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Homomorphism of distributive lattices as restriction of congruences

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Homomorphism of distributive lattices as restriction of congruences

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Dedicated to the memory of András Huhn

1. Introduction. Let I be an ideal of a lattice L . Then the map $q: \Theta \rightarrow \Theta_I$, restricting a congruence relation Θ to I is a 0 and 1 preserving lattice-homomorphism of the congruence lattice $\text{Con } L$ into $\text{Con } I$. G. GRÄTZER and H. LAKSER [1] have proved the converse for finite lattices:

Theorem A. *Let D and E be finite distributive lattices, and let $\varphi: D \rightarrow E$ be a 0 and 1 preserving homomorphism of D into E . Then there exist a finite lattice L , and an ideal I of L , such that there are isomorphisms $\alpha: D \rightarrow \text{Con } L$, $\beta: E \rightarrow \text{Con } I$, satisfying $\beta\varphi = q\alpha$, where $q: \Theta \rightarrow \Theta_I$ is the restriction of $\Theta \in \text{Con } L$ to I . (See Figure 1.)*

$$\begin{array}{ccc} D & \xrightarrow{\varphi} & I \\ \alpha \downarrow & & \downarrow \beta \\ \text{Con } L & \xrightarrow{q} & \text{Con } I \end{array}$$

Figure 1

The purpose of this paper is twofold. Firstly, we generalize Theorem A for distributive algebraic lattices satisfying the following condition

(*) for all compact x , $x \vee \bigwedge (x_i | i \in I) = \bigwedge (x \vee x_i | i \in I)$,

which is a weaker form of the infinite meet distributivity. Secondly, we win a short proof of Theorem A, which uses a construction given in SCHMIDT [3] and [4].

2. Dual Heyting algebras. Let L be a lattice. The dual pseudocomplement of a relative to b is an element $a * b$ of L satisfying $a \vee x \leq b$ iff $x \leq a * b$. A *dual Heyting algebra* is a distributive lattice with 1 in which $a * b$ exists for all $a, b \in L$. The subset

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of all compact elements of an algebraic lattice A is denoted by $K(A)$. This $K(A)$ is a join-subsemilattice with smallest element 0. A lattice A is called *arithmetic* iff it is algebraic and $K(A)$ is a sublattice of L .

Lemma 1. *Let L be a distributive arithmetic lattice, whose unit element is compact. L satisfies the condition (*) if and only if $K(L)$ is a dual Heyting algebra.*

Proof. First, let $K(L)$ be a dual Heyting algebra. Then L is isomorphic to the lattice of all ideals of $K(L)$, and the compact elements of the ideal lattice are precisely the principal ideals. Therefore we have to show that

$$(x] \vee \bigwedge (J_i | i \in I) = \bigwedge ((x] \vee J_i | i \in I)$$

where the J_i -s are ideals of $K(L)$. It is enough to verify that the right side is contained in the left side. Let $a \in \bigwedge ((x] \vee J_i | i \in I)$, then $a \in (x] \vee J_i$ for all $i \in I$, i.e., $a \leq x \vee y_i$ for suitable $y_i \in J_i$. $K(L)$ is a dual Heyting algebra, therefore $x * a$ exists and $x * a \leq y_i$ implies $x * a \in J_i$, $i \in I$. Thus $x * a \in \bigwedge (J_i | i \in I)$. By the definition of $x * a$ we have $a \leq x \vee (x * a)$, i.e. $a \in (x] \vee \bigwedge (J_i | i \in I)$.

By assumption L is a distributive arithmetic lattice with compact unit element, thus $K(L)$ is a bounded distributive lattice. Consider all u_i -s such that $a \vee u_i \leq b$. Then $b \in \bigwedge ((a] \vee (u_i])$. Applying (*) we obtain $b \in (a] \vee \bigwedge (u_i]$, i.e. there exists a $z \in \bigwedge (u_i]$ such that $a \vee z \leq b$. Obviously $z = a * b$.

By Lemma 1, we can work with dual Heyting algebras, namely L is determined by $K(L)$.

Let L be a $\{0, 1\}$ -sublattice of the Boolean lattice B . Then L is said to *R-generate* B if L generates B as a ring. The following lemma is due to H. M. MacNeille (see G. GRÄTZER [2]).

Lemma 2. *Let B be R-generated by L . Then every $a \in B$ can be expressed in the form*

$$a_0 + a_1 + \dots + a_{n-1}, \quad a_0 \leq a_1 \leq \dots \leq a_{n-1}, \quad a_0, \dots, a_{n-1} \in L.$$

A sublattice L' of a dual Heyting algebra L is called a *subalgebra* if for every $x \in L$ there exists a smallest $\bar{x} \in L'$ that $x \leq \bar{x}$.

Lemma 3. *A subalgebra of a dual Heyting algebra is a dual Heyting algebra.*

Proof. Let L' be a subalgebra of L and let $a, b \in L'$, $a \leq b$. Then $a * b$ exists in L , and it is easy to verify that $\overline{a * b}$ is the dual pseudocomplement of a relative to b in L' . It is clear that if the dual pseudocomplement exists for comparable pairs then there exists for arbitrary pairs.

For a bounded distributive lattice L we shall denote by $B(L)$ the Boolean lattice R-generated by L .

Lemma 4. *Let L be a dual Heyting algebra. Then L is a subalgebra of $B(L)$.*

Proof. Let L be a dual Heyting algebra. Then by Lemma 2 every x can be expressed in the form $x = a_0 + \dots + a_{n-1}$. We prove the existence of \bar{x} by induction on n . If $n=1$, i.e. $x = a_0$ then $x \in L$ hence $\bar{x} = x$. For $n=2$, i.e. $x = a_0 + a_1$, x is the relative complement of a_0 in the interval $[0, a_1]$. Then $a_0 \vee y \equiv a_1$ and $y \in L$ imply $y \equiv a_0 + a_1$, thus $\overline{a_0 + a_1}$ exists and $\overline{a_0 + a_1} = a_0 * a_1$. Let us assume that $n > 2$. $a_{n-2} + a_{n-1}$ is the relative complement of a_{n-2} in the interval $[0, a_{n-1}]$, hence $a_{n-3} \wedge (a_{n-2} + a_{n-1}) \equiv a_{n-2} \wedge (a_{n-2} + a_{n-1}) = 0$. Obviously $a_0 + \dots + a_{n-3} \leq a_{n-2}$, thus $(a_0 + \dots + a_{n-3}) \wedge (a_{n-2} + a_{n-1}) = 0$. This implies that $a_0 + \dots + a_{n-1} = (a_0 + \dots + a_{n-3}) + (a_{n-2} + a_{n-1}) = (a_0 + \dots + a_{n-3}) \vee (a_{n-2} + a_{n-1})$.

Let x, y be arbitrary elements of $B(L)$ such that \bar{x} and \bar{y} exist. We prove that $\overline{x \vee y}$ exists and $\overline{x \vee y} = \bar{x} \vee \bar{y}$. Let $x \vee y \leq z \leq \bar{x} \vee \bar{y}$, $z \in L$. Then we get from $x, y \leq x \vee y$ that $x \leq \bar{x} \wedge z \in L$, $y \leq \bar{y} \wedge z \in L$, and we conclude that $\bar{x} \leq z$, $\bar{y} \leq z$, i.e. $z = \bar{x} \vee \bar{y}$, which proves $\overline{x \vee y} = \bar{x} \vee \bar{y}$. Applying this equality for $x = a_0 + \dots + a_{n-3}$ and $y = a_{n-2} + a_{n-1}$ we obtain that $\overline{x \vee y} = \bar{x} + \bar{y}$ exists.

Lemma 5. *Let Θ be a compact congruence relation of a dual Heyting algebra L . Then L/Θ is a dual Heyting algebra.*

Proof. The compact congruence relations are exactly the finite joins of principal congruence relations. To prove the lemma, by the Second Isomorphism Theorem we may assume that Θ is a principal congruence relation, i.e. $\Theta = \Theta(u, v)$, $u \leq v$.

Let L be a dual Heyting algebra. We prove that each congruence class of $\Theta(u, v)$ contains a smallest element. In distributive lattices $\Theta(u, v)$ has the following description (see [2], p. 74): $a \equiv b$ ($\Theta(u, v)$) iff $v \vee a = v \vee b$ and $u \wedge a = u \wedge b$. Let b be a fixed element of L . Then $v \vee a = v \vee b$ implies that $a \equiv v * (v \vee b)$. Therefore $v * (v \vee b)$ is the least element of the $\Theta(u, v)$ -class containing b . Now, let $a < b$ and let c denote the least element of the $\Theta(u, v)$ -class containing b . Let $[x]$ denote the $\Theta(u, v)$ -class containing x . Then obviously $[a] * [b] = [a * c]$.

Corollary. *Every Θ -class of a compact congruence relation Θ of a dual Heyting algebra contains a smallest element.*

3. The main theorem. In this section we formulate our main theorem and then we give two special representations of dual Heyting algebras.

Theorem B. *Let D and E be dual Heyting algebras, and let $\varphi: D \rightarrow E$ be a 0 and 1 preserving homomorphism of D into E such that the congruence kernel $\text{Ker } \varphi$ is a compact congruence relation and $F = \text{Im } \varphi$ is a subalgebra of E . Then there exist a lattice L , and a principal ideal I of L , such that there are isomorphisms $\alpha: D \rightarrow K(\text{Con } L)$, $\beta: E \rightarrow J(\text{Con } L)$ satisfying $\beta\varphi = \varrho\alpha$, where $\varrho: \Theta \rightarrow \Theta_I$ is the restriction of $\Theta \in \text{Con } L$ to I .*

If L_1 and L_2 are lattices with zero elements 0_1 resp. 0_2 then in the direct product $L_1 \times L_2$ the elements $\langle x, 0_2 \rangle$ ($x \in L_1$) form an ideal L'_1 isomorphic to L_1 . Therefore we can identify L_1 with L'_1 and similarly L_2 with the ideal $L'_2 = \{\langle 0_1, x \rangle\}$.

Let Θ be the congruence kernel of the homomorphism $\varphi: D \rightarrow E$. By our assumption Θ is a compact congruence relation of D . On the other hand D is a bounded distributive lattice, therefore the unit of $\text{Con } D$ is compact. The compact elements of $\text{Con } D$ form a Boolean lattice (see [2], p. 86, Exercise 41), consequently Θ has a complement Θ' in $\text{Con } D$. Then D is a subdirect product of D/Θ and D/Θ' , therefore $D \subseteq D/\Theta \times D/\Theta'$.

$F = \text{Im } \varphi$ is a $\{0, 1\}$ -sublattice of E and F is isomorphic to D/Θ ; we identify D/Θ and F . Hence we may consider D as a $\{0, 1\}$ -sublattice of $E \times D/\Theta'$. Let e be the unit of E and $\pi_1(x) = x \wedge e$ denotes the projection map of $E \times D/\Theta'$ onto E . Observe, that the restriction of π_1 to D ($\subseteq E \times D/\Theta'$) gives the homomorphism φ (see Figure 2).

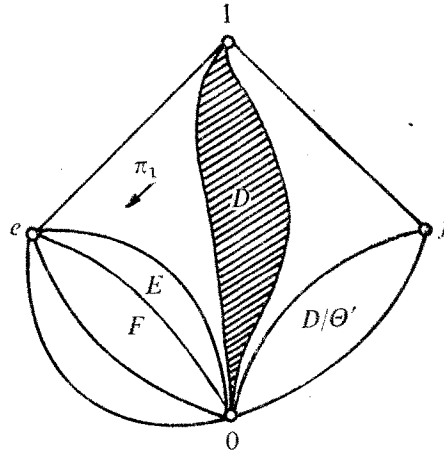


Figure 2

Lemma 6. $E \times D/\Theta'$ is a dual Heyting algebra and D is a subalgebra of $E \times D/\Theta'$.

Proof. By our assumptions D and E are dual Heyting algebras and Θ' is a compact congruence relation. Hence by Lemma 5 D/Θ' and thus $E \times D/\Theta'$ are dual Heyting algebras. $F = \text{Im } \varphi$ is a subalgebra of E , hence by Lemma 3 F and $F \times D/\Theta'$ are dual Heyting algebras.

We have seen that D is a subdirect product of F and D/Θ' . First we show that D is a subalgebra of the dual Heyting algebra $F \times D/\Theta'$. An arbitrary element of $F \times D/\Theta'$ can be written in the form $x = \langle [a]\Theta, [b]\Theta' \rangle$ where $a, b \in D$. By Corol-

lary of Lemma 5 the congruence classes of Θ and Θ' have smallest elements. Let now a_0 and b_0 be the smallest elements of $[a]\Theta$ resp. $[b]\Theta'$. Then $\langle [a_0 \wedge b_0]\Theta, [a_0 \wedge b_0]\Theta' \rangle \in D$. Obviously this element is in D , the smallest one which is greater or equal than x , i.e. \bar{x} exist. This proves that D is a subalgebra of $F \times D/\Theta'$. On the other hand F is a subalgebra of E , consequently $F \times D/\Theta'$ is a subalgebra of $E \times D/\Theta'$, which proves finally that D is a subalgebra of $E \times D/\Theta'$ (namely a subalgebra of a subalgebra is again a subalgebra).

In [3] (or see [4]) there was given a special lattice construction to prove that the lattice of all ideals of a dual Heyting algebra is isomorphic to the congruence lattice of some lattice. The most important properties of this construction are summarized in the following lemma.

Lemma 7. *Let K be a $\{0, 1\}$ -subalgebra of a Boolean lattice A , and let $\varepsilon: K \rightarrow A$ be the identity map. There exists a bounded lattice M with the following properties:*

- (i) *M contains three elements u, v, w such that $\{0, u, v, w, 1\}$ form a sublattice isomorphic to the diamond M_3 . There are isomorphisms $\mu: (u) \rightarrow K$ and $\tau: (v) \rightarrow A$. If for $x \in K$ $\pi(x)$ means $(x \vee w) \wedge v$ then $\tau\pi = \varepsilon\mu$.*
- (ii) *The map $x \rightarrow x \vee u$ ($x \in (v)$) is an isomorphism of (v) onto the filter $[u]$.*
- (iii) *A congruence relation $\Theta(0, x)$ of (v) can be extended to M iff $\tau(x) \in \varepsilon(K)$, and every compact congruence relation of M is the extension of a congruence relation $\Theta(0, x) \in \text{Con}(v)$.*

Remark. A is a Boolean lattice, therefore every compact congruence relation of A ($\cong(v)$) can be written in the form $\Theta(0, x)$. Condition (iii) implies that $\text{Con } M \cong \cong I(K)$, i.e. $K(\text{Con } M) \cong K$.

4. The proof of Theorem B. We apply Lemma 7 twice to get two lattices M_1 and M_2 . Then we use the so called Hall—Dilworth gluing construction which yields a lattice L having the properties required in the theorem.

By Lemma 6 D is a subalgebra of the dual Heyting algebra $E \times D/\Theta'$ and by Lemma 4 $E \times D/\Theta'$ is a subalgebra of $B(E \times D/\Theta')$. Consequently D is a subalgebra of $B(E \times D/\Theta')$. Then we can choose in Lemma 7 $K = D$ and $A = B(E \times D/\Theta')$. We obtain the lattice M_1 with a diamond $\{0_1, u_1, v_1, w_1, 1_1\}$ given in condition (i) of Lemma 7. In the second case we consider $K = E \times B(D/\Theta')$ and $A = B(E \times D/\Theta')$. By Lemma 4 E is a subalgebra of $B(E)$ hence $E \times B(D/\Theta')$ is a subalgebra of $B(E) \times B(D/\Theta') = B(E \times D/\Theta')$. The resulting lattice is M_2 with the diamond $\{0_2, u_2, v_2, w_2, 1_2\}$.

By condition (i) of Lemma 7 the ideal (v_1) of M_1 is isomorphic to $B(E \times D/\Theta')$. On the other hand by condition (ii) the filter $[u_2]$ of M_2 is isomorphic to $B(E \times D/\Theta')$. Consequently we have an isomorphism $\delta: [u_2] \rightarrow (v_1)$. We apply the Hall—Dilworth gluing construction which gives a lattice L having M_1 as a filter and M_2 as an ideal.

(L is the set of all $x \in M_1 \cup M_2$, we identify x with $\delta(x)$ for all $x \in \{u_2\}$; $x \leq y$ has unchanged meaning if $x, y \in M_1$ or $x, y \in M_2$ and $x < y$, $x, y \notin \{u_2\} = \{v_1\}$ iff $x \in M_2$, $y \in M_1$ and there exists a $z \in \{u_2\}$ such that $x < z$ in M_2 and $z < y$ in M_1 .) The lattice L is given by Figure 3 where $B = B(E \times D / \theta')$:

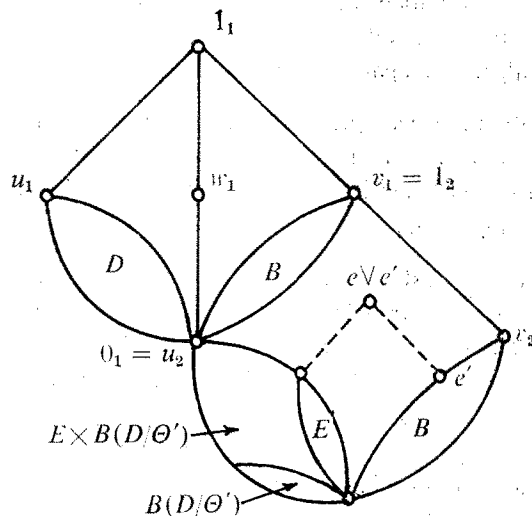


Figure 3

The function $\pi(x) = (x \vee w_2) \wedge v_2$ yields the element $e' = \pi(e) = (e \vee w_2) \wedge v_2 \leq v_2$. Let I be the principal ideal generated by $e \vee e'$. We have to prove that the pair L, I satisfies the properties given in Theorem B.

(1) First we prove that $\text{Con } L \cong I(D)$ i.e. D is isomorphic to the semilattice of all compact congruences of L . Every congruence relation θ of L is determined by its restrictions θ_{M_1} and θ_{M_2} to M_1 resp. M_2 . By condition (iii) of Lemma 7 θ_{M_1} is determined by its restriction to $\{v_1\}$ and similarly θ_{M_2} is determined by its restriction to $\{v_2\}$. But the interval $[0_1, v_1]$ is a transpose of $[0_2, v_2]$, hence we get that θ is determined by its restriction to the ideal $\{v_2\}$. This ideal is a Boolean lattice, thus every compact congruence relation of $\{v_2\}$ has the form $\theta(0_2, x)$, $x \in \{v_2\}$. Let now, $\theta(0_2, x)$ be a congruence relation of $\{v_2\}$. Under what conditions for x has this congruence relation an extension to L ? Condition (iii) of Lemma 7 gives the following isomorphisms:

$$\begin{aligned} \text{in } M_1, \quad \mu_1: \{u_1\} &\rightarrow D, \quad \tau_1: \{v_1\} \rightarrow B(E \times D / \theta'), \\ \text{in } M_2, \quad \mu_2: \{u_2\} &\rightarrow E \times B(D / \theta'), \quad \tau_2: \{v_2\} \rightarrow B(E \times D / \theta'). \end{aligned}$$

If $\varepsilon_1: D \rightarrow B(E \times D / \theta')$ and $\varepsilon_2: E \times B(D / \theta') \rightarrow B(E \times D / \theta')$ denote the identity maps, then $\tau_1 \pi_1 = \varepsilon_1 \mu_1$ and $\tau_2 \pi_2 = \varepsilon_2 \mu_2$ where $\pi_i(x) = (x \vee w_i) \wedge v_i$ ($i=1, 2$). By condition (iii) of

Lemma 7, the congruence relation $\Theta(0_2, x)$ can be extended to M_2 iff $\tau_2(x) \in \varepsilon_2(E \times B(D/\Theta'))$. Similarly, in M_1 we get that the congruence relation $\Theta(0_1, 0_1 \vee x)$ of $(v_1]$ can be extended to M_1 iff $\tau_1(0_1 \vee x) \in \varepsilon_1(D)$. Obviously the minimal extensions of $\Theta(0_2, x)$ and $\Theta(0_1, 0_1 \vee x)$ to L are the same and $\varepsilon_1(D)$ is a sublattice of $\varepsilon_2(E \times B(D/\Theta'))$, so we obtain that $\Theta(0_2, x)$ has an extension to L iff $\tau_1(0_1 \vee x) \in \varepsilon_1(D)$. This proves $\text{Con } L \cong I(D)$.

(2) Secondly we show $\text{Con } I \cong I(E)$. E is a direct factor of $(u_2]$ and $(e']$ is isomorphic to $B(E)$. Obviously $B(E \times B(D/\Theta')) = B(E) \times B(D/\Theta')$ hence the principal ideal $I = (e \vee e']$ is a direct factor of M_2 . This means that I is again a lattice given by Lemma 7, namely if $K = E$ and $A = B(E)$. Thus by condition (iii) we have $K(\text{Con } I) \cong E$, i.e. $\text{Con } I \cong I(E)$.

(3) Finally, let Θ be a compact congruence relation of L . We have seen that Θ is the extension of some $\Theta(0_2, x) \in \text{Con } (u_2]$ where $\tau_2(x) \in \varepsilon_1(D)$, i.e. the restriction $\Theta \rightarrow \Theta_I$ is determined by the projection $D \rightarrow E \times D/\Theta'$. As we have seen this is exactly the given homomorphism φ .

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