CONTRIBUTIONS TO GENERAL ALGEBRA 3
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PERFECT DISTRIBUTIVE LATTICES

Introduction

The motivation of this work is threefold. Firstly, it was proved in [14] by the last author that a finite distributive lattice L is a direct product of chains if and only if every 5-element (closed) interval of L is totally ordered. It is quite natural to ask for a similar characterization in the infinite case.

Secondly, the 5-element distributive lattices which are not chains are the smallest lattices which are not perfect in the sense of Fortunatov [6], i.e. it is possible to define on them a congruence Θ such that for some elements a,b either $[a]\Theta \vee [b]\Theta \neq [a \vee b]\Theta$ or $[a]\Theta \wedge [b]\Theta \neq [a \wedge b]\Theta$. In [13] it was shown that Boolean lattices and chains are perfect. Are the relatively complemented distributive lattices and the chains the only perfect distributive lattices? We prove that the answer is no.

Thirdly, there is a nice theorem of G. Grätzer and E.T. Schmidt [9] which says : a distributive lattice is relatively complemented if and only if it has no homomorphic image isomorphic to $\underline{\underline{3}}$, the 3-element chain. What about those distributive lattices which have no homomorphic image isomorphic to a 5-element distributive lattice which is not totally ordered? We shall see that in the finite case things behave quite nicely and that all reasonable conjectures are verified. The general case is somewhat more difficult.

Section 1 is devoted to the definitions of the main notions and symbols that will be used throughout this work. In Section 2 we characterize in two different ways the perfect distributive lattices: on the one hand via their posets of prime ideals or their homomorphic images; on the other hand in an axiomatic manner, so showing that the class of perfect distributive lattices is closed under the formation of direct products. In Section 3 we determine which partially ordered sets are isomorphic to the set of prime ideals of some perfect bounded distributive lattice. The relationships between perfect lattices, pseudocomplemented lattices and relatively pseudocomplemented lattices are analysed in Section 4. In particular we prove that the perfect double Heyting lattices are exactly the double relative Stone lattices. Section 5 is concerned with the notion of Boolean product. We show that every Boolean product of perfect lattices is perfect and that a lattice is a Boolean product of chains if and only if it is a perfect relative double Heyting lattice. We close this work with an example of a non-locally finite perfect lattice. We also mention some open questions.

1. Preliminaries

A congruence Θ on a lattice L is *join-perfect* if for all a,b $\in$ L holds $[a]\Theta \vee [b]\Theta = [a \vee b]\Theta$, where $[a]\Theta$ denotes the Θ-class containing a and where the symbol $\vee$ in the left-hand side has to be taken in the complex algebra of L. In other words, Θ is join-perfect if for every c such that $c \;\Theta\; a \vee b$ there exist a', b' $\in$ L satisfying $a' \;\Theta\; a$, $b' \;\Theta\; b$ and $a' \vee b' = c$. Dually one can define a *meet-perfect* congruence. Finally a congruence is *perfect* if it is join and meet-perfect. A lattice is join-perfect (resp. meet-perfect, perfect) if all its congruences are join-perfect (resp. meet-perfect, perfect). Clearly a lattice and its dual are simultaneously perfect or not.

The classes of all join-perfect, meet-perfect and perfect distributive lattices will be denoted by $\underline{\underline{P}}^{\vee}$, $\underline{\underline{P}}^{\wedge}$ and $\underline{\underline{P}}$ respectively, while $\underline{\underline{D}}$ will stand for the class of all distributive lattices. The subscripts o,1 on the right of the letters $\underline{\underline{P}}$ and $\underline{\underline{D}}$ will mean that the lattices under consideration are bounded.

The *cardinal sum* X + Y ot two posets X and Y is the set of all elements in X or Y, considered to be disjoint, the order relation on each of the sets X and Y is preserved, while an element of X is never comparable with an element of Y. The cardinal sum of arbitrarily many posets X_i $(i \in I)$ is defined in the same manner and denoted by ΣX_i.

The *ordinal sum* $X \oplus Y$ of two posets X and Y is the set of all elements in X or Y, considered to be disjoint, the order relation on each of the sets X and Y is preserved but each element of X is less than each element of Y. The 5-element lattices $\underline{\underline{2}}^2 \oplus \underline{\underline{1}}$ and $\underline{\underline{1}} \oplus \underline{\underline{2}}^2$ will be denoted by S_5 and S_5^d respectively.

We shall make a constant use of Priestley's duality, which is remarkably summarized in [12] . We emphasize the fact that in [12] $\underline{\underline{D}}$ represents the category of *bounded* distributive lattices. For all L $\in$ $\underline{\underline{D}}$ (bounded or not) X_L, ordered by set inclusion, is the poset of prime ideals of L. The empty set and the whole lattice are not considered to be prime ideals. In case L is finite,

X_L is nothing else than the poset M(L) of all non-unit meet-irreducible elements of L.

In this paper *we consider only distributive lattices* and very often we shall omit the adjective "distributive".

2. Characterizations of perfect lattices

We begin with two elementary lemmas.

2.1. LEMMA. *A finite direct product of lattices is perfect if and only if every factor is perfect.*

2.2. LEMMA. *Any homomorphic image of a perfect lattice is a perfect lattice.*

We leave the easy proofs of these lemmas to the reader and we point out that the finiteness hypothesis in 2.1 is superfluous, as we shall see later. We are now ready to prove

2.3. THEOREM. *Let* L *be a finite distributive lattice. Then the following are equivalent:*

(1) $L \in \underline{\underline{P}}$;

(2) L *has no homomorphic image isomorphic to* S_5 *or* S_5^d;

(3) X_L *is the cardinal sum of chains;*

(4) L *is a direct product of chains;*

(5) L *has no interval isomorphic to* S_5 *or* S_5^d.

Proof. The equivalence of (3), (4) and (5) was established in [14] .
(1) $\Rightarrow$ (2) since, as has already been noticed, neither S_5 nor S_5^d are perfect and it suffices to apply 2.2.
(2) $\Rightarrow$ (5). If L has an interval [a,b] isomorphic to S_5 or S_5^d , then the endomorphism $x \to (x \wedge b) \vee a$ yields a homomorphic image of L which contradicts (2).
(4) $\Rightarrow$ (1). Clear by 2.1 since a chain is of course a perfect lattice.

Let us observe that if $L \in \underline{\underline{P}}$ and L is finite, then L is self-dual. We shall see that this property is no longer true if the finiteness hypothesis is omitted. Moreover, although a nonatomic Boolean lattice B is perfect ([13], Corollary 1 of Theorem 2), B is not a direct product of chains; nevertheless B has no homomorphic image isomorphic to $\underline{\underline{3}}$ ([9]) , hence no homomorphic image isomorphic to S_5 or S_5^d. Moreover, the lattice $(\omega \times \underline{\underline{2}}) \oplus \underline{\underline{1}}$ shows that (5) does not imply (1). On the contrary we shall see in 2.8 that (4) always implies (1).

The equivalence of conditions (1), (2) and (3) of 2.3 holds in the general case, as shown by

2.4. THEOREM. *Let* $L \in \underline{\underline{D}}$. *Then the following are equivalent:*

(1) $L \in \underline{\underline{P}}$;

(2) L *has no homomorphic image isomorphic to* S_5 *or* S_5^d;

(3) X_L *is the cardinal sum of chains.*

Proof. (1) $\Rightarrow$ (2) has been already justified in 2.3.
(2) $\Rightarrow$ (1). Let us assume that the lattice L is not perfect; then there is a congruence Θ which has two classes A and B such that, for instance, $A \wedge B = \{x \wedge y \mid x \in A,\ y \in B\} \subset C$, where C is the infimum of A and B in L/Θ and $\subset$ means strict inclusion. Clearly $A \wedge B$ is a proper filter of C. Let $c \in C - (A \wedge B)$. There is a prime ideal P containing c and disjoint from $[A \wedge B)$. Note that the latter filter contains A and B.

Let us consider the ideal $I = P \vee (B]$. We claim that I does not meet $[A)$. Indeed, if not there are $p \in P$, $a \in A$ and $b \in B$ such that $p \vee b \geqslant a$, hence $(p \wedge a) \vee (b \wedge a) = a$. Of course we may assume $p \geqslant c$. Since $(c \wedge a, b \wedge a) \in \Theta$, we also have

$$((c \wedge a) \vee (p \wedge a),\ (b \wedge a) \vee (p \wedge a)) \in \Theta,$$

that is $(p \wedge a,\ a) \in \Theta$, hence $p \wedge a \in A$ and $p \in [A)$, which contradicts $P \cap [A) = \emptyset$. Consequently $I \cap [A) = \emptyset$ and there is a prime ideal Q which separates I and $[A)$. Similarly, one can prove the existence of a prime ideal R which separates $P \vee (A]$ and $[B)$. The prime ideals P,Q,R are such that $P \subset Q \cap R$ and $Q \parallel R$. They induce a 5-class congruence Φ with $L/\Phi \cong S_5^d$, which contradicts (2).
(2) $\Leftrightarrow$ (3). The condition "L has no homomorphic image isomorphic to S_5" is equivalent to " the join of any two uncomparable prime ideals of L is L" (see, for instance, [8]), while "L has no homomorphic image isomorphic to S_5^d "is equivalent to "any two uncomparable prime ideals of L are disjoint".

The condition "the join of any two uncomparable prime ideals of L is L" is equivalent to "the prime ideals contained in any given prime ideal of L form a chain" and also to " $\langle a,b \rangle \vee \langle b,a \rangle = L$ for any $a,b \in L$" where $\langle a,b \rangle$, the annihilator of a relative to b, is the ideal $\{x \in L \mid x \wedge a \leqslant b\}$. Each of the preceding conditions characterizes the *relatively normal* lattices, a notion which was first considered by M. Mandelker and then developped by W. Cornish ([11], Theorem 4, and [4] , Theorem 3.5). Hence the perfect lattices are the lattices which are relatively normal and dually relatively normal. But we emphasize the fact that we were led to the notion of perfect lattice by reasons quite different from those of M. Mandelker and W. Cornish.

To illustrate 2.4 we provide easy and well-known examples of perfect lattices.

a) X_L is totally unordered if and only if L is a distributive relatively complemented lattice (a Boolean lattice if L is bounded);

b) X_L is totally ordered if and only if L is a chain;

c) P-algebras, defined as in [5], are perfect; particular examples are generalized Post algebras.

Other examples of perfect lattices will be given later. But let us first characterize perfect lattices axiomatically.

2.5. LEMMA. *Let* $L \in \underline{\underline{D}}$. *Then* $L \in \underline{\underline{P}}^{\vee}$ *if and only if* L *satisfies* $(P_{\vee})$ *for all* $a,b,c \in L$ *there exist* $a',b' \in L$ *such that*

$$a' \wedge (a \vee b) = a,\ b' \wedge (a \vee b) = b \text{ and } a' \vee b' = a \vee b \vee c\ .$$

Proof. Let $L \in \underline{\underline{P}}^{\vee}$ and consider for a, b, $c \in L$ the congruence relation $\Theta = \Theta(a \vee b,\ a \vee b \vee c)$. Then from $a \vee b \vee c \in [a \vee b]\Theta$ it follows that for suitable $a' \in [a]\Theta$ and $b' \in [b]\Theta$ we have that $a' \vee b' = a \vee b \vee c$. On the other hand, $(a',a) \in \Theta$ implies $a' \wedge (a \vee b) = a \wedge (a \vee b) = a$. Similarly $b' \wedge (a \vee b) = b$.

Conversely, if L satisfies $(P_{\vee})$ then for every congruence Θ and for arbitrary elements a,b,c satisfying $a \vee b \leqslant c$ and $(c, a \vee b) \in \Theta$, we have that $(a' \wedge c,\ a' \wedge (a \vee b)) \in \Theta$, that is $(a',a) \in \Theta$ since $a' \wedge c = a'$. Similarly $(b',b) \in \Theta$. Since $a' \vee b' = c$, we have proved that $[a]\Theta \vee [b]\Theta = [a \vee b]\Theta$ and so L is join-perfect.

We could also deduce 2.5 from Theorem 2.4 of [4] . In fact, since a relatively normal lattice is a lattice every interval of which is normal, by condition (h) of the aforementioned theorem, a lattice L is relatively normal if and only if for all $a,b,c \in L$, there exist a', $b' \in L$ such that $a \wedge b' = a \wedge b = a' \wedge b$ and $a' \vee b' = a \vee b \vee c$. A routine computation shows that this condition is equivalent to $(P_{\vee})$.

Moreover the second part of the proof of 2.5 shows that in checking $(P_{\vee})$ one may restrict oneself to elements a,b,c satisfying $a \vee b \leqslant c$.

Of course the dual condition $(P_{\wedge})$ characterizes the meet-perfect lattices. The following theorem is then straightforward.

2.6. THEOREM. *Let* $L \in \underline{\underline{D}}$. *Then* $L \in \underline{\underline{P}}$ *if and only if* L *satisfies* $(P_{\vee})$ *and* $(P_{\wedge})$.

2.7. COROLLARY. *The notion of perfect distributive lattice is elementary.*

2.8. COROLLARY. *Let* $L \cong \prod_{i \in I} L_i \in D$. *Then* $L \in \underline{\underline{P}}$ *if and only if* $L_i \in \underline{\underline{P}}$ *for every* $i \in I$.

2.9. COROLLARY. *Any chain union of perfect distributive lattices is perfect* ([7] , Corollary page 280).

3. Posets representable over $\underline{\underline{P}}$

By 2.4 any $L \in \underline{\underline{D}}$ is perfect if and only if X_L is the cardinal sum of chains. But this does not mean that any cardinal sum of chains is isomorphic to the poset of prime ideals of a perfect lattice. We remind the reader that a poset S is said to be *representable over* some subclass $\underline{\underline{K}}$ of $\underline{\underline{D}}$ if there exists $L \in \underline{\underline{K}}$ such that $S \cong X_L$.

Starting from a poset S we can always form the new posets $S_0 = 0 \oplus S$, $S_1 = S \oplus 1$ and $S_{01} = 0 \oplus S \oplus 1$. With these notations we have

3.1. LEMMA. *A poset* S *is representable over* $\underline{\underline{D}}$ *if and only if* S_0 (resp. S_1, S_{01}) *is representable over* $\underline{\underline{D}}_0$ (*resp.* $\underline{\underline{D}}_1$, $\underline{\underline{D}}_{01}$).

Proof. Let S be representable over $\underline{\underline{D}}$, i.e. there exists $L \in \underline{\underline{D}}$ such that $X_L \cong S$. Then $L_0 = 0 \oplus L \in \underline{\underline{D}}_0$ and $X_{L_0} \cong S_0$.

Conversely, if $S_0 \cong X_{L_0}$ for some $L_0 \in \underline{\underline{D}}_0$, then the least prime ideal I_0 of L_0 is $\{0\}$. In fact, if $I_0 \neq \{0\}$, then $I_0 \ni a \neq 0$ and there is $I \in X_{L_0}$ such that $a \notin I$, which contradicts $I_0 \subseteq I$. It follows that $L = L_0 - \{0\} \in \underline{\underline{D}}$ and $X_L \cong S$.

The corresponding results in $\underline{\underline{D}}_1$ and $\underline{\underline{D}}_{01}$ admit similar proofs.

Let us recall that a chain is representable over $\underline{\underline{D}}_{01}$ if and only it is algebraic, i.e. it is complete and every proper interval contains a covering pair (that is, a pair $\{u,v\}$ with $u \prec v$) ([12], Section 6). As a consequence, a chain C is representable over $\underline{\underline{D}}$ (resp. $\underline{\underline{D}}_0$, $\underline{\underline{D}}_1$) if and only if C_{01} (resp. C_1, C_0) is algebraic.

It is rather easy to characterize those posets which are representable over $\underline{\underline{P}}_0$ or $\underline{\underline{P}}_1$.

3.2. LEMMA. *If* S *is a poset representable over* $\underline{\underline{P}}$ (*resp.* $\underline{\underline{P}}_0$, $\underline{\underline{P}}_1$, $\underline{\underline{P}}_{01}$) , *then* S *is the cardinal sum of chains representable over* $\underline{\underline{D}}$ (*resp.* $\underline{\underline{D}}_0$, $\underline{\underline{D}}_1$, $\underline{\underline{D}}_{01}$).

Proof. If S is representable over $\underline{\underline{P}}$, then $S = \sum_{i \in I} C_i$ by 2.4. For every $i \in I$, C_{i01} is representable over $\underline{\underline{D}}_{01}$. Indeed, properties (i) and (ii) of [12], Section 6, are inherited from S_{01} by each C_{i01} and are sufficient for C_{i01} to be representable over $\underline{\underline{D}}_{01}$.

Suppose now that S is representable over $\underline{\underline{P}}_0$. Then each element of S follows a minimal one, and each C_i has a least element. Every C_{i1} is representable over $\underline{\underline{D}}_{01}$, whence every C_i is representable over $\underline{\underline{D}}_0$.

Similar arguments hold if S is representable over $\underline{\underline{P}}_1$ or $\underline{\underline{P}}_{01}$.

3.3. THEOREM. *A poset* S *is representable over* $\underline{\underline{P}}_0$ *(resp.* $\underline{\underline{P}}_1$*) if and only if* S *is the cardinal sum of chains* C_i *such that* C_{i1} *(resp.* C_{i0}*) are algebraic.*

Proof. By 3.2 it remains to prove the sufficiency of the condition. Let $S = \sum_{i\in I} C_i$ with C_{i1} algebraic for each $i \in I$. There exist chains $D_i \in \underline{\underline{D}}_0$ such that $X_{D_i} \cong C_i$ $(i \in I)$. Let L be the discrete direct product of all D_i $(i \in I)$, i.e., $L = \{x \in \prod_{i\in I} D_i \mid x_i = 0 \text{ for all but a finite number of } i\}$. We will show that $X_L \cong S$.

Let $A \in X_L$. Then there exists exactly one $i \in I$ such that $pr_i A \neq D_i$. First, if $x^{(i)} \notin pr_i A$, $y^{(j)} \notin pr_j A$ and $i \neq j$, and if x (resp. y) is defined by $x_k = 0$ for $k \neq i$ and $x_i = x^{(i)}$ (resp. $y_k = 0$ for $k \neq j$ and $y_j = y^{(j)}$), then $0 = x \wedge y \notin A$, a contradiction. Also, suppose $pr_i A = D_i$ for each i and let $x \notin A$. For each $i \in I$ such that $x_i \neq 0$, let $x^{(i)} \in A$ be such that $pr_i x^{(i)} = x_i$. Then $x \leq \bigvee_i x^{(i)} \in A$, another contradiction. Now the mapping $A \to pr_i A$ is clearly an isomorphism from X_L onto $\sum_{i\in I} X_{D_i}$.

3.4. REMARK. 1) In [1] , Theorem 3, sufficient conditions are given for a poset S to be representable over $\underline{\underline{D}}$, and these conditions are shared by cardinal sums of chains representable over $\underline{\underline{D}}$ and bounded above. This provides another proof of Theorem 3.3, but no really new information is obtained since the lattice used in the proof of [1] , Theorem 3, is nothing else than the one we built in Theorem 3.3.

2) If I is infinite, the discrete direct product L of the proof of Theorem 3.3 is definitely not bounded above, even though all D_i are so. Moreover this construction cannot be performed if some chain D_i is not bounded below. However Theorem 3.3 is of some help in determining posets representable over $\underline{\underline{P}}$, as seen in the following result.

3.5. COROLLARY. *Let* S *be a cardinal sum of chains representable over* $\underline{\underline{D}}$. *Suppose that all but a finite number of them are bounded either above or below. Then* S *is representable over* $\underline{\underline{P}}$.

4. Pseudocomplemented perfect lattices

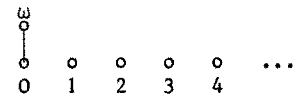
We recall that a lattice L with 0 is *pseudocomplemented* if for every $a \in L$ there is an element a^* such that $a \wedge a^* = 0$, and $a \wedge x = 0$ implies that $x \leq a^*$. It follows that L is also bounded above.

First we give an example which shows that some (necessarily infinite) bounded perfect lattices are not pseudocomplemented.

4.1. EXAMPLE. Let L be the subset of $P(\omega)$ consisting of all subsets of ω which are either finite, or cofinite and contain 0. Then L, ordered by inclusion, is a perfect bounded distributive lattice which is not pseudocomplemented.

In fact, $\{0\}^*$ does not exist. Moreover, X_L is nothing else than the one-point Alexandroff compactification $\omega + 1 = \omega \cup \{\omega\}$ of ω, with the order generated by $0 < \omega$. Hence L is perfect.

X_L



This example is very instructive in many respects and we shall somewhat analyse it. The center of L is the union of the prime ideal P whose elements are the finite subsets of ω not containing 0 and the prime filter Q whose elements are the cofinite subsets containing 0. For the induced order P is a generalized Boolean lattice, Q is a dual generalized Boolean lattice and $P \cup Q$ is a Boolean lattice. Moreover $L - (P \cup Q)$ is a generalized Boolean lattice with least element $\{0\}$. Note also that L-Q is a non-principal maximal ideal and P is the only prime ideal which is not maximal.

There are finitely many non-trivial decompositions of L into two factors; all of them have the form $(q] \times (p]$ or $[q) \times [p)$, where $q \in Q - \{\omega\}$, $p \in P - \{\emptyset\}$, $q = p'$. They are of course pairwise isomorphic. But the most remarkable fact is that in every decomposition one of the factors is always isomorphic to L itself and the other is a finite Boolean lattice. We thus have that $L \cong L \times \underline{\underline{2}}^n$ for every $n \in \omega$.

Let us recall that a lattice L is said to be (non-trivially directly) *indecomposable* if

$$L \cong \prod_{i \in I} L_i \Rightarrow |L_i| = 1 \text{ for all } i \text{ but one },$$

and (directly) *pseudoindecomposable* if

$$L \cong \prod_{i \in I} L_i \Rightarrow \exists\, i \in I \mid L_i \cong L.$$

With this terminology the lattice L of the preceding example is pseudo-indecomposable and we suggest the following questions :

1. Is an indecomposable perfect distributive lattice necessarily a chain?
2. How to characterize the pseudoindecomposable perfect distributive lattices?
3. Is every perfect distributive lattice isomorphic to a direct product of pseudoindecomposable lattices?

The preceding example highlights also the following facts :

1) although X_L has infinitely many connected components, any non-trivial decomposition of L has finitely many factors;
2) although L is perfect, it is not self-dual: one verifies easily that L is dually pseudocomplemented but, as already observed, not pseudocomplemented.

If we denote by L^d the (order theoretical) dual of L, then $L \times L^d$ provides an example of a lattice belonging to $\underline{\underline{P}}_{0,1}$ which is neither pseudocomplemented nor dually pseudocomplemented.

When dealing with perfect lattices, there is no great loss of generality in restricting oneself to the bounded case, as shown by

4.2. LEMMA. *Let* $L \in \underline{\underline{D}}$. *Then* $L \in \underline{\underline{P}}$ *if and only if each interval of* L *is perfect.*

Proof. Since each interval of L is a homomorphic image of L, the condition is necessary by 2.2. Suppose now that each interval of L is perfect and let ϕ be a homomorphism from L onto S_5 or S_5^d . Let $a \in \phi^{-1}(0)$ and $b \in [a) \cap \phi^{-1}(1)$. Then $\phi|_{[a,b]}$ is a homomorphism from [a,b] onto S_5 or S_5^d, contradicting the fact that [a,b] is perfect.

Despite what has been said at the beginning of this section, there are nice examples of pseudocomplemented lattices which belong to $\underline{\underline{P}}_{0,1}$. At this point we have to remind the reader a few concepts, more especially as the terminology is far from being standard.

A *relatively pseudocomplemented lattice* is a lattice every interval of which is pseudocomplemented. Note that we do not require the existence of an element 1. A *Stone lattice* is a pseudocomplemented distributive lattice which satisfies $x^* \vee x^{**} = 1$ identically. A *relative Stone lattice* is a (distributive) lattice all intervals of which are Stone lattices. A *Brouwerian lattice* L is a (distributive) lattice in which, for all $a,b \in L$, there exists $a * b \in L$ such that $a \wedge x \leqslant b \Leftrightarrow x \leqslant a * b$. A *Heyting lattice* is a Brouwerian lattice with zero. An L-*algebra* is a Heyting lattice satisfying $(x * y) \vee (y * x) = 1$.

The following property is straightforward.

4.3. THEOREM. *If* $L \in \underline{\underline{P}}^{\vee}$ *is pseudocomplemented, then* L *is a Stone lattice. If* $L \in \underline{\underline{P}}$ *is double pseudocomplemented, then* L *is a double Stone lattice.*

Proof. In fact, it is well known that for a pseudocomplemented lattice to be a Stone lattice, it suffices that the Glivenko congruence Φ (defined by $(a,b) \in \Phi$ if and only if $a^* = b^*$) be join-perfect ([13] , Théorème 3).

4.4. THEOREM. *Let* $L \in \underline{\underline{D}}$. *Then the following are equivalent:*

(1) $L \in \underline{\underline{P}}^{\vee}$ *and* L *is relatively pseudocomplemented;*

(2) L *is a relative Stone lattice;*

moreover, if L *is bounded above, we can add:*

(3) L *is an* L*-algebra;*

(4) L *is isomorphic to a* **-sublattice of a direct product of chains bounded above;*

(5) $L \in \underline{\underline{P}}^{\vee}$ *and* L *is a Brouwerian lattice.*

Proof. (1) $\Leftrightarrow$ (2) is in fact Theorem 3 of [8] .

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) is Theorem 3.11 of [2].

(1) $\Leftrightarrow$ (5) is well-known.

Then the following theorem is straightforward.

4.5. THEOREM. *Let* $L \in \underline{\underline{D}}$. *Then the following are equivalent*:

(1) $L \in \underline{\underline{P}}$ *and* L *is double relatively pseudocomplemented;*

(2) L *is a double relative Stone lattice;*

moreover, if L *is bounded, we can add* :

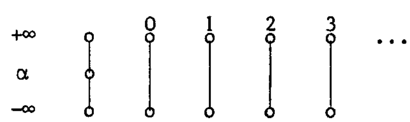
(3) L *is a double* L*-algebra;*

(4) L *is isomorphic to* $a^{\star +}$*-sublattice of a direct product of bounded chains;*

(5) $L \in \underline{\underline{P}}$ *and* L *is a double Heyting lattice.*

To close this section we show by an example that even in a bounded perfect lattice pseudocomplementation does not imply relative pseudocomplementation.

4.6. EXAMPLE. Let ω^{+} be the Alexandroff compactification of ω , where ω is discretely topologized. We denote by X the space $(\omega^{+} \times 2) + 1$ (1 is the space which has the only point α), ordered as follows:



Then X is a Priestley space whose dual L is a perfect double Stone lattice, but neither a Heyting lattice nor a dual Heyting lattice. Indeed, one easily verifies that X is a Priestley space. By 2.4 L is perfect. If U is a clopen set of Min X (that is, U is cofinite and contains $-\infty$, or U is finite and does not contain $-\infty$), then [U) is a clopen set of X. It follows that L is pseudocomplemented ([12] , Section 8), hence a Stone lattice by 4.3. Since L is self-dual, L is a double Stone lattice. Nevertheless $U = \{\alpha\}$ is a clopen set and neither [U) nor (U] are open, hence L is neither a Heyting lattice nor a dual Heyting lattice ([12], Section 8).

The axiomatic characterizations of join-perfectness and perfectness given in 2.5 and 2.6 respectively shed some light on the connections between the latter properties and that of relative pseudocomplementation.

Let $L \in \underline{\underline{P}}^{\vee}$. Then for all $a,b,c \in L$ there exist $a',b' \in L$ which satisfy

$$(1)\ a' \wedge (a \vee b) = a,$$
$$(2)\ b' \wedge (a \vee b) = b,$$
$$(3)\ a' \vee b' = a \vee b \vee c.$$

Of course the ordered pair (a',b') is not unique. If a'', b'' satisfy the above conditions as well, then by distributivity so does the pair $(a' \vee a'', b' \vee b'')$. Thus in case L is a finite join-perfect lattice there always exists a greatest pair which satisfies the above conditions. The question naturally arises : what is the meaning for a join-perfect lattice of the existence for all a,b,c of a greatest pair (a',b') satisfying (1),(2),(3)?

For brevity we shall denote this pair by $(f(a,b,c),f(b,a,c))$. Then for every (a',b') which satisfy (1),(2),(3) we have $a' \leqslant f(a,b,c)$ and $b' \leqslant f(b,a,c)$. The answer to the above question is very simple and by no means surprising.

4.7. THEOREM. *Let* $L \in \underline{\underline{P}}^{\vee}$. *Then* $f(a,b,c)$ *exists for all* $a,b,c \in L$ *if and only if* L *is relatively pseudocomplemented.*

Proof. Let L be a join-perfect and relatively pseudocomplemented lattice. Let us consider any three elements a,b,c, of L such that $c \geqslant a \vee b$. The pseudocomplement of $a \vee b$ in $[a,c]$, denoted by $(a \vee b)^{\star}_{[a,c]}$, is the greatest element in $[a,c]$ such that its meet with $a \vee b$ is a. So for any a',b' satisfying (1),(2), (3) we have $(a \vee b)^{\star}_{[a,c]} \geqslant a'$ and similarly $(a \vee b)^{\star}_{[b,c]} \geqslant b'$. It follows that $f(a,b,c) = (a \vee b)^{\star}_{[a,c]}$ and $f(b,a,c) = (a \vee b)^{\star}_{[b,c]}$.

Now let L be a join-perfect lattice in which $f(a,b,c)$ and $f(b,a,c)$ exist for all $a,b,c \in L$. Let us consider any three elements x,u,v of L such that $u \leqslant x \leqslant v$. From $v \geqslant u \vee x$ we deduce the existence of a greatest pair (u',x') such that

$$u' \wedge x = u,$$
$$u' \vee x' = v \text{ (hence } u \leqslant u' \leqslant v).$$

Thus $u' = f(u,x,v)$ is the greatest element of $[u,v]$ such that its meet with x is u, that is, the pseudocomplement of x in $[u,v]$. So we nave shown that L is relatively pseudocomplemented.

The dual of 4.7 is easy to formulate. We shall denote the least pair (a',b') satisfying the duals of conditions (1),(2),(3) by $(g(a,b,c),g(b,a,c))$.

Combining 4.7 and its dual we obtain

4.8. COROLLARY. *Let* $L \in \underline{\underline{P}}$. *Then* $f(a,b,c)$ *and* $g(a,b,c)$ *exist for all* $a,b,c \in L$ *if and only if* L *is double relatively pseudocomplemented.*

5. Boolean products

As already noticed in Section 2, not every Boolean lattice is the direct product of chains, but every Boolean lattice is the Boolean power of the two-element chain. Generalizing the notion of Boolean power, S. Burris and H. Werner have introduced the concept of Boolean product (which in the finite case coïncides with that of direct product). We refer the reader to [3] for the theory of Boolean products.

We first show that the class $\underline{\underline{P}}$ is closed under the formation of Boolean products.

5.1. THEOREM. *Every Boolean product of perfect lattices is perfect.*

Proof. Let L be a Boolean product of a family $(L_x)_{x \in X}$, $X \neq \emptyset$, of perfect lattices. Let us first suppose that L is bounded. Then every L_x enjoys the same property. By Proposition 1 of [10] the dual of L (i.e., the poset of prime ideals of L) is order isomorphic to the cardinal sum of the duals of the L_x, $x \in X$, and L is perfect.

If L is not bounded and if $a,b \in L$, $a \leqslant b$, then $[a,b]$ is the Boolean product of the $[a_x,b_x]$, $x \in X$. By the first part $[a,b] \in \underline{\underline{P}}$ and it suffices to apply 4.2 to conclude.

Now we use the notion of discriminator variety to extend the list of equivalences given in 4.5. and so we show that not all perfect lattices are Boolean product of chains.

5.2. THEOREM. *Let* $L \in \underline{\underline{D}}_{0,1}$. *Then the following are equivalent:*

(4) L *is isomorphic to a double Heyting subalgebra (i.e., a sublattice closed under the binary operations* $\star$ *and* + *) of a direct product of bounded chains;*

(5) $L \in \underline{\underline{P}}_{0,1}$ *and* L *is a double Heyting lattice;*

(6) L *is a Boolean product of bounded chains.*

Proof. (4) $\Leftrightarrow$ (5) has been proved in 4.5.

(4) $\Rightarrow$ (6). Any bounded chain, considered as a double Heyting algebra, has the discriminator term

$$t(x,y,z) = (((x\star y) \wedge (y\star x))^{+} \wedge x) \vee (((x+y) \vee (y+x))^{\star} \wedge z).$$

In fact,

if $x = y$, then $x \star y = y \star x = 1$ and $x + y = y + x = 0$;
if $x < y$, then $x \star y = 1$, $y \star x = x$, $x + y = y$, $y + x = 0$, $x^{+} = 1$ and $y^{\star} = 0$;
if $x > y$, then $x \star y = y$, $y \star x = 1$, $x + y = 0$, $y + x = x$, $x^{\star} = 0$ and $y^{+} = 1$.

Hence it is a simple algebra. Condition (4) implies that L is a member of the variety generated by the class of bounded chains. Hence by Theorem 9.4 of [3],L is isomorphic to a Boolean product of bounded chains.

(6) ⇒ (5). By 5.1 we have that $L \in \underline{\underline{P}}_{0,1}$. Furthermore, L is a double Heyting lattice since, if L is a Boolean product on X of the bounded chains $C_x (x \in X)$ and if $a,b \in L$, then $a \star b$, taken in the direct product ΠC_x, is equal to

$1|_N \cup b|_{X-N}$, where N is the clopen set

$[\![a \wedge b = a]\!] = [\![a \leqslant b]\!] = [\![a \star b = 1]\!]$, and belongs to L by the patchwork property.

The preceding theorem shows that the subclass of double Heyting algebras which are perfect in so far as lattices is a discriminator variety.

In case L is not bounded, one can easily extend 5.2 using "relative" notions. Then relative double Heyting algebras are algebras $L = \langle L; \vee,\wedge,\star:;+: \rangle$ of type $\langle 2,2,4,4 \rangle$ where the quaternary operation $(x,y,a,b) \to (x \star y)_a^b$ (resp. $(x + y)_a^b$) is the relative pseudocomplement (resp.the dual relative pseudocomplement) of

$$(x \vee (a \wedge b)) \wedge (a \vee b)$$

with respect to

$$(y \vee (a \wedge b)) \wedge (a \vee b)$$

in the interval $[a \wedge b, a \vee b]$.

Another consequence of 5.2 is that among the lattices which belong to $\underline{\underline{P}}_{0,1}$ and cannot be decomposed into a Boolean product of bounded chains are those which are not pseudocomplemented (see Example 4.1) but also those which are double pseudocomplemented (hence double Stone lattices) without being double Heyting lattices (see Example 4.6). Every Boolean decomposition of the lattice L of Example 4.1 has a factor isomorphic to L. In other words, L is "Booleanly indecomposable". The two problems should be worth being considered:

1) describe all Booleanly indecomposable perfect bounded lattices;
2) is every perfect bounded lattice isomorphic to a Boolean product of Booleanly indecomposable perfect bounded lattices?

It is clear that Boolean products are not sufficient to provide a representation of all perfect lattices as subdirect products of chains. Moreover a subdirect product of chains is not necessarily perfect. We define the notion of

E-subdirect product, which generalizes the concept of Boolean product, and we prove that every E-subdirect product of chains is join-perfect.

5.3. LEMMA. *Let* L *be a subdirect product of chains* D_i $(i \in I)$. *Assume that* L *is join-perfect. Let* $f,g,h \in L$ *with* $h \geq f \vee g$ *, and consider the following subsets of* I *:*

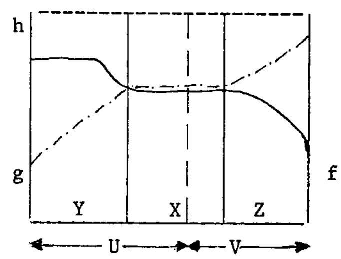
$$X = [\![f=g]\!] \ , \quad Y = [\![f > g]\!] \ , \quad Z = [\![f < g]\!] \ .$$

Then there exist a partition $\{U,V\}$ *of* I *and* $f',g' \in L$ *such that*

$$Y \subseteq U \subseteq X \cup Y$$

and

$$f'|U = h|U, \quad f'|Z = f|Z, \quad g'|V = h|V \ , \quad g'|X = g|X.$$



It is easy to show that this condition is in fact equivalent to (P_V).

5.4. DEFINITION. Let E be the family of all subsets of I which are equalizers, i.e. $[\![f = g]\!]$ for some $f,g \in L$. We say that the subdirect product L satisfies the E-patchwork property (shorter, that L is an E-*subdirect product*) if for all $f, g \in L$ and $T \in E$, $f|_T \vee g|_{I-T} \in L$.

The proof of the following theorem is then straightforward.

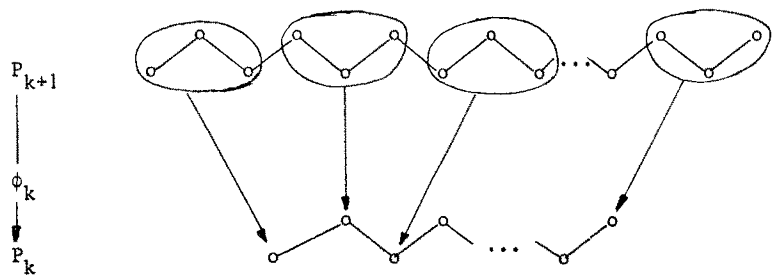
5.5. THEOREM. *If* L *is an* E-*subdirect product of chains, then* L *is join-perfect.*

6. Perfectness is not a locally finite property

We prove the statement of the title by an example. More precisely:

THEOREM. *There exist a perfect lattice* L *and a finite subset* X *of* L *such that no finite sublattice of* L *containing* X *is perfect.*

Proof. A) *Construction of* L. We first construct the dual of L.
For $k \geq 1 (k \in \mathbb{N})$ let P_k be the set$\{a/3^k | a \in \mathbb{N},\ 0 \leq a < 4.3^k\}$, ordered by $a/3^k \overset{(k)}{\leq} b/3^k$ if and only if $|a-b| = 1$ and a even, or $a = b$.



For every $k \geqslant 1$ we define the mapping $\phi_k : P_{k+1} \to P_k$ by $\phi_k(a/3^{k+1}) = [a/3]/3^k$, where $[x]$ denotes the largest integer not exceeding x. We first show ϕ_k is order preserving. Indeed let us suppose that $a/3^{k+1} \overset{(k+1)}{<} b/3^{k+1}$: a is even and $|b - a| = 1$. If $[a/3] = [b/3]$ then $\phi_k(a/3^{k+1}) = \phi_k(b/3^{k+1})$. If $[a/3] \neq [b/3]$ and if $a < b$, then $b = 3c$ and $a = 3c - 1$ for some $c \in \mathbb{N}$. Since a is even, so is $c - 1$ and $\phi_k(a/3^{k+1}) = (c-1)/3^k \overset{k}{\leqslant} c/3^k = \phi_k(b/3^{k+1})$. If $b < a$ the conclusion is the same.

Of course the $\phi_k : P_{k+1} \to P_k$ give rise to an inverse family of sets $(P_k, \phi_{k\ell} | k \geqslant \ell)$ where $\phi_{kk} = \mathrm{id}$ and $\phi_{k\ell} = \phi_k \circ \ldots \circ \phi_{\ell-1} : P_\ell \to P_k$ for $\ell > k$.

Let P be the inverse limit $\varprojlim (P_k, \phi_{k\ell})$. An element α of P is a sequence $\alpha = (\alpha_k | k \geqslant 1)$ such that $\alpha_k \in P_k$ (hence $\alpha_k = a_k/3^k$) and $\phi_k(\alpha_{k+1}) = \alpha_k$ for every k. The last equality implies

$$\alpha_{k+1} = a_{k+1}/3^{k+1} \geqslant [a_{k+1}/3] \;/\; 3^k = \phi_k(\alpha_{k+1}) = \alpha_k .$$

Since $\alpha_k \in [0,4[$ for every k, α is an increasing bounded sequence and converges in $\mathbb{R}$ to its limit $\bar{\alpha}$.

Now let us prove that $\bar{\alpha} - \alpha_k \leqslant 1/3^k$ for every k. Indeed, from $a_k = [a_{k+1}/3]$ we deduce $a_k \leqslant a_{k+1}/3 < a_k + 1$, hence $3a_k \leqslant a_{k+1} \leqslant 3a_k + 2$.
It follows $\alpha_k \leqslant \alpha_{k+1} \leqslant \alpha_k + 2/3^{k+1}$ and $\alpha_{k+1} - \alpha_k \leqslant 2/3^{k+1}$.
Finally $\bar{\alpha} - \alpha_k = \sum_{\ell \geqslant k} (\alpha_{\ell+1} - \alpha_\ell) \leqslant \sum_{\ell \geqslant k} 2/3^{\ell+1} = 1/3^k$.

The characterization of P follows from the following two observations.
1) *Let $\alpha, \beta \in P$ be such that $\bar{\alpha} \neq \bar{\beta}$. Then α and β are uncomparable in* P .

Let us suppose that $\alpha \overset{P}{\lneqq} \beta$. Then $\alpha_\ell \overset{(\ell)}{\lneqq} \beta_\ell$ for every ℓ. Let ℓ be large enough for $\alpha_\ell \overset{(\ell)}{<} \beta_\ell$ and $2/3^\ell < | \bar{\beta} - \bar{\alpha} |$. From $\alpha_\ell \overset{(\ell)}{<} \beta_\ell$ we deduce $|\beta_\ell - \alpha_\ell| = 1/3^\ell$. On the other hand, if we suppose $\beta_\ell > \alpha_\ell$ in $\mathbb{R}$, then $\beta_\ell - \alpha_\ell \geqslant \beta_\ell - \bar{\alpha} \geqslant \bar{\beta} - 1/3^\ell - \bar{\alpha} > 2/3^\ell - 1/3^\ell = 1/3^\ell$, a contradiction.

2) *If* $\alpha,\beta \in P$ *are such that* $\bar{\alpha} = \bar{\beta}$ *, then there exists a unique* ℓ *such that* $\alpha_k = \beta_k$ *if* $k < \ell$ *,* $\beta_k = \beta_\ell = \bar{\beta}$ *and* $\alpha_k = \bar{\beta} - 1/3^k$ *if* $k \geqslant \ell$ *(or the symmetric situation). In particular* α *and* β *are comparable in* P.

Let ℓ be the least index for which $\alpha_\ell \neq \beta_\ell$, say $\alpha_\ell < \beta_\ell$ in $\mathbb{R}$. We notice that if $u, v \in P_\ell$, then $|u - v|$ is a multiple of $1/3^\ell$. Now let us prove that $\alpha_k < \beta_\ell$ for every $k \geqslant \ell$, and more precisely that $\beta_\ell - \alpha_k \geqslant 1/3^k$ for every $k \geqslant \ell$. It is true for $k = \ell$. Moreover

$$\beta_\ell - \alpha_{k+1} = (\beta_\ell - \alpha_k) - (\alpha_{k+1} - \alpha_k) \geqslant 1/3^k - 2/3^{k+1} = 1/3^{k+1} .$$

From $\alpha_k < \beta_\ell$ for $k \geqslant \ell$ follows $\bar{\alpha} \leqslant \beta_\ell \leqslant \bar{\beta}$, hence $\beta_k = \beta_\ell = \bar{\beta}$ for every $k \geqslant \ell$. Since $\beta_k - \alpha_k = \beta_\ell - \alpha_k \geqslant 1/3^k$ and $\alpha_k \geqslant \bar{\alpha} - 1/3^k = \beta_k - 1/3^k$, we obtain $\alpha_k = \beta_k - 1/3^k = \beta_\ell - 1/3^k$.

We are now able to prove that P is the cardinal sum of ω two-element chains and 2^ω one-element chains. Indeed, if $r \in [0,4[$ there exist one or two sequences $\alpha \in P$ such that $\bar{\alpha} = r$. To show this, let us write r in the form $r = n + \sum_{\ell=1}^{\infty} p_\ell/3^\ell$ with $n \in \{0,1,2,3\}$ and $p_k \in \{0,1,2\}$. Then it suffices to take $\alpha_k = n + \sum_{\ell=1}^{k} p_\ell/3$. The numbers r of the form $n + \sum_{\ell=1}^{k} p_\ell/3_\ell$ admit two writings ($n + \sum_{\ell=1}^{k} p_\ell/3^\ell = n + \sum_{\ell=1}^{k-1} p_\ell/3_\ell + (p_k - 1)/3^k + \sum_{\ell>k} 2/3^\ell$ for $k \neq 0$ and $p_k \neq 0$), whereas the other ones admit a unique writing.

It suffices to dualize the preceding construction to obtain the lattice L that we desire. For $k \geqslant 1$, we denote by L_k the dual of P_k ($P_k = X_{L_k}$) and by $\psi_{k\ell} : L_k \to L_\ell$ the mapping that is the dual of $\phi_{k\ell} : P_\ell \to P_k$ ($\psi_{k\ell} = \phi_{k\ell}^{-1}$). Then $(L_k, \psi_{k\ell})$ is direct family of lattices and $L = \varinjlim (L_k, \psi_{k\ell})$. By duality theory we have that $X_L = \varprojlim (P_k, \phi_{k\ell}) = P$ and L is perfect.

B) <u>Conclusions</u>. Since the $\phi_{k\ell}$ are surjections, the $\psi_{k\ell}$ are injections and every L_k can be considered as a sublattice of L. We note that the L_k form a nested family of finite sublattices of L whose union is L.

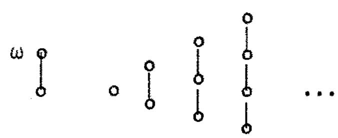
Take for X the sublattice L_1 of L. Let D be a finite sublattice of L containing L_1. There is k such that $D \subseteq L_k$. By duality the inclusions $L_1 \to D \to L_k$ give surjections $P_1 \leftarrow X_D \leftarrow P_k$. Since X_D is the image of P_k, X_D is connected. It contains uncomparable elements since P_1 does so.

Consequently D is not perfect.

Addendum

Quite recently Hans Dobbertin communicated us an example of an indecomposable perfect lattice which is not a chain, so answering in the negative Question 1 of Section 4. Recall (see [G-J] : L. Gillman and M. Jerison, Rings of Continuous Functions, Van Nostrand, 1960) that a Tychonoff space X is called an *F-space* if every finitely generated ideal of C(X), the ring of all continuous real valued functions on X, is principal. If X is an F-space then by [G-J, 14.25(2) and 2.12(a)] , the lattice Z(X) of all zero-sets of X is perfect. If moreover X is connected, then Z(X) contains no non-trivial clopen sets, whence it is indecomposable. Now, by [G-J , 14.2], there exist connected F-spaces such that Z(X) is not a chain.

The following example, due also to Hans Dobbertin, answers negatively Question 3 of Section 4 and Question 2 of Section 5. Let X_L be the one-point Alexandroff compactification of ω, endowed with the following order



It is easy to see that the only Boolean representations of this lattice are precisely the decompositions into a finite direct product.

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