Algebra Universalis, **18** (1984) 386–395 0003–5240/84/030386–10$01.50+0.20/0

Congruence lattices of complemented modular lattices

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1. Introduction

The congruence lattice of a finite modular lattice is a Boolean algebra. In contrast to this, I have proved in [6] that every finite distributive lattice D is isomorphic to the congruence lattice $\mathrm{Con}(K)$ of some modular lattice K. (R. Freese [1] has shown that there is a finitely generated K.) In this paper we prove the related result for complemented modular lattices.

THEOREM. *For every finite distributive lattice D there exists a complemented modular lattice K such that the congruence lattice of K is isomorphic to D and K is a sublattice of the lattice of all subspaces of a countably infinite dimensional vector space over the two element field.*

A modular lattice M will be called a *locally finite complemented modular lattice* if for every finite subset X of M there exists a finite complemented sublattice M' such that $X \subseteq M'$, in other words M is the colimit of its finite complemented sublattices. In the proof of our theorem we construct such a K. The method of the proof can be easily extended to infinite distributive algebraic lattices in which every element is the finite join of join-irreducible elements. My conjecture is that every distributive algebraic lattice is representable with a complemented modular lattice. In the last section we present some ideas concerning this conjecture.

First we introduce some notations. If V is a vector space then $\mathbf{L}(V)$ denotes the lattice of all subspaces of V. $\mathscr{V}$ will always denote the countably infinite dimensional vector space over the two-element field, $\mathbf{2}$. $\mathscr{V}_n$ is the n-dimensional vector space over $\mathbf{2}$, i.e. $\mathscr{V}_n \cong \mathbf{2}^n$. Further, $\mathscr{L} = \mathbf{L}(\mathscr{V})$ and $\mathscr{L}_n = \mathbf{L}(\mathscr{V}_n)$. The finite dimensional subspaces of $\mathscr{V}$ form an ideal $\mathscr{L}^f$ of $\mathscr{L}$. A subspace X of $\mathscr{V}$ is cofinite dimensional iff $\mathscr{V}/X$ is finite dimensional. These form a filter $\mathscr{L}^{cf}$ of $\mathscr{L}$. Obviously, $\mathscr{L}^f \cap \mathscr{L}^{cf} = \varnothing$. Let $\mathscr{L}^* = \mathscr{L}^f \cup \mathscr{L}^{cf}$, then $\mathscr{L}^*$ is a $\{0, 1\}$-sublattice of $\mathscr{L}$ and $\mathscr{L}^*$ is a locally finite complemented modular lattice. It is easy to show that $\mathrm{Con}\,(\mathscr{L}^*) \cong \mathbf{3}$,

Presented by R. Freese. Received November 25, 1981. Accepted for publication in final form June 21, 1983.

where **3** denotes the three-element chain. In the construction of the complemented modular lattice K with a given congruence lattice, a number of copies of $\mathscr{L}^*$ are "hooked together".

Let L be a complemented sublattice of $\mathscr{L}$. A prime quotient a/b of L is not necessarily a prime quotient of $\mathscr{L}$. Consider in the interval $[b, a]$ a $\{0, 1\}$-sublattice M. Then $L \cap M = \{a, b\}$. $L \cup M$ is a partial lattice (relative sublattice of $\mathscr{L}$). We would like to determine the sublattice generated by this partial lattice in a special case. This sublattice will be denoted briefly by $L[M]$ and is called the extension of L by M (in $\mathscr{L}$). Let us assume that L is a sublattice of $\mathscr{L}$, $L \cong \mathscr{L}^*$. If a/b is a prime quotient of L and M is a $\{0, 1\}$-sublattice of a/b then it is easy to show that $\mathrm{Con}\,(L[M]) \cong \mathrm{Con}\,(M)+1$. On this way we can easily represent all finite chains. In this paper we shall generalize this construction to represent all finite distributive lattices.

2. Normalized frames

To describe the extension $L[M]$ in $\mathscr{L}$ we need the concept of a normalized frame, which was introduced by J. von Neumann [4].

The elements $a_1, a_2, \ldots, a_n$ and c_{jk} $(j \neq k, j, k = 1, 2, \ldots, n)$ of a modular lattice K form a *normalized frame of order n* in K if the following relations hold:

(i) $(a_i; i = 1, 2, \ldots)$ is independent sequence.

(ii) $\{0, a_i, a_j, c_{ij}, a_i \vee a_j\}$ $(i \neq j)$ is a diamond, $c_{ij} = c_{ji}$ and $c_{ij} = (c_{ik} \vee c_{jk}) \wedge (a_i \vee a_j)$ for $i, j, k = 1, 2, \ldots, n$, $i \neq j \neq k \neq i$.

We denote this frame by $\mathscr{F} = (a_i, c_{jk})$. The sequence $(a_i; i = 1, 2, \ldots, n)$ is called the basis of $\mathscr{F}$. Let $(a_i; i = 1, 2, \ldots)$ be a denumerably infinite independent sequence and c_{ij} $(i \neq j, i, j = 1, 2, \ldots)$ elements of K. If these satisfy (ii) then $\mathscr{F} = (a_i, c_{jk})$ is called a normalized frame of order ∞.

Two elements a_i and a_j in K are perspective, in symbol $a_i \sim a_j$ if a_i and a_j have a common complement c. In this case $\{0, a_i, a_j, c \wedge (a_i \vee a_j), a_i \vee a_j\}$ is a diamond. Conversely, if $\{0, a_i, a_j, c', a_i \vee a_j\}$ is a diamond with suitable c' then $a_i \sim a_j$. The independent sequence $(a_i; i = 1, 2, \ldots)$ finite or denumerably infinite is called *homogeneous* if for every i, j $(i \neq j)$, $a_i \sim a_j$ (see [4] Definition 3.1 Part II). If $\mathscr{F} = (a_i, c_{jk})$ is a frame then the basis $(a_i, i = 1, 2, \ldots)$ is obviously homogeneous. Conversely, every homogeneous sequence $(a_i, i = 1, 2, \ldots)$ of K can be complete to a normalized frame, as follows:

LEMMA 1. ([4], Lemma 5.3, Part II). *Let $(a_i; i = 1, 2, \ldots)$ be a homogeneous sequence in a modular lattice K. There exists a normalized frame $\mathscr{F} = (a_i, c_{jk})$; the elements c_{1i} $(i = 1, 2, \ldots)$ may be chosen arbitrarily such that $\{0, a_1, a_i, c_{1i}, a_1 \vee a_i\}$*

is a diamond and the reaming $c_{jk}-s$ are the uniquely determined by $c_{jk}=(c_{1j}\vee c_{1k})\wedge(a_j\vee a_k)$.

In the lattice $\mathscr{L}_n$ we consider a maximal independent set of atoms, $\{a_1, a_2, \ldots, a_n\}$. This is obviously a homogeneous sequence. By Lemma 1 there exists a normalized frame (a_i, c_{jk}) of order n. Then $\{a_i, c_{jk}; i, j, k=1, 2, \ldots, n\}$ is a generating set for $\mathscr{L}_n$. Let $\langle\mathscr{F}\rangle$ be the sublattice of $\mathscr{L}$ generated by a frame $\mathscr{F}$. C. Hermann and A. P. Huhn [2] have proved the following:

LEMMA 2. *Let $\mathscr{F}$ be a normalized frame of order n or ∞ in $\mathscr{L}$. Then the sublattice $\langle\mathscr{F}\rangle$ generated by $\mathscr{F}$ is isomorphic to $\mathscr{L}_n$ resp. $\mathscr{L}^f$. The elements of $\mathscr{F}$ are atoms of $\langle\mathscr{F}\rangle$.*

Let $\mathscr{F}=(a_i, c_{jk}; i, j, k=1, 2, \ldots, n)$ be a normalized frame in $\mathscr{L}$, and let $b\le a_1$. We define the following elements:

$$b^1=b, \qquad b^i=(b\vee c_{1i})\wedge a_i, \qquad d^{1i}=(b\vee a_i)\wedge c_{1i}.$$

An easy computation shows (see [3]) that $\{0, b^1, b^i, d^{1i}, b^1\vee b^i\}$ is a diamond. Consequently $(b^i; i=1, 2, \ldots, n\}$ is a homogeneous sequence, and by Lemma 1 there exists a normalized frame $\mathscr{F}^b=(b^i, d^{ik})$ of order n, where $d^{jk}\le c_{jk}$ and $b^i\le a_i$.

Let $\mathscr{G}=(b_r, d_{st}; r, s, t=1, 2, \ldots, m)$ another normalized frame such that $b_r\le a_1$ for $r=1, 2, \ldots, m$, i.e. $\mathscr{G}$ is contained in the principal ideal $(a_1]$. We will define a new normalized frame $\mathscr{F}^{\mathscr{G}}$. First we determine the basis: $b_r^i=(b_r)^i=(b_r\vee c_{1i})\wedge a_i$ $(i=1, 2, \ldots, n;\ r=1, 2, \ldots, m)$. We write these elements in a matrix

$$B=\begin{pmatrix} b_1^1 & \cdots & b_m^1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ b_1^n & & b_m^n \end{pmatrix}$$

The elements of the first row form a homogeneous independent sequence, consequently the same holds for all rows. On the other hand, $a_1, \ldots, a_n$ is an independent sequence, thus by Theorem I.4.2. of [4] the elements of B form an independent set. The column $b_r^1, b_r^2, \ldots, b_r^n$ is the basis of the frame $\mathscr{F}^{b_r}$, consequently is again a homogeneous sequence. Thus we get that the elements of B form a homogeneous set. We complete this to a normalized frame of order $m\cdot n$ by defining the elements d_{1r}^{1i} $(i=1, 2, \ldots, m)$ having the property that

$\{0, b_1^1, b_r^i, d_{1r}^{1i}, b_1^1 \vee b_r^i\}$ is a diamond (see Lemma 1). Let d_{1r}^{11} be the element $d_{1r} \in \mathscr{G}$. In the frame $\mathscr{F}^{b_r}$ we have $d_{1r}^{1i} = (b_r \vee a_i) \wedge c_{1i}$. Let $d_{rr}^{1i} = d_r^{1i}$ and finally we define d_{1r}^{1i} to be $(d_{1r}^{11} \vee d_{rr}^{1i}) \wedge (b_1^1 \vee b_r^i)$. The elements of B together with the elements d_{1r}^{1i} determine the normalized frame $\mathscr{F}^{\mathscr{G}}$.

By Lemma 2, $\langle \mathscr{F}^{\mathscr{G}} \rangle$ is isomorphic to $\mathscr{L}_{mn}$. First we assume that $a_1 = \bigvee_{r=1}^m b_r$, i.e. the lattice $\langle \mathscr{G} \rangle$ generated by $\mathscr{G}$ is a $\{0, 1\}$-sublattice of the principal ideal $(a_1]$. Then we have the isomorphism $\varphi(x) = (x \vee c_{1i}) \wedge a_i$ from $(a_1]$ onto $(a_i]$, hence $a_i = \bigvee_{r=1}^m b_r^i$. Similarly, by inspection we get $c_{1i} = \bigvee_{r=1}^m d_{rr}^{1i}$, i.e. $\mathscr{F}$ is contained in $\langle \mathscr{F}^{\mathscr{G}} \rangle$. The principal ideal $(a_1]$ of $\langle \mathscr{F}^{\mathscr{G}} \rangle$ contains $\langle \mathscr{G} \rangle \cong \mathscr{L}_m$. But $\langle \mathscr{F}^{\mathscr{G}} \rangle \cong \mathscr{L}_{m \cdot n}$, consequently the principal ideal $(a_1]$ of $\langle \mathscr{F}^{\mathscr{G}} \rangle$ must be isomorphic to $\mathscr{L}_m$.

Now, we return to the description of $L[M]$ in the case of $L \cong \mathscr{L}_n$ and M is a locally finite complemented modular lattice. L is generated by a normalized frame $\mathscr{F} = (a_i, c_{jk})$ or order n, and $a_1/0$ is a prime quotient of L. First assume, that M is a finite complemented modular $\{0, 1\}$-sublattice of $(a_1]$ in L. Then M is the direct product of projective geometries, i.e. $M = M_1 \times \cdots \times M_t$ where each M_i is generated by a normalized frame $\mathscr{G}_i$ of order m_i. Similarly as before we can extend each $\mathscr{G}_i$ to a normalized frame $\mathscr{F}^{\mathscr{G}_i}$ $(i = 1, 2, \ldots, t)$ of order $n \cdot m_i$. The $\mathscr{G}_1, \ldots, \mathscr{G}_t$ form an independent set of frames in the sense that the join of the corresponding bases is independent. Thus we get that $\mathscr{F}^{\mathscr{G}_1}, \ldots, \mathscr{F}^{\mathscr{G}_n}$ is again an independent sequence of frames, i.e. if $\langle \mathscr{F}^{\mathscr{G}_1}, \ldots, \mathscr{F}^{\mathscr{G}_n} \rangle$ denotes the sublattice generated by these frames then

$$L[M] = \langle \mathscr{F}^{\mathscr{G}_1}, \ldots, \mathscr{F}^{\mathscr{G}_t} \rangle \cong \prod_{i=1}^{t} \langle \mathscr{F}^{\mathscr{G}_i} \rangle.$$

Consequently the ideal $(a_1]$ of $L[M]$ is isomorphic to M and for every congruence relation θ of $(a_1] = M$ there exists exactly one congruence relation $\bar{\theta}$ of $L[M]$ such that for $a, b \in M$ $a \equiv b(\bar{\theta})$ iff $a \equiv b(\theta)$ in M.

Now we are ready to prove:

LEMMA 3. *Let L be a sublattice of $\mathscr{L}$ isomorphic to $\mathscr{L}^f$ or $\mathscr{L}_n$ and let a/b be a prime quotient of L. If M is a locally finite complemented $\{0, 1\}$-sublattice of $[b, a]$ then $[b, a] \cap L[M] = M$. For every congruence relation θ of M there exists exactly one congruence relation $\bar{\theta}$ of $L[M]$ such that for $a, b \in M$ $a \equiv b(\bar{\theta})$ iff $a \equiv b(\theta)$ in M.*

Proof. L is isomorphic to $\mathscr{L}^f$ (or $\mathscr{L}_n$). M is locally finite, i.e. M is the colimit of finite complemented modular $\{0, 1\}$-sublattices M_t, $t \in T$. Then $L[M]$ is the colimit of the lattices $L[M_t]$. Each congruence relation θ of M is determined by the system $\{\theta \mid M_t, t \in T\}$, i.e. θ can be extended to $L[M]$ iff the same is satisfied for all $\theta \mid M_t$ in $L[M_t]$, which is already proved.

If we replace in the definition of the independent set resp. normalized frame the two lattice operations we get the notation of the *dually independent set* resp. *normalized dual frame.*

In the countably infinite dimensional vector space $\mathcal{V}$ we choose a basis in matrix form:

$$\begin{matrix} e_{11}e_{12}e_{13}\cdots \\ e_{21}e_{22}e_{23}\cdots \\ \cdot\ \ \cdot\ \ \cdot \\ \cdot\ \ \cdot\ \ \cdot \\ \cdot\ \ \cdot\ \ \cdot \end{matrix}$$

where this matrix has infinite many rows and columns. Let a_i be the subspace spanned by the i-th row, i.e. $a_i=[e_{i1}, e_{i2}, \ldots]$. Then $(a_i; i=1,2,\ldots)$ is a denumerably infinite independent set. Let c_{ij} be the subspace $[e_{i1}+e_{j1}, e_{i2}+e_{j2}, \ldots]$ then it is easy to show that $\{0, a_i, a_j, c_{ij}, a_i \vee a_j\}$ is a diamond, i.e. $\mathcal{F}=(a_i, c_{jk})$ is a normalized frame of order ∞ in $\mathcal{L}$.

The principal ideal $(a_i]$ of $\mathcal{L}$ is again isomorphic to $\mathcal{L}$. In the following $\mathcal{F}$ will denote always this special frame.

Let a_i' be subspace of $\mathcal{V}$ spanned by $[e_{jk}; j=1,2,\ldots, k=1,2,\ldots, j\neq i]$. Then a_i' is a complement of a_i in the lattice $\mathcal{L}$ (a_i has many other complements, but if we fix a basis then a_i determines a_i' uniquely).

Let c_{ij}' defined by $c_{ij}\vee(a_i'\wedge a_j')$ then $\mathcal{F}'=(a_i', c_{jk}')$ is a dual frame. This $\mathcal{F}'$ is uniquely determined by $\mathcal{F}$ (in the given basis) and is called the complement of $\mathcal{F}$. The sublattice $L=\langle\mathcal{F}\rangle$ generated by $\mathcal{F}$ is isomorphic to $\mathcal{L}^f$ (Lemma 2) and $\mathcal{F}\cup\mathcal{F}'$ generates a sublattice L^* isomorphic to $\mathcal{L}^*$.

3. The construction of a complemented lattice

Let $\mathcal{F}=(a_i, c_{jk})$ be our given frame in $\mathcal{L}$ and let M be a locally finite complemented $\{0,1\}$-sublattice of the principal ideal $(a_1]$. Every congruence relation θ of M is determined by its kernel I, which is a neutral ideal of M. By Lemma 3 the congruence relation $\theta\in\operatorname{Con}(M)$ has exactly one extension to $L[M]$ (L denotes $\langle\mathcal{F}\rangle$), i.e. $\bar\theta\mid(a_1]=\theta$.

$L[M]$ is a relatively complemented lattice, hence $\bar\theta$ is determined by its kernel $\bar I$. Then $\bar\theta\mid(a_1]=\theta$ yields that $\bar I\cap(a_1]=I$. We say that $\bar I$ is the extension of I ($\subseteq M$) to $L[M]$.

$(a_1]$ and $\bar I$ are two ideals of $L[M]$ such that $I=(a_1]\cap\bar I$ is a neutral ideal of

$(a_1]$. Let S be the sublattice of $L[M]$ generated by $(a_1]$ and $\bar{I}$. We prove that every element s of S has a representation $s = a \vee x$, where $a \leq a_1$, $x \in \bar{I}$, $a_1 \wedge s = a$. By assumption M is a locally finite complemented modular lattice, which implies that we may assume, M is finite. Then I is a principal ideal, say $I = (u]$. Let v be the complement of u in M then I is a neutral ideal, hence I is a direct factor, i.e. $M = (u] \times (v]$, and $S = \bar{I}x(v]$, i.e. every element $s \in S$ has the given representation.

Every congruence relation θ of S is the join of $\theta_1 \in \mathrm{Con}\,(M)$ and $\theta_2 \in \mathrm{Con}\,(\bar{I})$. By Lemma 3 θ_2 is determined by its restriction to I, i.e. θ is determined by its restriction to I, i.e. θ is determined by a congruence relation of M. This proves $\mathrm{Con}\,(S) \cong \mathrm{Con}\,(M)$.

Let I' be the cokernel of θ in M, i.e. $I' = \{x;\, x \in M, x \equiv 1(\theta)\}$. Then I' is a dual ideal of M. Consider the sublattice L' generated by the dual frame $\mathscr{F}'$. The quotients $a_1/0$ and $1/a_1'$ are transposed, hence we can assume that M is a $\{0, 1\}$-sublattice of $[a_1', 1]$. Similarly as before we can extend I' to a filter $\bar{I}'$ of $L'[M]$. Let S' be the sublattice generated by $[a_1')$ and $\bar{I}'$, then $\mathrm{Con}\,(S') \cong \mathrm{Con}\,(M)$. Obviously $S \cap S' = \varnothing$, and $K = S \cup S'$ is a locally finite complemented modular lattice. (Figure 1). This K is a sublattice of $L^*[M]$ where L^* is the sublattice of $\mathscr{L}$ generated by $\mathscr{F} \cup \mathscr{F}'$.

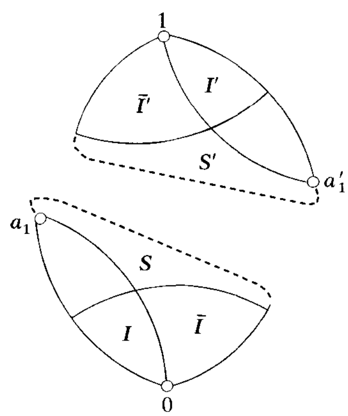

Fig. 1.

4. The proof of the Theorem

Let D be a finite distributive lattice. $J(D)$ denotes the poset of all nonzero join-irreducible elements of D. For an arbitrary poset P, call $A \subseteq P$, hereditary iff $x \in A$, $y \leq x$ imply that $y \in A$. The set $\mathscr{H}(P)$ of all hereditary subsets of P is a distributive lattice. It is well-known that D is isomorphic to $\mathscr{H}(J(D))$, i.e. D is determined by $J(D)$. Therefore we have to prove that for every finite poset P there exists a locally finite complemented modular lattice $K(P)$ such that $\mathrm{Con}(K(P))$ is isomorphic to $\mathscr{H}(P)$. To construct $K(P)$ we induct on the size of P. If P has only one element, $P=1$ then $K(1)$ is the two-element lattice, which is obviously a $\{0, 1\}$-sublattice of $\mathscr{L}$.

Now, suppose $|P|\geq 2$ and let m be a maximal element of P. Let m covered by $m_1, \ldots, m_k$ in P. We denote by Q the poset $P\setminus\{m\}$. $|P|\geq 2$ implies that Q is not empty. By induction, there exists a complemented modular lattice $K(Q)$ and an isomorphism

$$\varphi : \mathscr{H}(Q) \to \mathrm{Con}(K(Q)).$$

This $K(Q)$ is a $\{0, 1\}$-sublattice of $\mathscr{L}$. Let $(m_1, m_2, \ldots, m_k)$ be the hereditary subset of Q generated by $m_1, m_2, \ldots, m_k$. Then $\varphi((m_1, m_2, \ldots, m_k)) = \theta$ is a congruence relation of $K(Q)$. The lattice $K(Q)$ is complemented, i.e. θ is determined by its kernel I. This I is an ideal of $K(Q)$. Let I' by the cokernel of θ, i.e. $I' = \{x; x\in K(Q),\ x\equiv 1(\theta)\}$.

Let M be the lattice $K(Q)$. We assume that M is a $\{0, 1\}$-sublattice of $(a_1]$ where $\mathscr{F} = (a_i, c_{jk})$ is our fixed frame. In the previous section we constructed from $\mathscr{F}\cup\mathscr{F}'$, M and I a lattice K. We define $K(P)$ to be this lattice K. Then we have to prove that $\mathrm{Con}(K(P)) \cong \mathscr{H}(P)$.

$K(P)$ is a complemented lattice, therefore every principal congruence relation has the form $\theta(0, u)$. If $u\in S$ then $\theta(0, u)$ is the extension of a congruence relation of $M = K(Q)$. If $u\in S'$, then $u = a'\wedge x$, for suitable $a'\geq a_1'$, $x\in \bar{I}'$ and $a' = u\vee a_1'$. Thus we get $\theta(u\wedge a_1', u) = \theta(a_1', u\vee a_1') = \theta(a_1', a') = \theta(a_1, a)$, i.e. $\theta(u\wedge a_1', u)$ is again the extension of a congruence relation of M. Further, $\theta(0, u) = \theta(0, u\wedge a_1')\vee\theta(u\wedge a_1', u)$ implies that $\theta(0, u)$ is the join of $\theta(0, u\wedge a_1')$ with a congruence relation $(\theta(u\wedge a_1', u))$ which is the extension of a congruence relation of M. The congruence relation $\theta(0, u\wedge a_1')$ is equal $\theta(0, a_1')$ for arbitrary $u\in S'$, i.e. every join-irreducible congruence relation of $K(P)$ is either the extension of a join-irreducible congruence relation of M or it is $\theta(0, a_1')$. It is clear that $\theta(0, a')\geq\bar{\theta}$, where $\theta\in\mathrm{Con}(M)$ if and only if $\theta\leq\varphi((m_i))$, which proves that $J(\mathrm{Con}(K(P))$ is isomorphic to P.

5. On the representation of infinite distributive lattices

In the introduction we have mentioned the following

Conjecture. Every distributive algebraic lattice is isomorphic to the congruence lattice of some complemented modular lattice.

In this section we make some comments to this conjecture. In [7] I have proved the following theorem: the ideal lattice of a distributive lattice is the congruence lattice of a lattice. Later P. Pudlák [5] has given a new proof for this theorem. His proof reduces the characterization problem to investigation of the representations of finite distributive lattices. First we present this approach.

The compact congruences of a lattice L form a distributive semilattice $\mathrm{Con}^c(L)$ whose ideal lattice is $\mathrm{Con}(L)$. (A semilattice S is called distributive iff $a \leq b_0 \vee b_1$ implies the existence of $a_0, a_1 \in S$, $a_i \leq b_i$, $i = 0, 1$ with $a = a_0 \vee a_1$). Let L_i be a $\{0\}$-sublattice of L_j, and let $\lambda_{ij} : L_i \rightarrowtail L_j$ the identical embedding given by inclusion. Then λ_{ij} induces a homomorphism $\mathrm{Con}(\lambda_{ij})$ of the semilattice $\mathrm{Con}^c(L_i)$ into $\mathrm{Con}^c(L_j)$ which maps each $\theta \in \mathrm{Con}^c(L_i)$ to the smallest congruence of L_j that contains the image of θ. If S_i and S_j are semilattices with least elements then $S_i \subseteq S_j$ means that S_i is a $\{0\}$-subsemilattice of S_j. Let γ_{ij} be the identical embedding given by inclusion. Let D be a distributive algebraic lattice. The compact elements of D form a distributive semilattice S. P. Pudlák has proved that every finite subset of S is contained in a finite distributive subsemilattice of S, i.e. the distributive semilattices are locally finite. This implies that the system $\{S_i ; i \in I\}$ of all finite subsemilattices of S is a directed system. Let $\mathcal{S} = \{S_i, i \in I, \gamma_{ij} : S_i \rightarrowtail S_j, S_i \subseteq S_j\}$ be the category of all finite distributive $\{0\}$-subsemilattices of S, where γ_{ij} are the identical embeddings. Then S with the embeddings $\gamma_i : S_i \rightarrowtail S$ $i \in I$ is a colimit of $\mathcal{S}$. Let be noted that every finite (distributive) semilattice is a lattice.

Suppose that we have a "construction" $\mathcal{K}$ which assignes to each $S_i \in \mathcal{S}$ a lattice $L_i = \mathcal{K}(S_i)$ and an isomorphism $\iota_i : S_i \to \mathrm{Con}^c(L_i)$ such that for each semilattice-embedding $S_i \xrightarrow{\gamma_{ij}} S_j$ there is a lattice-embedding $L_i \xrightarrow{\lambda_{ij}} L_j$ which satisfies the following two properties:

(i) λ_{ii} is the identity on L_i and $\lambda_{jk} \circ \lambda_{ij} = \lambda_{ik}$ if $S_i \subseteq S_j \subseteq S_k$.

(ii) $\iota_j \circ \gamma_{ij} = \mathrm{Con}(\lambda_{ij}) \circ \iota_i$ (Fig. 2.)

P. Pudlák has proved that under these assumptions there exist a colimit L of the system $\bar{\mathcal{S}} = \{L_i, i \in I, L_i = \mathcal{K}(S_i), \lambda_{ij} : L_i \rightarrowtail L_j\}$ and $\mathrm{Con}^c(L) \cong S$, i.e. $\mathrm{Con}(L) \cong I(S) = D$. If the lattices L_i are complemented modular lattices then the same holds

$$\begin{tikzcd} S_i \arrow[r, tail, "\gamma_{ij}"] \arrow[d, "\iota_i"] & S_j \arrow[d, "\iota_j"] \\ \mathrm{Con}(L_i) \arrow[r, "\mathrm{Con}(\lambda_{ij})"] & \mathrm{Con}(L_j) \end{tikzcd}$$

Fig. 2.

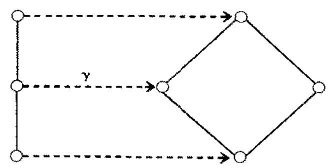

Fig. 3.

for the colimit L. Therefore to solve our conjecture we need to give a "construction" $\mathscr{K}$, which assings to each finite distributive lattice S_i a complemented modular lattice L_i satisfying the condition (i) and (ii).

We illustrate the idea of such a construction $\mathscr{K}$. Let $K(S)$ be the complemented modular lattice which is assigned to S by $\mathscr{K}$. Then $\mathrm{Con}\,(K(S)) \cong S$. Let $S = \mathbf{2}^n$ where $\mathbf{2}$ denotes the two-element lattice. $\mathrm{Con}\,(K(\mathbf{2}^n)) \cong \mathbf{2}^n$ implies that $K(\mathbf{2}^n)$ is the direct product of a simple complemented modular lattices. We may suppose that these direct factors are isomorphic. Let C denote this simple lattice. The three-element chain $\mathbf{3}$ is a sublattice of $\mathbf{2}^2$, thus we get that $K(\mathbf{3})$ is a $\{0\}$-sublattice of $K(\mathbf{2}^2) \cong C^2$. Since the congruence lattice of a finite modular lattice is a boolean lattice, $\mathrm{Con}\,(K(\mathbf{3})) \cong \mathbf{3}$ implies that C is an infinite simple complemented modular lattice. Such lattices are the continuous geometries, i.e. the infinite dimensional, continuous, complemented modular lattices.

Consider the following embedding, $\mathbf{3} \xrightarrow{\gamma} \mathbf{2}^2$ (Figure 3).

Let I be a nonprincipal ideal of C and let I' be the "complement" of I, i.e. $I' = \{y;\, y = x', \text{ where } x \in I\}$. Then $I \cap I' = \varnothing$. We define L to be $I \cup I'$. Every interval of a continuous geometry is again a continuous geometry, consequently a simple lattice. Thus we get, $\mathrm{Con}\,(L) \cong \mathbf{3}$. We give an embedding δ of L into C^2 as follows: if $x \in I$ then $\delta(x) = (x, 0) \in C^2$ and for $y \in I'$ $\delta(y) = (1, y)$. It is easy to see that the condition (ii) is satisfied for $K(\mathbf{3}) = L$, $K(\mathbf{2}^2) = C^2$. (In the definition of $K(\mathbf{3})$ we considered only the embedding γ, the correct definition of $K(\mathbf{3})$ is more complicated).

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