# CONGRUENCE RELATIONS RELATED TO A GIVEN AUTOMORPHISM GROUP OF A BOOLEAN LATTICE

By

E. T. SCHMIDT

Mathematical Institute of the Hungarian Academy of Sciences, Budapest

*(Received March 7, 1984)*

## 1. Introduction

Let $A$ be an algebra. Aut $(A)$ denotes the group of all automorphisms of $A$. The subgroups of Aut $(A)$ are called the automorphism groups of $A$. If $\alpha \in \mathrm{Aut}\,(A)$ then a congruence relation $\Theta$ of $A$ is called $\alpha$-admissible if $x \equiv y(\Theta)$ implies $\alpha x \equiv \alpha y(\Theta)$. Similary if $G$ is an automorphism group of $A$ then $\Theta$ is called $G$-admissible if $\Theta$ is $\alpha$- admissible for every $\alpha \in G$. It is easy to see that $G$-admissible congruence relations of $A$ form a $\{0, 1\}$-sublattice $\mathrm{Con}_G(A)$ of the congruence lattice Con $(A)$. We have therefore a mapping from the lattice of all subgroups of Aut $(A)$ into the lattice of all $\{0, 1\}$-sublattices of Con $(A)$. This mapping naturally arises a sequence of representation problems. The purpose of this note is to prove a special representation if $A$ is a Boolean lattice. Then $\mathrm{Con}_G(A)$ is a distributive lattice. We prove the following:

THEOREM. *Let $D$ be a finite distributive lattice. Then there exist a Boolean lattice $B$ and an automorphism group $G$ of $B$ such that $\mathrm{Con}_G(B)$ is isomorphic to $D$.*

## 2. Preliminaries

Let $D$ be a finite distributive lattice. $\mathcal{J}(D)$ denotes the poset of all non-zero join-irreducible elements of $D$. For an arbitrary poset $P$, call $A \subset P$ hereditary iff $x \in A$, $y \leq x$ imply that $y \in A$. The set $\mathcal{H}(P)$ of all hereditary subsets of $P$ is a distributive lattice. It is well-known that $D$ is isomorphic to $\mathcal{H}(\mathcal{J}(D))$, i.e. $D$ is determined by $\mathcal{J}(D)$. Therefore we have to prove that for every finite poset $P$ there exists a Boolean lattice $B(P)$ such that for suitable automorphism group $G_P$ of $B(P)$, $\mathrm{Con}_{G_P}(B(P)) \cong \mathcal{H}(P)$. To construct $B(P)$ we induct on the size of $P$. $\mathbf{B}$ will denote the Boolean lattice containing all subsets of a countable infinite set $X$. All finite subsets of $X$ form an ideal $\mathbf{J}$ of $\mathbf{B}$. Similarly the cofinite subsets form a liter $\mathbf{D}$. $\mathbf{B}_f = \mathbf{J} \cup \mathbf{D}$ is a Boolean sublattice of $\mathbf{B}$. Our construction will give that $B(Q)$ is a $\{0, 1\}$-sublattice of $\mathbf{B}$.

It is well known that every congruence relation $\Theta$ of a Boolean lattice is determined by the $\Theta$-class containing 0, which is an ideal. The ideal $I$ of the Boolean lattice $B$ is called $G$-admissible (where $G$ is an automorphism group of $B$) if $x \in I$ implies $\alpha x \in I$ for every $\alpha \in G$. Obviously $I$ is $G$-admissible iff $\Theta[I]$ is $G$-admissible. ($\Theta[I]$ denotes the congruence relation $\Theta$ with the $\Theta$-class $I$.)

If $G$ is an arbitrary automorphism group of a finite Boolean lattice $B$ then $\mathrm{Con}_G(B)$ is clearly a Boolean lattice. Therefore to represent a non Boolean lattice we must construct an infinite Boolean lattice. The simplest case is the representation of the three-element chain $3$: if $G_3 = \mathrm{Aut}(\mathbf{B}_f)$ then $\mathrm{Con}_{G_3}(\mathbf{B}_f) \cong 3$, i.e. $\mathbf{B}_f$ has exactly one non trivial $G_3$-admissible ideal.

## 3. The construction of $B(P)$

Let $P$ be an arbitary finite poset. For $B(1)$ (i.e. $P = 1$) we choose the two-element Boolean lattice $2$, and let $G_2 = \mathrm{Aut}(2)$. Obviously $\mathrm{Con}_{G_2}(2) \cong 2 \cong \cong \mathscr{H}(1)$. $B(1)$ is a $\{0, 1\}$-sublattice of $\mathbf{B}$.

Now, suppose $|P| \geq 2$ and let $m$ be a maximal element of $P$. Let $m_1, \ldots, \ldots, m_k$ be the set of all elements in $P$ which are covered by $m$. (Of course $\{m_1, \ldots, m_k\}$ may be empty.) We denote by $Q$ the poset $P \setminus \{m\}$. The assumption $|P| \geq 2$ implies that $Q$ is not empty. By induction, there exist a Boolean lattice $B(Q)$ and an automorphism group $G_Q$ such that $\mathscr{H}(Q)$ is isomorphic to $\mathrm{Con}_{G_Q}(B(Q))$.

Let $\varphi : \mathscr{H}(Q) \to \mathrm{Con}_{G_Q}(B(Q))$ be a fixed isomorphism. Let $(m_1, \ldots, m_k]$ denote the hereditary subset of $Q$ generated by $m_1, \ldots, m_k$. Then $\varphi((m_1, \ldots, \ldots, m_k]) = \Phi$ is a $G$-admissible congruence relation of $B(Q)$. Let $I$ be the ideal-kernel of $\Phi$ in $B(Q)$. Then $I$ is $G_Q$-admissible.

Consider a countably infinite set $H$. The power-set algebra $B = P(H)$ is, clearly, isomorphic to $\mathbf{B}$. Let $A_1, A_2, \ldots$ be a countably infinite sequence of subsets of $H$ such that

(i) $A_i \cap A_j = \emptyset \quad (i \neq j)$;

(ii) $\bigcap\limits_{i=1}^{\infty} A_i = H$;

(iii) $|A_i| = \aleph_0 \, (i = 1, 2, \ldots)$.

The principal ideal $(A_i]$ generated by $A_i$ in $B = P(H)$ is, again, isomorphic to $\mathbf{B}$. For each $i$ ($i = 1, 2, \ldots$) we fix an isomorphism

$$\varphi_i : (A_1] \to (A_i]$$

such that $\varphi_1$ is the identity on $(A_1]$. Let $A_i'$ be the complement of $A_i$ in $B$, i.e. $A_i' = \bigcup (A_j; j \neq i)$. All these elements of $B$ generate a Boolean sublattice of $B$ which is obviously isomorphic to $\mathbf{B}_f$. The ideal of $B$ generated by $A_1$, $A_2, \ldots$ will be denoted by $\bar{J}$ and $\bar{D}$ is the filter generated by $A_1', A_2', \ldots$. Then $\bar{J} \cap \bar{D} = \emptyset$.

We consider $B(Q) = B_1(Q)$ as a $\{0, 1\}$-sublattice of $(A_1]$. If $B_i(Q)$ denotes $\varphi_i(B(Q))$ then $B_i(Q)$ is a $\{0, 1\}$-sublattice of $(A_i]$. The ideal $I$ of $B(Q)$ defines an ideal $I_k = \varphi_k(I)$ of $B_k(Q)$. Let $S$ be the direct sum of the lattices $B(Q)$ $(=B_1(Q))$, $I_2$, $I_3, \ldots$ All these lattices are relatively complemented, consequently the same holds for $S$. Finally let $S'$ be the "complement" of $S$ in $B$, i.e. $S' = \{x';\ x\in S\}$. Obviously $S\subset\bar{J}$, $S'\subset\bar{D}$, hence $S\cap S' = \emptyset$. We define $B(P)$ to be the complemented $\{0, 1\}$-sublattice $S\cup S'$.

The next step is the definition of an automorphism group $G_P$ of $B(P)$. By our assumption $B(Q)$ has an automorphism group $G_Q$ such that $\mathrm{Con}_{G_Q}(B(Q))\cong\mathcal{H}(Q)$. It is enough to define an automorphism group of $S$ – this can be extended uniquely to $B(P) = S\cup S'$.

The ideal $I_1$ is $G_Q$-admissible, i.e. the restriction of any $\alpha\in G_Q$ to $I_1$ is an automorphism of $I_1$. Then $\alpha$ induces an automorphism of $I_i$ (which will be denoted by the same letter $\alpha$) by the rule $\alpha(\varphi_i(x)) = \varphi_i(\alpha x)$. This proves that each $\alpha\in G_Q$ can be extended to $S$. All these automorphisms of $S$ form an automorphism group $\bar{G}(\cong G_Q)$ of $B(P)$. We define some other automorphisms. First we prove the following:

LEMMA. Let $L_1$ and $L_2$ be Boolean lattices and let $I_1$ be an ideal of $L_1$ such that $\varphi: I_1\to L_2$ is an isomorphism. There exists an automorphism $\alpha$ of $L = L_1\times L_2$ such that $\alpha x = \varphi(x)$ for each $x\in I_1$.

PROOF. $I_1$ and $L_2$ are isomorphic, hence $I_1$ has a unit element $i$. Let $i'$ be the complement of $i$ in $L_1$. Then $L_1 = (i]\times(i']$, i.e. $L = (i]\times(i']\times L_2$. Interchanging $(i]$ and $L_2$ gives the desired automorphism.

Let $i, j\geq 1$ and let $x$ be an arbitary element of $I_i\subset S$. $\varphi_{ij} = \varphi_j\varphi_i^{-1}$ is an isomorphism between $I_i$ and $I_j$, consequently $\varphi_{ij}((x]]$ is isomorphic to $(x]$. Applying the lemma we have an automorphism $\beta_{ij}^x$ on $I_i\times I_j$ (if $i = 1$ then we consider $B(Q)$ instead of $I_1$) such that $\beta_{ij}^x(x) = \varphi_{ij}(x)$. We can extend this automorphism to $S$. Let $G_P$ be the subgroup of $\mathrm{Aut}(S)$ (i.e. $\mathrm{Aut}(B(P))$) generated by $\bar{G}$ and all $\bar{\beta}_{ij}$ ($\bar{\beta}_{ij}$ is the extension of $\beta_{ij}$ to $B(P)$).

We are going to prove that $\mathrm{Con}_{G_P}(B(P))\cong\mathcal{H}(P)$.

If $a, b\in B(P)$ $(a\geq b)$, $\Theta(a, b)$ denotes the smallest $G_P$-admissible congruence relation under which $a\equiv b$. This congruence relation is called principal. We determine all join-irreducible, $G_P$-admissible principal congruence relations of $B(P)$. By the definition of $B(P)$, this lattice contains the prime ideal $S$ and the ultrafilter $S'$ such that $S\cup S' = B(P)$, $S\cap S' = \emptyset$. We distinguish two cases.

(1) $a, b\in S$ (or similarly $a, b\in S'$).

$S$ is the direct sum of the lattices $B(Q)$, $I_2$, $I_3, \ldots$, therefore $\Theta(a, b)$ is determined by its projections into these lattices. Let $\Pi_k$ be the projection onto $I_k$, and let $a_k = \Pi_k(a)$, $b_k = \Pi_k(b)$. Applying the automorphism $\beta = \beta_{1k}^{a_k}$ we get $a_k\equiv b_k(\Theta)$ iff $\beta(a_k)\equiv\beta(b_k)$ $(\Theta)$ for an arbitary congruence relation $\Theta$ of $B(P)$. But $\beta(a_k), \beta(b_k)\in I_1\subseteq B(Q)$, hence $\Theta(a, b)$ is determined by a congruence relation $\Theta$ of $B(Q)$. This $\Theta$ must be a $G_Q$-admissible congruence relation of

$B(Q)$. Conversely, if $\Theta$ is an arbitary $G_Q$-admissible congruence relation of $B(Q)$, then the relation $\bar{\Theta}$ defined by:

$$u \equiv v(\bar{\Theta}) \text{ iff } \beta(\Pi_k(u)) \equiv \beta(\Pi_k(v))\ (\Theta)$$

is obviously $G_P$-admissible, and this $\bar{\Theta}$ can be extended to the whole $B(P)$. Thus we have: $\Theta(a, b)$ $(a, b \in S)$ is a join-irreducible $G_Q$-admissible congruence relation of $B(P)$ if and only if its restriction to $B(Q)$ is a join-irreducible $G_Q$-admissible congruence relation.

(2) Let $a \in S'$, $b \in S$ (or $a \in S$, $b \in S'$).

We have denoted the congruence relation $\varphi((m_1, m_2 \dots m_k])$ by $\varphi$. Let $\bar{\varphi}$ be the extension of $\varphi$ to $B(P)$. $\varphi$ is $G_Q$-admissible, which implies $\bar{\varphi}$ is $G_T$-admissible (namely $\bar{\varphi}$ is $\beta_{ij}$-admissible). The kernel of $\bar{\varphi}$ contains every ideal $I_i$ $(i = 1, 2, \dots,)$, i.e. $\Theta(a, b) \geq \bar{\varphi}$. In the construction of $B(P)$, $A_1$ is the unit element of $B(Q)$. Let $A_1'$ be the complement of $A_1$ in $B(P)$ (i.e. $A_1' = \bigcup_{i=2}^{\infty} A_i$) and 0 be the zero element of $B(P)$. Then $\Theta(0, A_1')$ is a join-irreducible congruence relation, and $\Theta(0, A_1') \geq \bar{\Theta}$ $(\Theta \in \mathrm{Con}_G(B(Q)))$ if and only if $\Theta \leq \varphi((m_i])$ for some $i$. On the other hand it is easy to see that every $\Theta(a, b)$ has the representation $\Theta(0, A_1') \vee \bar{\Theta}$ where $\bar{\Theta}$ is the extension of a $G_Q$-admissible congruence relation of $B(Q)$. Thus we obtain $\mathcal{J}(\mathrm{Con}_G(B(P))) \cong P$, i.e. $\mathrm{Con}_G(B(P)) \cong \cong \mathcal{H}(P)$.