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Affine Complete Semilattices

By

K. Kaarli, L. Márki, and **E. T. Schmidt,** Budapest

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Abstract: This paper presents a structural characterization of affine complete and locally affine complete semilattices.

The notion of affine completeness arises naturally when investigating geometric properties of (universal) algebras. Their general study was initiated by H. WERNER [5]. However, from a different starting point and without using the term "affine complete", G. GRÄTZER already asked in [2], Problem 6 about characterizing affine complete algebras. Till now this problem has been solved only in some varieties with good congruence structure for the algebras (see e. g. the introduction of D. CLARK—H. WERNER [1] and K. KAARLI [3]). Here we present a solution for semilattices, i. e., in a variety in which congruences in general have no good behaviour but the operation can be easily handled. We also characterize locally affine complete semilattices.

1. Preliminaries

Throughout the paper meet semilattices will be considered. A function (of finite arity) in a semilattice S is said to be *compatible* if it preserves all congruences of S. Clearly, an n-ary function f in S is compatible iff $(f(x_1,\ldots,x_n),\ f(y_1,\ldots,y_n)) \in \bigvee_{i=1}^{n} \theta(x_i, y_i)$ for all $x_1,\ldots,x_n, y_1,\ldots,y_n \in S$, where $\theta(x, y)$ is the smallest congruence under which x and y are congruent (see e. g. W. NÖBAUER [4]; for unary functions this condition is equivalent to saying that f preserves all $\theta(x, y)$; the latter congruences are called the *principal congruences*).

By a *polynomial function* in S we mean a function of the form $a \wedge x_1 \ldots \wedge x_n$ where a is either the empty symbol or an element of S and the set of variables may also be empty. A *local polynomial function* in S is a function whose restriction to every finite subset of

S is a polynomial function. An n-ary function f is said to be *essentially n-ary* if it depends on all its variables (i. e., for every $i = 1, \ldots, n$ there exist elements $a_1, \ldots, a_n, a_i'$ (depending on i) such that

$$f(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_n)).$$

S is (*locally*) *affine complete* if every compatible function in S is a (local) polynomial function. S is called (*locally*) *n-affine complete* if every compatible n-ary function in S is a (local) polynomial function. (Note that some authors use the term "locally affine complete" in a different sense.)

We have noticed already that when investigating compatible unary functions it suffices to consider principal congruences only. Moreover, for arbitrary elements x, y in a semilattice S we have $\theta(x, y) = \theta(x \wedge y, x) \vee \theta(x \wedge y, y)$, hence we conclude that *a unary function in S is compatible iff it preserves all principal congruences of the form $\theta(b, a)$, $b < a$.* Next we describe these congruences.

Lemma. *Let $b < a$ in a semilattice S. Then for any $c, d \in S$, $(c, d) \in \theta(b, a)$ iff $c = d$ or $c, d \leqslant a$ and $b \wedge c = b \wedge d$.*

Proof. Put $\theta = \theta(b, a)$ and

$$\Phi = \{(c, d) \mid c = d \text{ or } c, d \leqslant a \text{ and } b \wedge d = b \wedge c\}.$$

Clearly, Φ is an equivalence relation. Let $c \Phi d$ and $e \in S$ be arbitrary, then it is also clear that $c \wedge e \Phi d \wedge e$. Hence Φ is a congruence and $(a, b) \in \Phi$, so $\theta \subseteq \Phi$. Conversely, let $(c, d) \in \Phi$, $c \neq d$. Then $c, d \leqslant a$ and $b \wedge c = b \wedge d$, hence $c = a \wedge c \theta b \wedge c = b \wedge d \theta a \wedge d = d$, therefore $\Phi \subseteq \theta$ and we are done.

Some notions and notations which will be important for us: $a \succ\!\!- b$ if $a > b$ and there are no elements between a and b; $a \parallel b$ means that a and b are incomparable.

A unary function f in a semilattice S is called a *contraction* if $f(s) \leqslant s$ for all $s \in S$. A function f is *monotonous* if $f(x_1, \ldots, x_n) \leqslant f(y_1, \ldots, y_n)$ whenever $x_1 \leqslant y_1, \ldots, x_n \leqslant y_n$.

By an *ideal* of S we mean a non-empty subset I such that for all $x \in I$ and $s \in S$, $s \leqslant x$ implies $s \in I$. For an ideal I of S, θ_I will denote the *Rees congruence* belonging to I, i. e., $a \theta_I b$ iff $a = b$ or $a \in I$ and $b \in I$.

Let $s, t \in S$. By (s) we denote the principal ideal generated by s, i. e., $(s) = \{x \in S \mid x \leqslant s\}$. For $t \leqslant s$ we denote by $[t, s]$ the interval $\{x \in S \mid t \leqslant x \leqslant s\}$.

By a *filter* of S we mean a non-empty subet D such that for all $x, y \in D$ and $s \in S$, we have $x \wedge y \in D$ and $x \leqslant s$ implies $s \in D$.

An *inductive subset* of S is an $A \subseteq S$ such that any two elements of A have a common upper bound in A.

An element $s \in S$ is said to be *thick* if there is no $t < s$, $t \neq 0$, such that $(t) \cup [t, s] = (s)$.

For any congruence θ and $a \in S$, $[a]_\theta$ stands for the θ-class containing a.

The one-element semilattice, which is trivially affine complete, will be disregarded in what follows.

2. Results

Our main results are the following.

Theorem 1. *A semilattice S is affine complete if and only if it satisfies the following conditions:*

(i) *it has not atoms,*

(ii) *each of its elements is thick,*

(iii) *every proper ideal I of S such that $(a) \cap I$ is a principal ideal for all $a \in S$, is itself a principal ideal.*

Theorem 2. *A semilattice S is locally affine complete if and only if it satisfies conditions* (i), (ii), *and*

(iii′) *if I is a proper ideal of S such that $(a) \cap I$ is a principal ideal for all $a \in S$, then I is inductive.*

We shall prove these theorems in several steps, which will be formulated as separate propositions. Firstly we describe unary compatible functions.

Proposition 1. *A unary function $f(x)$ in S is compatible iff it is of one of the following forms.*

(α) *$f(x)$ is constant.*

(β) *$f(x)$ is a monotonous contraction: there is an ideal $I \lhd S$ such that $(x) \cap I = (f(x))$ for all $x \in S$.*

(γ) *$f(x)$ is monotonous but not a contraction: there are an element $0 \neq a \in S$ and an ideal I of the filter $\{x \in S \mid x \geqslant a\}$ such that if $u > a$ and $u \in I$ then all $z \leqslant u$ $(z \in S)$ are comparable with a, and we have $f(x) = a$ for $x \ngeqslant a$ and $(x) \cap I = (f(x))$ for $x \geqslant a$.*

(δ) *$f(x)$ is not monotonous: there are elements $a, b \in S$ such that $a \succ\!\!- b$, for all $c \not\geqslant a$ we have $c \wedge a \leqslant b$, and*

$$f(x) = \begin{cases} b & \text{if } x \geqslant a \\ a & \text{otherwise.} \end{cases}$$

Proof. Put $D = \{x \mid f(x) \leqslant x\}$. Now $x \in D, y > x$ implies $y \in D$, since otherwise $(f(x), f(y)) \in \theta(x, y)$ would not hold. Hence if $x, y \notin D$ then $x \wedge y \notin D$ either. By $(f(x \wedge y), f(x)) \in \theta(x \wedge y, x)$ and $f(x) \not\leqslant x$ we must have $f(x) = f(x \wedge y)$, similarly $f(y) = f(x \wedge y)$, so $f(x) = f(y)$. Therefore, if $f(x) \notin D$ then $f(f(x)) = f(x)$, i. e., $f(x) \in D$. Thus for any $x, y \notin D$ we have $f(x) = f(y) \in D$. Consequently, D is not empty (for S has more than one element). Further, if $x, y \in D, x \| y$, then by $(f(x), f(x \wedge y)) \in \theta(x, x \wedge y)$ we have $f(x \wedge y) \leqslant x$. Similarly, $f(x \wedge y) \leqslant y$ and therefore $f(x \wedge y) \leqslant x \wedge y$, so $x \wedge y \in D$. All this proves that D is a filter in S.

Let J denote the ideal in S generated by the $f(x)$, $x \in D$. For any $x \in D$, $(x) \cap J = (f(x))$ must hold; in fact, otherwise there were $y \in D$ and $z \in S$ such that $y \geqslant f(y) \geqslant z \not\leqslant f(x)$ and $x \geqslant z$, so, letting θ denote the smallest congruence which collapse the principal filter generated by z, we would have $(z, f(y)) \in \theta$ and, by $(x, y) \in \theta$, also $(f(x), f(y)) \in \theta$, although $f(x) \notin [z]_\theta$.

Now we distinguish three cases:

1. $D = S$. Then we have $J = I$ and (β) holds.

2. *$D \neq S$ and $I = J \cap D \neq \emptyset$.* We prove first $f(D) \subseteq D$. By the assumption there is an $x \in D$ such that $f(x) \in D$. For any $y \in D$, $y \neq x$, we have $(f(x), f(y)) \in \theta(x, y)$, hence $f(x) \wedge x \wedge y = f(y) \wedge x \wedge y$. Here the left-hand side belongs to the filter D, therefore $f(y) \in D$. Take an $x_0 \notin D$ and put $a = f(x_0)$. Then we have, as was shown above, $a = f(z)$ for all $z \notin D$. Next we prove that D is the principal filter generated by a. In fact, otherwise there were a $c < a$, $c \in D$; then $x_0 \wedge c < c$ (for $x_0 \notin D$), and since $a > c$, $(f(c), a) = (f(c), f(x_0 \wedge c)) \in \theta(x_0 \wedge c, c)$ is possible only if $f(c) = a (> c)$, which contradicts $c \in D$.

We have seen that $(x) \cap J = (f(x))$ if $x \in D$ and $f(y) = a$ if $y \notin D$. Now, if $I = \{a\}$ then f is constant and we are in case (α). Otherwise there is a $u \in I$, $u > a$. Suppose that for a $z \in S$ we have $z \leqslant u$, $z \| a$. Then $f(z) = a$ and $f(u) = u$, hence $(a, u) = (f(z), f(u)) \in \theta(z, u)$, which contradicts $z \| a$. Hence we are in case (γ).

3. *$D \neq S$ and $I = \emptyset$.* Take an $x \notin D$ and put $a = f(x), b = f(a)$. By $a \in D$ we have $f(a) \leqslant a$ but $f(a) \neq a$ for $I = \emptyset$, hence $b < a$. If there were a c such that $b < c < a$, then $f(c) = a$ for $c \notin D$, hence $(b, a) = (f(a), f(c)) \in \theta(a, c)$, which is impossible. This proves that $b \prec a$. Let $c \not\geqslant a$, then $f(c) = a$ and $(a, b) = (f(c), f(a)) \in \theta(c, a)$, hence $c \wedge a \leqslant b$.

Let now $c \geqslant a$. Then $(b, f(c)) = (f(a), f(c)) \in \theta(a, c)$, hence $f(c) \wedge a = b \wedge a = b$. Furthermore, $(f(c), f(f(c))) \in \theta(f(c), c)$ but $f(c) \notin D$ hence $f(f(c)) = a$, so $(f(c), b) = (f(c) \wedge f(c),\ a \wedge f(c)) \in \in \theta(f(c), c)$, and by $b \leqslant f(c)$ this implies $f(c) = b$. Thus we are in case (δ).

Conversely, it is easily seen that the functions occuring in (α)—(δ) are compatible, and this completes the proof of Proposition 1.

Proposition 2. *A semilattice S is (locally) 1-affine complete iff it satisfies the conditions of Theorem 1 (Theorem 2).*

Proof. Let $p \in S$ be an atom. Then we define a compatible function $f(x)$ by putting $a = p$ and $b = 0$ in Proposition 1 (δ). This f is not a local polynomial function because it is not monotonous. Next, suppose $s \in S$ is not thick: for some $0 \neq t < s$ we have $(s) = (t) \cup [t, s]$. Then we define D as the principal filter generated by t, put $J = (s)$, and by Proposition 1 (γ) we obtain a compatible function $f(x)$ which is not a local polynomial function for it is not a contraction. Further, let $I \lhd S$ be a proper ideal such that $(x) \cap I = (x')$ for all $x \in S$. Putting $f(x) = x'$ we obtain a compatible function by Proposition 1 (β). If $f(x)$ is a polynomial function, say $f(x) = a \wedge x$, then $I = (a)$. Suppose now that $f(x)$ is a local polynomial function and take arbitrary elements $a, b \in I$ and $c \in S \setminus I$. Then there is a polynomial function $p(x)$ such that $p(a) = f(a) = a, p(b) = f(b) = b, p(c) = f(c)$. Since $c \notin I$, $p(c) \neq c$, hence p is not the identical mapping of S. Also, p is not constant, so $p(x) = x \wedge u$ for some $u \in S$. Now $p(a) = a, p(b) = b$ imply $a, b \leqslant u$, hence also $a, b \leqslant f(u) \in I$, which proves the inductivity of I. Thus a (locally) 1-affine complete semilattice must satisfy the conditions of Theorem 1 (Theorem 2).

Conversely, suppose that S satisfies (i)—(iii) and consider an arbitrary compatible function $f(x)$. If f is a contraction then by (β) there is an ideal $I \lhd S$ such that $(x) \cap I = (f(x))$. If $I = S$ then $f(x) = x$, if $I \neq S$ then by (iii) $I = (a)$ for some $a \in S$, hence $f(x) = a \wedge x$. If f is monotonous but not a contraction, and $|I| \geqslant 2$ in (γ), then $(u) = (a) \cup [a, u]$, which contradicts (ii), so $|I| = 1$ and f

is constant. Finally, if f is not monotonous then we are in case (δ); if here $b = 0$ then a is an atom, contrary to (i); if $b \neq 0$ then $(a) = (b) \cup [b, a]$ follows from the condition in (δ), and this contradicts (ii).

If S satisfies (i), (ii), (iii′), then the only case which remains unsettled is when $f(x)$ is a contraction, $(f(x)) = (x) \cap I$ for a proper ideal $I \lhd S$. Take now any elements $s_1, \ldots, s_n \in S$. The condition (iii′) implies that any finite subset of I has an upper bound in I, hence there is a $c \in I$ such that for $i = 1, \ldots, n$, $f(s_i) \leqslant c$, and therefore $f(s_i) \leqslant c \wedge s_i$; on the other hand, $c \wedge s_i \in (f(s_i))$, so $f(s_i) = c \wedge s_i$, $i = 1, \ldots, n$. Thus we see that S is locally 1-affine complete.

Next we turn to the investigation of functions in several variables.

Observation 1. If a local polynomial function $f(x_1, \ldots, x_n)$ in S properly depends on x_i then $f(a_1, \ldots, a_n) \leqslant a_i$ for all $a_1, \ldots, a_n \in S$. In fact, we have $b_1, \ldots, b_n, b_i'$ such that

$$f(b_1, \ldots, b_n) \neq f(b_1, \ldots, b_{i-1}, b_1', b_{i+1}, \ldots, b_n),$$

so there is a polynomial function $p(x_1, \ldots x_n)$ which contains the variable x_i and coincides with f on

$$(a_1, \ldots, a_n), (b_1, \ldots, b_n), (b_1, \ldots, b_{i-1}, b_i', b_{i+1}, \ldots, b_n),$$

whence the assertion.

Observation 2. If S is locally 1-affine complete, and $f: S^n \to S$ is a compatible function for which there exist $a_1, \ldots, a_n \in S$ such that $f(a_1, \ldots, a_n) \nleqslant a_i$, $i = 1, \ldots, n$, then f is constant. Indeed, this is obvious for $n = 1$. Suppose it is true for $k < n$. Then the function $f(a_1, x_2, \ldots, x_n)$ is constant by the induction hypothesis, and likewise the function $f(x_1, b_2, \ldots, b_n)$ for arbitrary $b_2, \ldots, b_n$. Putting the two parts together we obtain the assertion.

Observation 3. If S is locally 1-affine complete, $f: S^n \to S$ is a compatible function and $f(a_1, \ldots, a_n) = u$, then, since $f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$ is a local polynomial function, we have $f(a_1, \ldots, a_{i-1}, u, a_{i+1}, \ldots, a_n) = u$ for $i = 1, \ldots, n$; in particular, $f(u, \ldots, u) = u$.

Proposition 3. *If S is (locally) 1-affine complete then it is (locally) affine complete.*

Proof. Suppose that S is locally $(n-1)$-affine complete, we have

to prove that it is n-affine complete. The heart of the matter is to show that if f is an essentially n-ary compatible function in S then $f(a_1, \ldots, a_n) \leqslant a_1 \wedge \ldots \wedge a_n$ for all $a_1, \ldots, a_n \in S$. This is what we start with.

Suppose that there exist $a_1, \ldots, a_n \in S$ such that $f(a_1, \ldots, a_n) \nleqslant a_j$ for some j. Choose $a_1, \ldots, a_n \in S$ for which

$$|\{j \mid f(a_1, \ldots, a_n) \nleqslant a_j\}| = m$$

is maximal. Note that by Observation 2, $m \neq n$.

Let now $k = n - m$, $f(a_1, \ldots, a_n) = u$ and assume, without loss of generality, that $u \leqslant a_1, \ldots, u \leqslant a_k, u \nleqslant a_{k+1}, \ldots, u \nleqslant a_n$. By the induction hypothesis, the function $f(a_1, x_2, \ldots, x_n)$ is locally polynomial, so by Observation 1 it does not depend on $x_{k+1}, \ldots, x_n$.

Denote by A the set of all those elements $a \in S$ for which the function $f(a, x_2, \ldots, x_n)$ is not constantly zero and does not depend on $x_{k+1} \ldots, x_n$. Since $a_1 \in A$, $A \neq \emptyset$. Our aim is to prove $S = A \cup S_0$ where $S_0 = \{b \in S \mid f(b, x_2, \ldots, x_n) \equiv 0\}$, which contradicts the assumption that f is essentially n-ary. We need several steps to obtain this result.

I. $a \in A \Rightarrow f(a, x_2, \ldots x_n) \leqslant a$. This is clear by the definition of A and m.

II. $\left.\begin{array}{r} a \in A, c_2, \ldots, c_n \in S \\ 0 \neq w \leqslant v = f(a, c_2 \ldots, c_n) \end{array}\right\} \Rightarrow w \in A.$

Suppose $f(w, x_2, \ldots, x_n)$ depends, say, on x_n. Take an element $z < w$. Since f is compatible,

$$(f(w, c_2, \ldots, c_{n-1}, z), f(a, c_2, \ldots, c_{n-1}, z)) \in \theta(w, a). \tag{1}$$

By the Lemma we have

$$f(w, c_2, \ldots, c_{n-1}, z) \wedge w = f(a, c_2, \ldots, c_{n-1}, z) \wedge w =$$
$$= f(a, c_2, \ldots, c_n) \wedge w = w,$$

which contradicts $f(w, c_2, \ldots, c_{n-1}, z) \leqslant z < w$. Consider now the unary compatible function $g(x) = f(x, c_2, \ldots, c_n)$. If it is constant then $f(w, c_2 \ldots, c_n) = v \neq 0$; otherwise $g(x)$ is a non-constant local polynomial function and $g(a) = v \geqslant w$, hence $f(w, c_2, \ldots, c_n) = g(w) = w \neq 0$. So $f(w, x_2, \ldots, x_n)$ is not constantly zero, and we have $w \in A$.

III. $\forall s \in S \quad f(s, x_2 \ldots, x_n) \leqslant s$.

If $v = f(s, b_2 \ldots, b_n) \nleqslant s$ for some $b_2, \ldots, b_n \in S$ then $f(x, b_2, \ldots, b_n)$ is a

constant function. In particular, $0 \neq v = f(a_1, b_2, \ldots, b_n)$ and for any $0 \neq w < v$ also $f(w, b_2, \ldots, b_n) = v$. But $a_1 \in A$ and II imply $w \in A$, so $f(w, b_2, \ldots, b_n) \leqslant w < v$ by I, a contradiction. Hence there is no $w \in S$ with $0 \nleqslant w < v$ and v must be an atom, a contradiction.

IV. $a \in A$, $a < b \Rightarrow b \in A$. Since $a \in A$, there is $0 \neq w \in S$ such that

$$v = f(a, b_2, \ldots b_k, w, \ldots, w) \nleqslant w$$

for some $b_2, \ldots, b_k \in S$. Suppose $z = f(b, b_2, \ldots, b_k, w, \ldots, w) \leqslant w$. Then $(v, z) \in \theta(a, b)$. By the Lemma we have now $v \wedge a = z \wedge a$ and by III $v \wedge a = v$, hence $v \leqslant z \leqslant w$, a contradiction. This proves that $f(b, b_2, \ldots, b_k, \ldots, w) \nleqslant w$, thus $f(b, x_2, \ldots, x_n)$ does not depend on $x_{k+1}, \ldots, x_n$ and is not constantly zero.

V. $a \in A, b < a \Rightarrow b \in A \cup S_0$. We have to prove that, for any $b_2, \ldots b_k$, the local polynomial function $f(b, b_2, \ldots, b_k, x_{k+1}, \ldots, x_n)$ is constant. Since $a \in A$, $f(a, b_2, \ldots, b_k, x_{k+1}, \ldots, x_n) = v$ is constant. By III we have $v \leqslant a$ and $f(b, b_2, \ldots, b_k, x_{k+1}, \ldots, x_n) \leqslant b < a$, hence from

$$(f(a, b_2, \ldots, b_k, x_{k+1}, \ldots, x_n), f(b, b_2, \ldots, b_k, x_{k+1}, \ldots, x_n)) \in \theta(a, b)$$

it follows that

$$v \wedge b = f(b, b_2, \ldots, b_k, x_{k+1}, \ldots, x_n) \wedge b = f(b, b_2, \ldots, b_k, x_{k+1}, \ldots, x_n)$$

and the latter function is indeed constant.

Now we are ready to prove $S = A \cup S_0$. Suppose $b \in B = S \setminus (A \cup S_0)$. Then $f(b, x_2, \ldots, x_n)$ must properly depend on some of the variables $x_{k+1}, \ldots, x_n$, say on x_n. Let $f(b, b_2, \ldots, b_n) = c \neq 0$. Take $0 \neq d \in c$. Since the local polynomial function $f(b, x_2, \ldots, x_n)$ depends on x_n, the unary function $f(b, b_2, \ldots, b_{n-1}, x)$ is not constant (Observation 1). Since the latter is a local polynomial function which takes on the value c in b_n and $d < c$, we have $f(b, b_2, \ldots, b_{n-1}, d) = d$ and by Observation 3 $f(d, \ldots, d) = d \neq 0$. Thus $d \notin S_0$. Since by III $d < c \leqslant b$, IV yields that $d \in A$ is also impossible. Hence $d \in B$.

On the other hand, by II and Observation 3 we have for $u = f(a_1, \ldots, a_n)$ $f(u, \ldots, u) = u \in A$, and then $f(u, \ldots, u, d) = f(u, \ldots, u, u) = u$. Consider the unary function $f(x, \ldots, x, d)$. Since it takes different values on d and u, it is not constant. Since $f(b, x_2, \ldots, x_n)$ depends on x_n, Observation 1 yields $f(c, \ldots, c, d) \leqslant d < c$, hence $f(x, \ldots, x, d)$ is not the identity. So there exists a $t \in S$ such that

$$f(u,\ldots,u,d) = u = t \wedge u, \quad f(d,\ldots,d) = d = t \wedge d.$$

Now from $u = t \wedge u$ by IV we infer $t \in A$ and then from $d = t \wedge d$ by V it follows $d \in A \cup S_0$, a contradiction.

We have proved $S = A \cup S_0$, which contradicts the fact that f is essentially n-ary. This contradiction proves that

$$\forall i \forall a_1,\ldots,a_n \in S \quad f(a_1 \ldots, a_n) \leqslant a_i .$$

The rest will be short. Our next step is the proof of the implication

$$a_1 \wedge \ldots \wedge a_n = b_1 \wedge \ldots \wedge b_n \Rightarrow f(a_1,\ldots,a_n) = f(b_1 \ldots, b_n).$$

Let $a_1 \wedge \ldots \wedge a_n = d$. Then for any i, $d = a_1 \wedge \ldots \wedge a_n \wedge b_1 \wedge \wedge \ldots \wedge b_n \leqslant a_i \wedge b_i$. By the compatibility of f,

$$(f(a_1,\ldots,a_n), f(b_1,\ldots,b_n)) \in \theta(a_1,b_1) \vee \ldots \vee \theta(a_n,b_n). \tag{2}$$

Since

$$f(a_1,\ldots,a_n) \leqslant d \leqslant a_i \wedge b_i,$$

all equivalence classes $[f(b_1,\ldots,a_n)]_{\theta(a_i,b_i)}$ are one-element. Hence (2) implies $f(a_1,\ldots,a_n) = f(b_1,\ldots,b_n)$, and we can represent f as a composition $f = g \circ h$, where $h: S^n \to S$ is the polynomial function $h(x_1,\ldots,x_n) = x_1 \wedge \ldots \wedge x_n$, and $g: S \to S$. Since $g(x) = (g \circ h)(x,\ldots,x) = f(x,\ldots,x)$, g is a unary compatible function. Thus f is a (local) polynomial function if S is (locally) 1-affine complete, and Proposition 3 is proven.

Putting together Propositions 2 and 3, we obtain Theorems 1 and 2.

Remark. In Proposition 3 we have actually proven the following. If S has no atoms and $f: S^n \to S$ is a compatible function all of whose restrictions to less than n variables are (local) polynomial functions then f itself is a (local) polynomial function. One might be tempted to try to eliminate from here the condition on the structure of S (i. e., the lack of atoms) and obtain thereby a statement of a purely combinatorial kind on compatible functions. However, this is not possible, as shown by the following example.

Consider the following function $f(x,y)$ on the two element semilattice:

$$f(0,0) = 0, f(0,1) = f(1,0) = f(1,1) = 1.$$

It is easy to see that $f(x,y)$ is not a polynomial function though all its unary restrictions are polynomials.

3. Examples

In what follows we shall treat affine completeness only, the similar "local results" can be proven in the same way.

1. Finite semilattices are not affine complete for they have atoms.

2. Chains are not affine complete since only an atom can be thick in a chain.

3. **Theorem 3.** *A direct product* $S = \prod_{i \in I} S_i$ $(|I| \geqslant 2, |S_i| \geqslant 2)$ *of semilattices is affine complete iff each* S_i *has an identity and one of the following conditions is satisfied:*

1) *there are at least two components which have no zero;*

2) *there is a component which has no zero and has only thick elements;*

3) *each* S_i *has zero, no atoms, and only thick elements.*

Proof. Suppose S_j has no identity and fix any elements $a_i \in S_i (i \in I \setminus \{j\})$, $a_i \neq 1$. Then $X = \{(s_i)_{i \in I} \mid s_i \leqslant a_i, i \neq j\}$ is an ideal but not a principal ideal although for any $t = (t_i)_{i \in I}$ in S we have $(t) \cap X = (u)$, where $u_j = t_j$ and $u_i = a_i \wedge t_i$, $i \neq j$. Hence in this case S is not affine complete. Conversely, if all S_i have identity then for any ideal X we have $X = X \cap (1)$ where 1 is the identity of S, so that the ideal condition is trivially satisfied.

Let us consider the other two conditions. Clearly, S has an atom if and only if all S_i have zero and at least one of them has an atom. Hence S has no atoms iff either there is an S_i which has no zero or all S_i have zero but no atoms.

Next, consider any $a = (a_i \mid i \in I)$ with at least two components different from zero. Then a is thick in S. Indeed, let $0 \neq b = (b_i \mid i \in I) < a$, say, $b_j \neq 0$ (then of course $a_j \neq 0$) and $a_k \neq 0$, $j \neq k$. Now, if $b_k = a_k$ then for $d = (d_i \mid d_j = a_j, d_k < b_k, d_i \leqslant a_i$ for $i \neq j, k)$ d and b are incomparable; if, on the other hand, $b_k < a_k$ then for $c = (c_i \mid c_j < b_j, c_k = a_k, c_i \leqslant a_i$ for $i \neq j, k)$ c and b are incomparable. Finally, it is obvious that $a = (a_i \mid a_i = 0$ for $i \neq j)$ is thick iff a_j is thick.

Putting together our conclusions, we obtain the statement of Theorem 3.

Corollary 1. *A direct product of chains* S_i *is affine complete iff each* S_i *has an identity and at least two of the* S_i *have no zero.*

Corollary 2. *A direct product of affine complete semilattices S_i is affine complete iff each S_i has an identity.*

4. **Theorem 4.** *The free product of infinitely many semilattices is always affine complete.*

Proof. Let S denote the free product of the semilattices S_i ($i \in I, |I| \geqslant \aleph_0$). Since I is infinite, S has no zero, hence it cannot have atoms. Next, consider an arbitrary element $s = s_1 \wedge \ldots \wedge s_k$ of S, $s_j \in S_{i_j}$, $i_j \in I$, $i_j \neq i_{j'}$, if $j \neq j'$ (clearly, every element of S admits a unique decomposition of this form). For any $t \in S$, $t < s$, consider an $i \in I$ such that t has no component from S_i, and take any $x \in S_i$. Then $s \wedge x$ and t are incomparable, hence $(s) \neq (t) \cup [t, s]$, which proves that s is thick.

Let X be an ideal of S. We call a finite subset $J \subset I$ an X-set if there is an $s \in X$ whose components (in the canonical decomposition above) are exactly from the $S_j, j \in J$. We call J a minimal X-set if it is an X-set but none of its proper subsets is so. Obviously, minimal X-sets exists for every ideal $X \lhd S$.

We show that if $(a) \cap X$ is a principal ideal for all $a \in S$ then there is a unique minimal X-set. Suppose first that J and J' are minimal X-sets such that $J \neq J'$, $J \cap J' = K \neq \emptyset$. Let $p, q \in X$ be represented by J and J', respectively: $p = \bigwedge_{i \in J} s_i, q = \bigwedge_{i \in J'} s'_i$, and put $r = \bigwedge_{i \in K} (s_i \wedge s'_i)$. Then $X \cap (r)$ is not a principal ideal because it contains $p \wedge r$ and $q \wedge r$, and the latter two elements have no common upper bounds in $X \cap (r)$. Let now J and J' be disjoint minimal X-sets, $p, p' \in X$ be elements represented by J and J', respectively. Take any $r \in S \setminus X$, then $p \wedge r \in X$ and $p' \cap r \in X$, and $r \notin X$ is the smallest upper bound of $p \wedge r$ and $p' \wedge r$. Hence the latter two elements have no common upper bound in X and therefore $X \cap (r)$ is not a principal ideal. This proves our assertion.

Let the ideal X be as before, and let J be the minimal X-set. We prove that for every $x \in X$ there is a $z \in X$ represented by J such that $x \leqslant z$. Let $s \in X$ be an element which is represented by J, take an index $i \in I \setminus J$, which does not occur in the canonical decomposition of x, and a $y \in S_i$. Now $(y) \cap X$ is a principal ideal generated by u, say. For this u we have $u = u \wedge y \geqslant s \wedge y$, $u \geqslant x \wedge y$. Here $u = u \wedge y \geqslant s \wedge y$ implies, by the choice of y, that in the canonical decomposition of u the i-component is y; denoting by v the meet of the other components of u, we have $v \geqslant s$ and similarly $v \geqslant x$. Now $(v) \cap X$ is a principal

ideal generated by some z, for which $z \geqslant x$ and, in view of $z \geqslant s$ and $z \in X$, z is represented by J.

Fix now an index $j \in J$ and denote by X_j the set of those $a_j \in S_j$ which occur as j-components for some elements from X. Obviously, $X_j \lhd S_j$. Take an index $i \notin S_j$. Take an index $i \notin J$ and an $s_i \in S_i$. Then $X \cap (s_i)$ is a principal ideal in S; let y stand for its generator. Here $y \in X$ and y must have a j-component $y_j \in X_j$ since $j \in J$ and J is the minimal X-set. Take now any $z_j \in X_j$; z_j is the j-component of some $z \in X$. Then $z \wedge s_i \in X \cap (s_i) = (y)$, hence $z_j \leqslant y_j$. This shows that $X_j = (y_j)$. Moreover, notice that the choice of y did not depend on j, therefore $X_j = (y_j)$ for all $j \in J$. Finally, by the foregoing we know the existence of a $y' \in X$, $y' \geqslant y$, which is represented by J, and this y' can only be $\bigwedge_{j \in J} y_j$ as $X_j = (y_j)$ for all $j \in J$. Now we have $X = (y')$, and the proof is complete.

Corollary 3. *Every infinite free semilattice is affine complete.*

Remark. One can also treat the free product of finitely many semilattices, but the result misses the elegance of the infinite case because too many conditions get involved in the formulation. Therefore this will not be done here. As an example let it only be mentioned that if $S_1, \ldots, S_n$ are affine complete semilattices with zeros $0_i \in S_i$ and S_{n+1} is affine complete with identity 1_{n+1} then the free product of the $S_1, \ldots, S_{n+1}$ is not affine complete because for $a = \bigwedge_{i=1}^{n} 0_i$ and $b = a \wedge 1_{n+1}$ we have $(a) = (b) \cup [b, a]$.

Finally we exhibit two easy ways of constructing new affine complete semilattices from given ones.

5. *Every principal ideal of an affine complete semilattice is affine complete.* (This is an immediate consequence of Theorem 1.)

6. *If S is an affine complete semilattice without identity then S^1 is also affine complete* (S^1 is obtained by adjoining an identity to S). In fact, S^1 obviously satisfies conditions (i) and (iii), and as for condition (ii), we only have to show that 1 is a thick element. Take any $s \in S$. Since S has no identity, there is a $t \in S$, $s < t$. Now, if s were a "cutting point" for 1 then it would be the same for t, which is impossible as t is thick.

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K. KAARLI
Faculty of Mathematics
Tartu State University
Tartu, Estonian SSR, USSR

L. MÁRKI and E. T. SCHMIDT
Mathematical Institute
Hungarian Academy of Sciences
H-1364 Budapest, Pf. 127, Hungary

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