$. It is easy to see, that $h|_K\colon K \to D$ can be extend to a join-homomorphism $h\colon \mathbf{B} \to D$, and h is determined by its restriction to K, i.e., by $h|_K$.

We have to prove that h is a distributive join-homomorphism. Assume that for $p \in K \cap G_K$ and $q \in K$, $p \not\geq q$ and $h(p) \geq h(q)$, i.e., $h(p \vee q) = h(q)$. Then in the free Boolean algebra $F(G)$

$$s_{p,q}(p) = p \vee q (= C_{p,q}(p))$$

and, for an arbitary $r \in K \cap G_K$, $s_{p,q}(r) = r$. Let us consider $K \cap G_{K+1}$. If $p \wedge q \in K \cap G_{K+1}$ for some $g = (a, b)$ then, by the definition of K, we have $a \vee b \geq h(p)$. By our assumption, $h(p) \geq h(q)$, thus $a \vee b \geq h(q)$. This implies $q \wedge g \in K$ and $(p \wedge q) \vee (q \wedge g) \in \mathbf{B}$. But

$$C_{p\wedge g, q\wedge g}(p \wedge g) = (p \wedge g) \vee (q \wedge g),$$

hence

$$s_{p,q}(p \wedge g) = C_{p\wedge g, q\wedge g}(p \wedge g) \in \mathbf{B}.$$

Thus $s_{p,q}(x) \in \mathbf{B}$ for every $x \in \mathbf{B}$; i.e., $s_{p,q}$ is a topological closure operation on $\mathbf{B}$, and $\ker h \supseteq \ker s_{p,q}$. Consequently, $\ker h = \bigvee \ker s_{p,q}$, i.e., h is distributive.

REFERENCES

[1] H. DOBBERTIN, *Vaught measures and their applications in lattice theory*, Universität Hannover, 1985. (Thesis)

[2] E. T. SCHMIDT, The ideal lattice of a distributive lattice with 0 is the congruence lattice of a lattice, *Acta Sci. Math.* (Szeged) **43** (1981), 153–168. *MR* **82**g: 06015

(Received January 13, 1986)

MTA MATEMATIKAI KUTATÓINTÉZET
H–1364 BUDAPEST
P.O. BOX 127.
REÁLTANODA U. 13–15.
HUNGARY