ON THE LATTICE OF ALL JOIN-ENDOMORPHISMS OF A LATTICE

G. GRÄTZER AND E. T. SCHMIDT

A many-one correspondence Θ of the lattice L into itself is called a join-endomorphism, if it satisfies

$$\Theta(x \cup y) = \Theta x \cup \Theta y \tag{1}$$

for all $x, y \in L$.

It is easily shown that every join-endomorphism is an isotone correspondence (i.e. $x \geqq y$ implies $\Theta x \geqq \Theta y$). It is also easy to see that the set I of antecedents of 0 under any join-endomorphism is an ideal. (For these and other facts used in the sequel we refer to the textbook of G. Birkhoff, *Lattice theory*, rev. ed., New York, 1948, henceforth cited as LT.)

G. Birkhoff states in LT (p. 208, Example 4) that all join-endomorphisms Θ of any lattice L form an l-semigroup, where the join of two join-endomorphisms Θ and Φ satisfies

$$(\Theta \cup \Phi)x = \Theta x \cup \Phi x \tag{2}$$

for all $x \in L$.

But this statement is not true in general; indeed, we shall show that there exists a lattice whose join-endomorphisms do not form an l-semigroup under the join stated.

We shall also deal with the following question proposed by G. Birkhoff in LT.

Problem 93. Is the lattice of all join-endomorphisms of an arbitrary lattice semi-modular?

We shall show that the answer is negative.[1] Namely, restricting ourselves to finite lattices, we shall prove that there exists no lattice whose lattice of all join-endomorphisms is semi-modular and not distributive; furthermore, the lattice of all join-endomorphisms of any lattice L is distributive if and only if L is distributive.

We give also some generalizations of these results.

1. On the existence of the lattice of all join-endomorphisms. In what follows $L_\cup$ will denote the set of all join-endomorphisms of a lattice L, where the join of two elements Θ and Φ of $L_\cup$ is defined by

Received by the editors July 12, 1957; and, in revised form, November 22, 1957.

[1] This much was remarked by R. P. Dilworth in his review of LT, Bull. Amer. Math. Soc. vol. 56 (1950) pp. 204–206.

(2); the join operation gives rise to a partial ordering in $L_\cup$; $\Theta \leqq \Phi$ if and only if $\Theta x \leqq \Phi x$ for all $x \in L$.

Firstly we show by an example that $L_\cup$ is in general not a lattice.

Let V be the chain of all real numbers of the closed interval $[0, 1]$ with the usual ordering. Let us consider in $V \cdot V$ (the cardinal product of V by itself, in the sense of LT p. 7) the sublattice L consisting of all elements of $V \cdot V$, with the exception of $(1, 0)$. L is actually a sublattice of $V \cdot V$, for $(1, 0)$ is join- and meet-irreducible.

Let us consider the following mappings of L into itself:

$$\Theta = \{(1, y) \to (1, 1) \text{ if } y \neq 0;\ (x, y) \to (0, 0) \text{ if } x \neq 1\},$$

$$\Phi = \{(x, 0) \to (0, 0) \text{ if } x \neq 1;\ (x, y) \to (1, y) \text{ if } y \neq 0\}.$$

It is routine to check that Θ, $\Phi \in L_\cup$. For all $\psi \in L_\cup$ satisfying $\psi \leqq \Theta$ and $\psi \leqq \Phi$ we have $\psi(x, y) \leqq \Theta(x, y) = (0, 0)$ if $x \neq 1$, and for this reason $\psi(1, y) = \psi(1, z)$ holds for all $y \neq 0$ and $z \neq 0$. Let $\psi(1, y) = (a, b)(y \neq 0)$. From $\Phi(1, y) = (1, y)$ and $\psi \leqq \Phi$ it follows that $(a, b) \leqq (1, y)$ for all $y \neq 0$, that is $b = 0$. Thus each ψ is of the form $\psi = \{(x, y) \to (0, 0) \text{ if } x \neq 1;\ (1, y) \to (a, 0) \text{ if } y \neq 0\}$. Since among these ψ there is clearly no greatest one, $a = 1$ being impossible, it follows that $L_\cup$ is not a lattice.

In what follows we shall need the following sufficient condition for $L_\cup$ to be a lattice.

THEOREM 1. *If L is a complete lattice, then $L_\cup$ is also a complete lattice.*

PROOF. Let $\Theta_\alpha \in L_\cup$ $(\alpha \in A)$ and define Θ by $\Theta x = \bigvee_{\alpha \in A} \Theta_\alpha x$. From the general associative law we get

$$\Theta x \cup \Theta y = \bigvee_{\alpha \in A} \Theta_\alpha x \cup \bigvee_{\alpha \in A} \Theta_\alpha y = \bigvee_{\alpha \in A} (\Theta_\alpha x \cup \Theta_\alpha y)$$

$$= \bigvee_{\alpha \in A} \Theta_\alpha(x \cup y) = \Theta(x \cup y)$$

i.e., $\Theta \in L_\cup$. Clearly $\Theta \geqq \Theta_\alpha$; moreover if $\Phi \geqq \Theta_\alpha$ for all $\alpha \in A$, then $\Phi x \geqq \bigvee_{\alpha \in A} \Theta_\alpha x$ whence $\Phi \geqq \Theta$ and any subset of $L_\cup$ has a join. $L_\cup$ has a zero-element, for L has a 0 and the mapping $x \to 0$ for all $x \in L$ is a join-endomorphism of L. Hence $L_\cup$ is a partly ordered set with zero-element and complete joins, consequently $L_\cup$ is a complete lattice (LT, p. 49).

COROLLARY (ZACHER'S THEOREM).[2] *If L is a finite lattice then all*

[2] Giovanni Zacher, *Sugli emiomorfismi superiori ed inferiori*, Atti del Quarto Congresso dell'Unione Matematica Italiano, Taormina, vol. 2 (1951) pp. 251–252, and by the same author, with the same title in Rend. Accad. Sci. Fis. Mat. Napoli (4) vol. 19 (1952) pp. 45–56 (1953).

join-endomorphisms of L form a lattice.

2. **Nondistributive lattices with finite bounded chains.**[3] If the lattice L is not distributive, then it contains as a sublattice one of the lattices

S: the elements of S include a, b, c, i, o; $a\cup b=b\cup c=c\cup a=i$, $a\cap b=b\cap c=c\cap a=o$;

T: the elements of T include a, b, c, i, o; $a\cup c=b\cup c=i$, $a\cap c=b\cap c=o$, $a\cap b=a$.

a. Let L be a modular, but not distributive lattice with finite bounded chains. Then L contains the lattice S as a sublattice with the further condition that a, b, c cover o (LT, p. 134).

Let us consider in L the following join-endomorphisms:

$$\Theta = \{(a] \to o;\ L-(a] \to a\},$$
$$\Phi = \{(o] \to o;\ (b]-(o] \to b;\ (c]-(o] \to c;\ L-(b]-(c] \to i\},$$
$$\Psi = \{(o] \to o;\ (b]-(o] \to b;\ L-(b] \to i\},$$
$$\Omega = \{(i] \to o;\ L-(i] \to a\}.$$

Then Θ covers Ω and $\Theta\cup\Phi=\{(o]\to o;\ L-(o)\to i\}$. Clearly $\Omega\cup\Phi=\Phi<\Psi<\Theta\cup\Phi$ so that $L_\cup$ is not semi-modular.

b. If L is a nonmodular lattice with finite bounded chains, then L contains as a sublattice the lattice T. It is clear that there exist in L an x and y, such that x covers $x\cap y$ and $x\cap y\neq y$. The mappings ($\mathfrak{J}$ denotes a maximal ideal, such that $c\in\mathfrak{J}$, $a\notin\mathfrak{J}$)

$$\Theta = \{\mathfrak{J} \to x\cap y;\ L-\mathfrak{J} \to x\},$$
$$\Phi = \{(o] \to x\cap y;\ (c]-(o] \to x;\ (b]-(o] \to y;\ L-(c]-(b] \to x\cup y\},$$
$$\Psi = \{(o] \to x\cap y;\ (c]-(o] \to x;\ (a]-(o] \to y;\ L-(c]-(a] \to x\cup y\},$$
$$\Omega = \{(i] \to x\cap y;\ L-(i] \to x\}$$

are join-endomorphisms and it may be easily checked that Θ covers Ω, yet $\Theta\cup\Phi>\Psi>\Omega\cup\Phi=\Phi$, i.e. $L_\cup$ is not semi-modular. Thus we have the following

THEOREM 2. *The lattice of all join-endomorphisms of a nondistributive lattice with finite bounded chains is not semi-modular.*

3. **The case of finite distributive lattices.** Let L be a distributive lattice with 0 and I, of finite length. It is known that L is itself finite

[3] By a bounded chain we mean a chain with a least and a greatest element. $(a]$ denotes the ideal generated by a.

and in L every element is the join of join-irreducible elements. Conversely, if L has exactly k join-irreducible elements $a_1, a_2, \cdots, a_k$ then $\Theta \in L_\cup$ is completely determined by the elements $b_i = \Theta a_i$ $(i = 1, 2, \cdots, k)$. Thus we may write Θ in the form $\Theta = (b_1, b_2, \cdots, b_k)$, where evidently $a_i \geqq a_j$ implies $b_i \geqq b_j$. Moreover

LEMMA 1. *If $a_1, \cdots, a_k$ are all the join-irreducible elements of L, and $b_1, \cdots, b_k$ are arbitrary in L, then a necessary and sufficient condition for the existence of a join-endomorphism Θ with $\Theta a_i = b_i$ $(i = 1, 2, \cdots, k)$ is the fulfillment of the condition: $a_i \geqq a_j$ implies $b_i \geqq b_j$.*

PROOF. Let us denote by $r(a)$ the set of all join-irreducible elements contained in a; then $a = \bigcup_{a_i \in r(a)} a_i$. We need only prove the sufficiency of the condition. If the b_i satisfy the stated condition and we define Θ as $\Theta a = \bigcup_{a_i \in r(a)} \Theta a_i$, then we have to verify that $\Theta(a \cup b) = \Theta a \cup \Theta b$ for all $a, b \in L$. Only $r(a) \vee r(b) = r(a \cup b)$ need be proved, where $\vee$ means the set-theoretical union. Evidently, $r(a) \vee r(b) \subseteq r(a \cup b)$. On the other hand, if $x \in r(a \cup b)$ and $x \notin r(a), r(b)$, then $x \cap a < x$, $x \cap b < x$ and by the distributive law $x = (x \cap a) \cup (x \cap b)$ which is a contradiction; $\Theta a_i = b_i$ is obvious, completing the proof.

LEMMA 2. *Let L be a finite distributive lattice and k the number of its join-irreducible elements. Then $L_\cup$ may be imbedded as a sublattice in the cardinal product of k lattices isomorphic with L.*

PROOF. Let $\Theta = (b_1, \cdots, b_k)$ and $\Phi = (c_1, \cdots, c_k)$ be join-endomorphisms of the lattices L. Let us consider $\Xi = (b_1 \cap c_1, \cdots, b_k \cap c_k)$ and $\mathrm{H} = (b_1 \cup c_1, \cdots, b_k \cup c_k)$. These are join-endomorphisms in view of Lemma 1. Evidently, $\Xi = \Theta \cap \Phi$ and $\mathrm{H} = \Theta \cup \Phi$. Therefore the join-endomorphisms form a lattice isomorphic to a sublattice of the cardinal product of k lattices isomorphic with L. q.e.d.

Consequently, $L_\cup$ is distributive, for it is a sublattice of a distributive lattice.

On the other hand, if $L_\cup$ is distributive, then so is L, because L may be imbedded in $L_\cup$ by identifying $a \in L$ with the join-endomorphism $\Theta = (a, a, \cdots, a)$.

Summarizing the above statements, we arrive at

THEOREM 3. *All join-endomorphisms of a finite lattice L form a distributive lattice if and only if L is distributive.*

Obviously, our results may be generalized to lattices with 0 elements and finite bounded chains.

LEMMA 2'. *Let L be a distributive lattice with* 0 *element and with finite bounded chains.* $L_{\cup}$ *may be imbedded in the discrete cardinal product of as many copies of L as there exist join-irreducible elements in L.*

THEOREM 3'. *Let L be a lattice with* 0 *element and with finite bounded chains. All join-endomorphisms of L form a distributive lattice if and only if L is distributive.*

BUDAPEST, HUNGARY