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ON LOCALLY ORDER-POLYNOMIALLY COMPLETE MODULAR LATTICES

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1. Introduction

Let L be a lattice. $F_k(L)=L^{(L^k)}$ is the set of all k-place functions on L. If we define pointwise meet and join operation on $F_k(L)$, then $F_k(L)$ becomes a lattice. The elements of the sublattice $P_k(L)$ of $F_k(L)$ generated by the projections and the constant functions will be called k-place polynomial functions on L. If $f\in F_k(L)$ has the property that for every finite subset $M\subseteq L^k$ there exists a $p\in P_k(L)$ such that f and p coincide on M, then p is called a local polynomial function. $f\in F_k(L)$ is called order-preserving if $a_i\leqq b_i$, $i=1, \ldots, k$ implies $f(a_1, \ldots, a_k)\leqq f(b_1, \ldots, b_k)$. L is called (locally) order-polynomially complete iff every order-preserving function on L is a (local) polynomial function.

The first characterization of finite order-polynomially complete lattices was given in Wille [6]. For finite modular lattices he proved the following

THEOREM A. *A finite modular lattice L is (locally) order-polynomially complete if and only if L is simple and relatively complemented (i.e. an irreducible projective geometry).*

This theorem suggests the following question: is every locally order-polynomially complete modular lattice relatively complemented? In [2], Fried proved that the answer is yes if L has locally finite length. Our main result is a construction of a locally order-polynomially complete modular lattice wich is not relatively complemented. To prove that our example is locally order-polynomially complete we need the following useful result of Dorninger [1]:

THEOREM B. *A lattice L is locally order-polynomially complete if and only if* (1) *L is simple and* (2) *for all $a, b\in L$, $a\leqq b$ and all 1-place polynomial functions p, q with $p(b)=q(a)$ there exists a 1-place polynomial function r such that $r(a)\leqq p(a)$ and $r(b)\geqq q(b)$.*

It is an easy consequence of this theorem that every simple, relatively complemented lattice is locally order-polynomially complete (for finite lattices see Wille [7]).

2. Preliminaries

By a 1-translation of a lattice L we mean a unary polynomial-function on L that is either the identity function id (x) or a constant function or is obtained from one of the two lattice operations by fixing one of the arguments. By a translation of L we mean a unary polynomial that is the composition of 1-translations. Therefore if $t(x)$ is a translation but not a constant function then $t(x)$ may be written in the form $t(x)=(\dots((x\vee a_1)\wedge a_2)\vee a_3\dots)\wedge a_n$ where each $a_i\in L$ or is the empty symbol. We say that a translation is a unary polynomial function of degree 1. If $f(x)$ and $g(x)$ are unary polynomial functions of degree n resp. m, then the degree of the functions $f(x)\vee g(x)$ and $f(x)\wedge g(x)$ is $n+m$.

If two intervals $[a, b]$ and $[c, d]$ in a lattice are such that $a=b\wedge c$ and $d=b\vee c$, then each is said to be transpose (perspective) of the other. $[a, b]$ and $[c, d]$ are said to be projective if there exists intervals $[a, b]=[x_0, y_0], [x_1, y_1], \dots, [x_n, y_n]=[c, d]$ such that any two successive intervals are transposes of each other. In a modular lattice, any two projective intervals are isomorphic. A well-known property of modular lattices (see [3], p. 133) is expressed in the following

LEMMA 1. *Let $t(x)$ be a translation of a modular lattice L and let $a, b\in L$, $a<b$ such that $t(a)\neq t(b)$. Then there exists a proper subinterval $[a', b']$ of $[a, b]$ such that $[a', b']$ and $[t(a), t(b)]$ are projective.*

Let D be an arbitrary distributive lattice with 0 and 1. Take the subposet of D^3 consisting of all ordered triples (a, b, c) such that $a\wedge b=a\wedge c=b\wedge c$. This poset is a modular lattice $M_3[D]$ (see Schmidt [4]). The elements $i=(1, 1, 1)$, $u=(1, 0, 0,)$ $v=(0, 1, 0)$, $w=(0, 0, 1)$ and $o=(0, 0, 0)$ form a diamond, M_3. The interval $[0, u]$, i.e. the ideal $(u]$ is isomorphic to D. Similarly, $(v]\cong(w]\cong D$.

Let us take two bounded lattices L_1 and L_2. Suppose that L_1 has a principal dual ideal J_1, L_2 has a principal ideal J_2 and $J_1\cong J_2$. Let $\varphi: x\to x'$ denote this isomorphism. We can construct a lattice L as follows: L is the set of all $x\in L_1$ and $x\in L_2$; we identify x with x' for all $x\in J_1$; $x\leqq y$ has unchanged meaning if $x, y\in L_1$ or $x, y\in L_2$ and $x<y$, $x, y\notin J_1=J_2$ iff $x\in L_1$, $y\in L_2$ and there exists a $z\in J$ such that $x<z$ in L_1, and $z<y$ in L_2. It is easy to see that L is a modular lattice if so are L_1 and L_2. This is the so-called Hall—Dilworth construction.

3. Modular lattices of finite length

By Theorem B, a locally order-polynomially complete lattice is simple. A direct proof is the following (see Wille [7]): let Θ be a non trivial congruence relation of L and $a, b, c, d\in L$ such that $a\nleqq b$, $c>d$, $(a, b)\in\Theta$ and $(c, d)\notin\Theta$. We define a mapping $f\colon L\to L$ by

$$f(x):=\begin{cases} c, & \text{if} \quad a\leqq x\\ d, & \text{if} \quad a\nleqq x\end{cases}$$

then f is an order-preserving function and it cannot be a local polynomial function, namely $(a, b)\in\Theta$ but $(c, d)=(f(a), f(b))\notin\Theta$.

PROPOSITION. *Let $p(x)$ be a polynomial function on a modular lattice L of locally finite length. If the interval $[u, v]$ is a complemented sublattice then $[p(u), p(v)]$ is complemented, too.*

PROOF. We prove this statement by induction on the degree of $p(x)$. (The degree of $p(x)$ is the number of occurrences of the variable x; the constant function has degree 0.) If $p(x)$ has degree 1 (i.e. it is a translation) then by Lemma 1 $[p(u), p(v)]$ is isomorphic to a subinterval of a complemented modular lattice $[u, v]$, hence $[p(u), p(v)]$ is complemented. Assume that the assertion is proved for polynomials of degree $<n$ $(n>1)$, and let $p(x)$ be a polynomial function of degree n. Then $p(x)$ has one of the following two decompositions, $p(x)=q(x)\vee r(x)$ or $p(x)=q(x)\wedge r(x)$ where the degree of q and r is less than n. Denote $p(u)$ by s then $q'(x)=q(x)\vee s$ is a polynomial function and has the same degree as q. Then by our assumption $[q'(u), q'(v)]$ is complemented. On the other hand $q'(u)=q(u)\vee s=q(u)\vee p(u)=p(u)$ and $q'(v)=q(v)\vee s=q(v)\vee p(u)$, hence $[p(u), p(u)\vee q(v)]$ is a complemented interval of finite length. Then the unit element (i.e. $p(u)\vee q(v)$) is the join of atoms of this interval. Similarly, $p(u)\vee r(v)$ is the join of atoms in $[p(u), p(u)\vee r(v)]$. We claim that $p(v)=(p(u)\vee q(v))\vee(p(u)\vee r(v))$ is the join of atoms in $[p(u), p(v)]$, hence this interval is complemented.

THEOREM C (Fried [2]). *Let L be a modular lattice of locally finite length. L is locally order-polynomially complete iff each interval of L is an irreducible projective geometry.*

PROOF. If each interval is an irreducible projective geometry then L is a relatively complemented, simple lattice; hence L is locally order-polynomially complete.

Conversely, let us assume that L is locally order-polynomially complete. If L is not relatively complemented then L contains a triple a, b, c such that $a\prec b\prec c$ and b has no relative complement in $[a, c]$. Let $u\prec v$ be any two elements of L and define

$$f(x) := \begin{cases} a, & \text{if } x \leqq u \\ c, & \text{if } x \not\leqq u. \end{cases}$$

Then $f(x)$ is an order-preserving function and by the Proposition, f cannot be a local polynomial function. Let $[a, b]$ be an interval of L and let c_1, c_2 $(c_1\neq c_2)$ be two atoms of $[a, b]$. To prove that $[a, b]$ is an irreducible projective geometry, it is enough to show that there exists an atom d of $[a, b]$ such that $d\neq c_1, c_2$ and $d\leqq c_1\vee c_2$. Since L is a simple modular lattice, the intervals $[a, c_1]$ and $[a, c_2]$ are projective in L, hence they are projective in some interval $[\bar{a}, \bar{b}]$, where $\bar{a}\leqq a<b\leqq\bar{b}$. The interval $[\bar{a}, \bar{b}]$ is again a complemented modular lattice of finite length, therefore $[\bar{a}, \bar{b}]$ is the direct product of irreducible projective geometries ([3], p. 212). The projectivity of $[a, c_1]$ and $[a, c_2]$ in $[\bar{a}, \bar{b}]$ yields that these intervals belong to the same irreducible component, i.e. $[a, c_1\vee c_2]$ is a subinterval of an irreducible projective geometry, therefore there exists a d with $a<d<c_1\vee c_2$.

4. The construction

We prove the following:

THEOREM D. *There exists a locally order-polynomially complete modular lattice which is not relatively complemented.*

Let Q be the interval $[0, 1]$ of rational numbers. First, we define two unary operations on Q:

$$f(x) := \begin{cases} 2x, & \text{if } 0 \leqq x \leqq \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} < x, \end{cases} \qquad g(x) := \begin{cases} 2x-1, & \text{if } \frac{1}{2} \leqq x \leqq 1 \\ 0, & \text{if } x < \frac{1}{2}, \end{cases}$$

and we consider the unary algebra $\mathbf{Q}=\langle Q; f, g\rangle$. Let id be the identity map on Q and define $f^\circ=g^\circ=\mathrm{id}$. Then apart from the constant maps the polynomial functions on $\mathbf{Q}$ are of the form $p(x)=g^{k_1}f^{l_1}g^{k_2}\dots g^{k_r}f^{k_r}$, where $k_i\geqq 0$, $l_i\geqq 0$.

LEMMA 2. *To each $a, b\in Q$, $a<b$ there exists a 1-place polynomial function $p(x)$ on $\mathbf{Q}$ such that $p(a)=0$ and $p(b)=1$.*

PROOF. If $0\leqq a<b\leqq 1$ then for suitable k and n $(n\geqq 1,\ k=0, \dots, 2^{n-1})$ $a\leqq\frac{k}{2^n}<\frac{k+1}{2^n}\leqq b$. Therefore if we have a $p(x)$ such that $p\left(\frac{k}{2^n}\right)=0$ and $p\left(\frac{k+1}{2^n}\right)=1$ then by the order-preserving property of polynomials we get $p(a)=0$, $p(b)=1$, i.e. we can assume that $a=\frac{k}{2^n}$ and $b=\frac{k+1}{2^n}$. We prove the lemma by induction on n. If $n=1$ then we have:

$$f\colon \left[0, \frac{1}{2}\right]\to[0, 1] \quad\text{and}\quad g\colon\left[\frac{1}{2}, 1\right]\to[0, 1].$$

Assume that the statement is proved for $n-1$. The following two cases arise:

$$\text{(1)}\qquad b=\frac{k+1}{2^n}\leqq\frac{1}{2},\quad\text{then}\quad f(a)=\frac{k}{2^{n-1}},\quad f(b)=\frac{k+1}{2^{n-1}}\leqq 1.$$

By our assumption there exists a polynomial function $p(x)$ such that $p\left(\frac{k}{2^{n-1}}\right)=0$, $p\left(\frac{k+1}{2^{n-1}}\right)=1$, thus $\bar p=pf$ satisfies $\bar p(a)=0$, $\bar p(b)=1$.

$$\text{(2)}\qquad a=\frac{k}{2^n}\geqq\frac{1}{2},\quad\text{then}\quad g(a)=\frac{k}{2^{n-1}}-1,\quad g(b)=\frac{k+1}{2^{n-1}}-1$$

and we have a polynomial function $p(x)$ such that $p\left(\frac{k}{2^{n-1}}-1\right)=0$, $p\left(\frac{k+1}{2^{n-1}}-1\right)=1$. Then $\bar p=pg$ satisfies $\bar p(a)=0$ and $\bar p(b)=1$.

Now we consider the modular lattice $M_3[Q]$. The zero resp. unit of this lattice is denoted by o resp. i. $M_3[Q]$ has three elements u, v, w such that o, u, v, w, i form a diamond, M_3.

We take $M_3[Q]$ in three pairwise disjoint copies L_1, L_2 and L_3. Then o_k, u_k, v_k, w_k, i_k denote the elements corresponding to o, u, v, w, i by the isomorphism $M_3[Q]\cong\cong L_k$.

$J_1=[u_1, i_1]$ is a principal dual ideal of L_1 and $J_1\cong Q$. Similarly, $J_2=[o_2, w_2]$ is a principal ideal of L_2 isomorphic to Q. Therefore $J_1\cong J_2$, we can apply the Hall—Dilworth construction and we get the following modular lattice $L_{1,2}$:

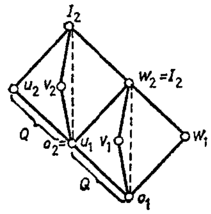

Fig. 1

Let S be the sublattice of L_{12} consisting of all elements $x \vee y$ where $x \leqq u_2$ and $y \leqq w_1$. The interval $[o_1, u_2]$ is isomorphic to Q hence $S \cong Q \times Q$. Similarly, L_3 contains a sublattice $T := \{x \vee y;\ x \leqq u_3,\ y \leqq w_3\}$ isomorphic to $Q \times Q$. Consequently we have an isomorphism $\varphi: S \to T$ with $\varphi(u_2) = u_3$, $\varphi(w_1) = w_3$. Now, we apply a gluing construction (similar to the Hall—Dilworth construction) by identifying the corresponding elements by φ. This construction was first defined in [4], see Fig. 2.

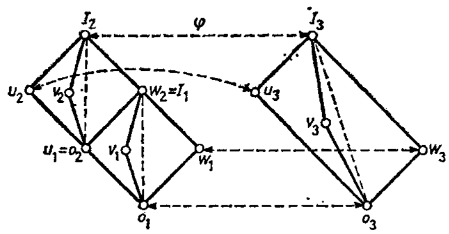

Fig. 2

In this way we get a modular lattice L.

L:

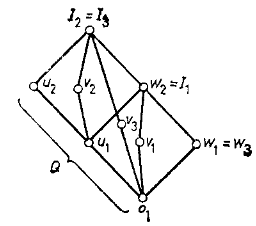

Fig. 3

To prove that L is modular, we have to show that L does not contain a pentagon generated by a, b, c. But L contains four sublattices generated by $\{u_2, v_1, x;\ 0 \leqq x \leqq v_3\}$, $\{u_2, w_1, x;\ 0 \leqq x \leqq v_3\}$, $\{v_2, v_1, x;\ 0 \leqq x \leqq v_3\}$ and $\{v_2, w_1, x;\ 0 \leqq x \leqq v_3\}$ which are all isomorphic to $M_3[Q]$, hence they are all modular sublattices. If L contains a pentagon generated by a, b, c then it is easy to see that a, b, c are contained in one of these sublattices, a contradiction.

The lattice L contains the diamonds $(o_k, u_k, v_k, w_k, i_k)$, $k = 1, 2, 3$. We identify $[o_1, u_2]$ with Q, and define two polynomial functions on L:

$$\bar{f}(x) = (((x \vee v_1) \wedge w_1) \vee v_3) \wedge u_2, \quad \bar{g}(x) = (((x \vee v_2) \wedge w_1) \vee v_3) \wedge u_2.$$

It is easy to show that the restrictions of these functions to Q are exactly the functions f and g defined above.

We prove that L is locally order-polynomially complete. Let $a, b \in L$, $a < b$. By the gluing construction there exists a $c \in L$ such that $a \leqq c \leqq b$ and $a, c \in L_{1,2}$, $c, b \in L_3$ or conversely $a, c \in L_3$, $c, b \in L_{1,2}$. On the other hand $a < b$ implies that either $a < c$ or $c < b$. If $a, c \in L_{1,2}$, $a < c$ then either $a \wedge u_2 < c \wedge u_2$ or $a \wedge w_1 < c \wedge w_1$. Similarly if $c, b \in L_3$, $c < b$ then either $c \wedge u_2 < b \wedge u_2$ or $c \wedge w_1 < b \wedge w_1$. The intervals

$[o_1, u_2]$ and $[o_1, w_1]$ are projective and therefore we have a 1-place polynomial function t on L such that $t(a) < t(b) \leqq u_2$. (The second case, $a, c \in L_3$ is similar.) By Lemma 2 there exists a polynomial function $s(x)$ satisfying $u_2 = st(b)$, $o_1 = st(a)$. In L we have the polynomial function $d(x) = (x \vee v_3) \wedge w_3$ which satisfies $d(u_2) = w_3$, $d(o_1) = o_1$. Finally let $r(x) = st(x) \vee dst(x)$, then we have

$$r(a) = st(a) \vee dst(a) = o_1 \vee o_1 = o_1,$$

$$r(b) = st(b) \vee dst(b) = u_2 \vee d(u_2) = u_2 \vee w_3 = i_2.$$

We have to show that the conditions of Theorem B are satisfied. Indeed, let Θ be a congruence relation of L such that $a \equiv b(\Theta)$. Then we obtain $o_1 = r(a) \equiv \equiv r(b) = i_2(\Theta)$, i.e. L is a simple lattice. If p, q are arbitary 1-place polynomial functions then with the given polynomial function $r(x)$ we get condition (2). The theorem is proved.

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